involve

a journal of mathematics

Continuous factorization of the identity matrix

Yuying Dai, Ankush Hore, Siqi Jiao, Tianxu Lan and Pavlos Motakis

msp

2020 vol. 13, no. 1

INVOLVE 13:1 (2020)
dx.doi.org/10.2140/involve.2020.13.149

Continuous factorization of the identity matrix

Yuying Dai, Ankush Hore, Siqi Jiao, Tianxu Lan and Pavlos Motakis

(Communicated by Kenneth S. Berenhaut)

We investigate conditions under which the identity matrix I_n can be continuously factorized through a continuous $N \times N$ matrix function A with domain in $\mathbb{R}$. We study the relationship of the dimension N, the diagonal entries of A, and the norm of A to the dimension n and the norms of the matrices that witness the factorization of I_n through A.

1. Introduction

The problem from which this paper draws motivation concerns the relation between the magnitude of the diagonal entries a_{ii} of an $N \times N$ matrix A, the norm of A, and the dimension n of a vector space that A preserves in a satisfying manner, as precisely described below.

Problem 1. Given $N \in \mathbb{N}$ and $\delta > 0$ find the largest $n \in \mathbb{N}$ with the following property: for every $N \times N$ matrix $A = (a_{ij})$, with $\|A\| \le 1$, the diagonal entries of which satisfy $|a_{ii}| \ge \delta$ for $1 \le i \le N$, there exist $n \times N$ and $N \times n$ matrices L and R so that $LAR = I_n$ and $\|L\|\|R\| \le 2/\delta$.

The upper bound imposed on the quantity $\|L\|\|R\|$ must necessarily be at least $1/\delta$ (see Remark 2.12). We use elementary combinatorics and linear algebra to study Problem 1. Subsequently, we allow the entries of A to vary continuously and study the corresponding problem in the solution of which it is additionally required that the preserved vector spaces vary continuously as well. In this article we are mainly concerned with the following.

Problem 2. Given $N \in \mathbb{N}$ and $\delta > 0$ find the largest $n \in \mathbb{N}$ with the following property: for every $N \times N$ continuous matrix function $A : \mathbb{R} \to M_N(\mathbb{R})$ with $\|A(t)\| \le 1$ and $|a_{ii}(t)| \ge \delta$ for $1 \le i \le N$ and all $t \in \mathbb{R}$, there exist continuous matrix functions $L : \mathbb{R} \to M_{n\times N}(\mathbb{R})$ and $R : \mathbb{R} \to M_{n\times N}(\mathbb{R})$ so that $L(t)A(t)R(t) = I_n$ and $\|L(t)\|\|R(t)\| \le 2/\delta$ for all $t \in \mathbb{R}$.

MSC2010: 15A23, 46B07.
Keywords: factorization of the identity, matrices with large diagonal.

We provide lower bounds for n in Problems 1 and 2. In particular, we show that in both cases the order of magnitude of n is at least $\delta^{4/3}N^{1/3}$ (see Theorems 2.10 and 3.9). In the continuous case, this is achieved by using the proof of our estimate for Problem 1 pointwise. In this fashion, we obtain an open cover of $\mathbb{R}$ consisting of intervals on each of which there are continuous matrix functions L and R factoring I_n through A. In the final step, we use these local solutions as building blocks to construct a continuous solution defined on the entire real line.

Although our approach is entirely Euclidean and finite-dimensional, this topic has origins that fit neither description. On a (generally infinite-dimensional) Banach space X with a coordinate system $(e_i)_i$ (e.g., a Schauder basis) every bounded linear operator $A : X \to X$ can be identified with an infinite matrix (a_{ij}). If this matrix has large diagonal, in the sense that $\inf_i |a_{ii}| > 0$, one may ask whether there exist bounded linear operators $L, R : X \to X$ so that $LAR = I_X$. A. D. Andrew [1979] first showed that the answer is yes if $X = L_p$, $1 < p < \infty$, and the coordinate system under consideration is the Haar system. Since then, a number of papers have contributed to the study of this general problem in a variety of infinite-dimensional Banach spaces X; see, e.g., [Laustsen et al. 2018; Lechner 2017; 2018a; 2018b; 2019c; Lechner et al. 2018]. The source of the finite-dimensional version of this problem can be traced to J. Bourgain and L. Tzafriri [1987]. Their paper, among other results, provides an estimate for n in Problem 1 which is of the order $\delta^2 N$ (see Remark 2.11). Within this context, other finite-dimensional non-Euclidean spaces have been studied by R. Lechner [2019a; 2019b]. To the best of our knowledge, the continuous matrix function case has not been considered before.

The paper is divided into two sections. In Section 2 we provide necessary estimates for the norm of a matrix, as well as estimates for the size of families of columns of a given matrix A with the property of being almost orthogonal to one another. Subsequently, we proceed to give an estimate of n for Problem 1 by defining matrices L and R. In Section 3 we explicitly use the definition of L and R of the constant case to find for each t in the domain of the matrix function A $L(t)$ and $R(t)$ as desired. We then extend these solutions continuously on a small interval around t. From there on, we synthesize these local solutions by taking appropriate convex combinations of them and we observe that the desired conclusion is satisfied.

In the sequel, for an $N \times N$ matrix $A = (a_{i,j}) = [a_1 \cdots a_N]$ we will consider the quantity $\theta = \min_i \|a_i\|$, instead of $\delta = \min_i |a_{i,i}|$. As $\delta \le \theta$, our results are slightly more general than already advertised. We have included proofs of some well-known facts and estimates in an effort to make this paper as self-contained as possible. Although all results are stated and proved for matrices with real entries, obvious modifications make them valid for matrices with complex entries as well.

2. The constant case

We use elementary counting tools and tools from linear algebra to factorize the identity matrix through a square matrix with large diagonal. The section is organized into three subsections. The first one includes simple estimates of the norm of a matrix, the second one presents combinatorial arguments that are used to find collections of columns of a matrix that are almost orthogonal to one another, and in the third one we present the construction of the factors L and R and prove their desired properties.

Let us recall some necessary notions used in this section. We identify $\mathbb{R}^n$ with the collection of $n \times 1$ matrices. Thus when we write $x = (x_1, \dots, x_n)$ in reality we mean $x = [x_1 \cdots x_n]^\top$. For $1 \le i \le n$ we denote by e_i the vector in $\mathbb{R}^n$ that has 1 in the i-th entry and 0 in all others. Recall that for a vector $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ we define its Euclidean norm to be the quantity

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}.$$

For two vectors $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ their inner product is the quantity

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i.$$

The Cauchy–Schwarz inequality states that for such x and y we have $|\langle x, y\rangle| \le \|x\|\|y\|$; see, e.g., [Meckes and Meckes 2018, Theorem 4.6]. For an $m \times n$ matrix $A = (a_{i,j})$ when we write $A = [a_1 \cdots a_n]$ we mean that for each $1 \le j \le n$ the entries of the j-th column of A form a_j, i.e., the vector $(a_{1,j}, \dots, a_{m,j})$ in $\mathbb{R}^m$ (a similar notation can be used for writing A with respect to its rows $\alpha_1^\top, \dots, \alpha_m^\top$). Then, for $n \in \mathbb{N}$ the $n \times n$ identity matrix I_n is the matrix $[e_1 \cdots e_n]$. Recall, if A is an $m \times n$ matrix with columns $a_1, \dots, a_n$ and B is a $k \times m$ matrix with rows $\beta_1^\top, \dots, \beta_k^\top$, then the i, j-th entry of the product matrix BA is $\langle \beta_i, a_j\rangle$. For an $m \times n$ matrix A we define its norm to be the quantity

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^n, \|x\| \le 1\}.$$

It is easy to see that for A and x of appropriate dimensions we have $\|Ax\| \le \|A\|\|x\|$. Similarly, by the association property of matrix multiplication, see, e.g., [Meckes and Meckes 2018, Theorem 2.10], for matrices A and B of appropriate dimensions we have $\|AB\| \le \|A\|\|B\|$. Finally, recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called convex if for every $0 \le \lambda \le 1$ and $s, t \in \mathbb{R}$ we have

$$f(\lambda s + (1-\lambda)t) \le \lambda f(s) + (1-\lambda) f(t).$$

A direct computation can be used to show that the square function $f(t) = t^2$ is a convex function.

2A. ***Upper bounds of matrix norms.*** The estimates in this subsection are elementary and well known, yet we include the simple proofs for completeness.

Proposition 2.1. *Let $m, n \in \mathbb{N}$ and $A = [a_1 \cdots a_n]$ be an $m \times n$ matrix. Set*

$$\Lambda = \max_{1\le i\le n} \|a_i\| \quad \textit{and} \quad \lambda = \max_{1\le i\neq j\le n} |\langle a_i, a_j\rangle|.$$

Then $\|A\| \le (\Lambda^2 + (n-1)\lambda)^{1/2}$.

Proof. Let $x = (x_1, \dots, x_m)$ be a vector of norm 1. By convexity of the square function we have

$$\left(\sum_{i=1}^{n} \frac{1}{n}|x_i|\right)^2 \le \frac{1}{n}\sum_{i=1}^{n} |x_i|^2,$$

or

$$\sum_{i=1}^{n} |x_i| \le n^{1/2}\|x\|.$$

Then,

$$\begin{aligned}\|Ax\|^2 = \langle Ax, Ax\rangle &= \sum_{i=1}^{m} x_i^2\|a_i\|^2 + \sum_{i\neq j} x_i x_j\langle a_i, a_j\rangle \\ &\le \Lambda^2\|x\|^2 + \lambda\sum_{i\neq j}|x_i x_j| = \Lambda^2 + \lambda\left(\sum_{i=1}^{n}|x_i|\sum_{i=j}^{n}|x_j| - \sum_{i=1}^{n}|x_i|^2\right) \\ &\le \Lambda^2 + \lambda(n-1).\end{aligned}$$

□

Corollary 2.2. *Let $n \in \mathbb{N}$ and $A = (a_{i,j})$ be an $m \times n$ matrix. Set $d = \max_{i,j} |a_{i,j}|$. Then $\|A\| \le dm^{1/2}n^{1/2}$.*

Proof. Every column of A has norm at most $dm^{1/2}$ and any two different columns have inner product with absolute value at most md^2. A direct application of Proposition 2.1 yields the desired bound. □

Corollary 2.3. *Let $N, n \in \mathbb{N}$ and $A = [a_1 \cdots a_n]$ be an $N \times n$ matrix. Set*

$$\lambda = \max_{1\le i\neq j\le n} |\langle a_i, a_j\rangle| \quad \textit{and} \quad \Delta = \max_{1\le i\le n} |\|a_i\|^2 - 1|.$$

Then $\|A^T A - I_n\| \le n \max\{\lambda, \Delta\}$.

Proof. The i, j entry of the matrix $A^T A - I$ is $\langle a_i, a_j\rangle$ if $i \neq j$ and $\|a_i\|^2 - 1$ if $i = j$. The result follows from applying Corollary 2.2. □

2B. ***Counting arguments.*** In this section we estimate the maximal number of columns of a norm-1 matrix that can have large inner product with a fixed column. This estimate is then used to find collections of columns which are almost orthogonal to one another.

Proposition 2.4. *Let $A = [a_1 \cdots a_M]$ be an $N \times M$ matrix and let $\varepsilon > 0$. Then for every $1 \le i \le M$ the set*

$$B_i^\varepsilon = \{1 \le j \le M : |\langle a_i, a_j \rangle| \ge \varepsilon\}$$

has at most $\|A\|^4/\varepsilon^2$ elements.

Proof. If a_i is the zero vector then the conclusion is obvious and we may therefore assume that it is not. Recall that for any matrix A we have $\|A\| = \|A^\top\|$. Indeed, if x is a norm-1 vector with $\|A\| = \|Ax\|$ then

$$\begin{aligned}\|A\|^2 &= \langle Ax, Ax \rangle = \langle x, A^\top Ax \rangle \\ &\le \|x\| \|A^\top Ax\| \le \|A^\top\| \|A\| \|x\|^2 = \|A^\top\| \|A\|\end{aligned}$$

and hence $\|A\| \le \|A^\top\|$. By symmetry of the argument we also have $\|A^\top\| \le \|A\|$. We calculate

$$\|A\|^2 = \|A^T\|^2 \ge \frac{1}{\|a_i\|^2} \|A^T a_i\|^2 = \frac{1}{\|Ae_i\|^2} \sum_{k=1}^M |\langle a_k, a_i \rangle|^2 \ge \frac{1}{\|A\|^2} \varepsilon^2 \# B_i^\varepsilon. \quad \square$$

Corollary 2.5. *Let $n \in \mathbb{N}$ with $n \ge 2$, $0 < \varepsilon < 1/(n-1)^{1/2}$, and $N \ge n/\varepsilon^2$. Then for any $L \in \mathbb{N}$ and $L \times N$ matrix $A = [a_1 \cdots a_N]$, with $\|A\| \le 1$, there exists $F \subset \{1, \dots, N\}$, with $\#F = n$, so that for $i \ne j \in F$ we have $|\langle a_i, a_j \rangle| < \varepsilon$.*

Proof. Set $i_1 = 1$ and inductively pick $i_2, \dots, i_n$ so that for $2 \le k \le n$

$$i_k \in \{1, \dots, N\} \setminus \big(\{i_1, \dots, i_{k-1}\} \cup \big(\textstyle\bigcup_{m=1}^{k-1} B_{i_m}^\varepsilon\big)\big).$$

This is possible because, by Proposition 2.4, in every inductive step $2 \le k \le n$ the set $\{1, \dots, N\} \setminus \big(\{i_1, \dots, i_{k-1}\} \cup \big(\bigcup_{m=1}^{k-1} B_{i_m}^\varepsilon\big)\big)$ has at least

$$N - \left(k - 1 + \frac{(k-1)}{\varepsilon^2}\right) \ge \frac{n}{\varepsilon^2} - (n-1)\left(1 + \frac{1}{\varepsilon^2}\right) = \frac{1}{\varepsilon^2} - (n-1) > 0$$

elements. $\square$

The following estimate will be used in Section 3. We include it here for consistency.

Corollary 2.6. *Let $n \in \mathbb{N}$ with $n \ge 2$, $0 < \varepsilon < 1/(n-1)^{1/2}$, and $N \ge 5n/\varepsilon^2$. Let $A = [a_1 \cdots a_N]$ be an $N \times N$ matrix with $\|A\| \le 1$. Then for every $F_1, F_2 \subset \{1, \dots, N\}$ with $\#F_1 = \#F_2 = n$ there exists $F_3 \subset \{1, \dots, N\}$ with $\#F_3 = n$ so that the following hold*:

(i) *F_3 is disjoint from $F_1 \cup F_2$.*

(ii) *For any $i \ne j \in F_3$ we have $|\langle a_i, a_j \rangle| < \varepsilon$.*

(iii) *For any $i \in F_3$, $j \in F_1 \cup F_2$, we have $|\langle a_i, a_j \rangle| < \varepsilon$.*

Proof. Define $G = \{1, \dots, N\} \setminus \big((F_1 \cup F_2) \cup \big(\bigcup_{i \in F_1 \cup F_2} B_i^\varepsilon\big)\big)$. Then

$$\#G \geq \frac{5n}{\varepsilon^2} - 2n - \frac{2n}{\varepsilon^2} = \frac{3n}{\varepsilon^2} - 2n \geq \frac{n}{\varepsilon^2}.$$

We now follow the exact same argument as in the proof of Corollary 2.5 to find $F_3 \subset G$ with $\#F_3 = n$ so that for all $i \neq j \in F_3$ we have $|\langle a_i, a_j\rangle| < \varepsilon$. The fact that $F_3 \subset G$ also yields (i) and (iii). □

2C. *The matrices L and R.* We next explicitly define the matrices L and R with the property $LAR = I_n$. For the definition of L and R we use the results from Section 2B. We then use the estimates provided in Section 2A to estimate the quantity $\|L\|\|R\|$.

We now introduce the matrices $L_{(A,F)}$, $R_{(A,F)}$ that are defined using A and a subset F of the columns of A. This dependence on F will also be important in the next section.

Definition 2.7. Let $n \leq N \in \mathbb{N}$, $A = [a_1 \cdots a_N]$ be an $N \times N$ matrix, and $F = \{i_1 < \cdots < i_n\}$ be a subset of $\{1, \dots, N\}$ with $\|a_i\| > 0$ for $i \in F$. For $k = 1, \dots, n$ set $r^k_{(A,F)} = e_{i_k}/\|a_{i_k}\|$; i.e., $r^k_{(A,F)}$ is the N-dimensional vector that has $1/\|a_{i_k}\|$ in the i_k-th entry and zero everywhere else. Define the $N \times n$ and $n \times N$ matrices

$$R_{(A,F)} = [r^1_{(A,F)} \cdots r^n_{(A,F)}] \quad \text{and} \quad L_{(A,F)} = (AR_{(A,F)})^T.$$

Remark 2.8. Observe that for $1 \leq k \leq n$ we have $Ar^k_{(A,F)} = a_{i_k}/\|a_{i_k}\|$ and thus

$$AR_{(A,F)} = \left[\frac{a_{i_1}}{\|a_{i_1}\|} \cdots \frac{a_{i_n}}{\|a_{i_n}\|}\right].$$

Here, we give estimates for the norms of the matrices $L_{(A,F)}$, $R_{(A,F)}$, and $L_{(A,F)}AR_{(A,F)} - I_n$.

Proposition 2.9. *Let $n \leq N \in \mathbb{N}$, A be an $N \times N$ matrix, and $F = \{i_1 < \cdots < i_n\}$ be a subset of $\{1, \dots, N\}$ with $\|a_i\| > 0$ for $i \in F$. Set*

$$\theta = \min_{i \in F} \|a_i\| \quad \textit{and} \quad \varepsilon = \max_{i \neq j \in F} |\langle a_i, a_j\rangle|.$$

Then we have

$$\|R_{(A,F)}\| \leq \theta^{-1}, \quad \|L_{(A,F)}\| \leq 1 + \frac{(n-1)^{1/2}\varepsilon^{1/2}}{\theta},$$

and

$$\|L_{(A,F)}AR_{(A,F)} - I_n\| \leq \frac{n\varepsilon}{\theta^2}.$$

Proof. The first two estimates follow from Proposition 2.1, whereas the third is a consequence of Corollary 2.3. For the first one observe that the columns of $R_{(A,F)}$

all have norm at most $1/\theta$ and they are all orthogonal to one another. For the second one, if we set $b_k = a_{i_k}/\|a_{i_k}\|$ for $1 \le k \le n$ then by Remark 2.8

$$AR_{(A,F)} = [b_1 \cdots b_n].$$

That is, all columns of $AR_{(A,F)}$ have norm 1 and for $1 \le k \ne m \le n$ we have $|\langle b_k, b_m\rangle| \le \varepsilon/\theta^2$. Recall that for $x \ge 0$ we have $(1+x)^{1/2} \le 1 + x^{1/2}$. Thus,

$$\|L_{(A,F)}\| = \|L^T_{(A,F)}\| \le \left(1 + \frac{(n-1)\varepsilon}{\theta^2}\right)^{1/2} \le 1 + \frac{(n-1)^{1/2}\varepsilon^{1/2}}{\theta}.$$

The last estimate follows from Corollary 2.3 directly applied to the matrix $AR_{(A,F)} = [b_1 \cdots b_n]$. □

The following is the main result of this section.

Theorem 2.10. *Let $N \in \mathbb{N}$ and let $A = [a_1 \cdots a_N]$ be an $N \times N$ matrix with $\|A\| \le 1$. If $\theta = \min_{1 \le i \le N} \|a_i\| > 0$ then for every $1 \le n \le \frac{1}{5}\theta^{4/3}N^{1/3}$ there exist $n \times N$ and $N \times n$ matrices L and R respectively so that $LAR = I_n$ and $\|L\|\|R\| \le 2/\theta$.*

Proof. If $n = 1$ the result easily follows by picking any column a_i and defining $R = e_i/\|a_i\|$ and $L = a_i^T/\|a_i\|$. We will therefore assume that $2 \le n \le \frac{1}{5}\theta^{4/3}N^{1/3}$. Define $\varepsilon = \theta^2/(9(n-1))$. This choice of ε ensures that

$$\frac{(n-1)^{1/2}\varepsilon^{1/2}}{\theta} = \frac{1}{3} \quad \text{and} \quad \frac{n\varepsilon}{\theta^2} \le \frac{1}{4}. \tag{1}$$

The two estimates above will be used as assumptions to apply Proposition 2.9; however, we will first use Corollary 2.5. For that purpose, the choice of ε ensures that

$$\frac{n}{\varepsilon^2} = \frac{81n(n-1)^2}{\theta^4} \le \frac{81}{\theta^4}n^3 \le \frac{81}{\theta^4}\frac{\theta^4 N}{125} \le N,$$

i.e., $N \ge n/\varepsilon^2$. It is also easily checked that $\varepsilon < 1/(n-1)^{1/2}$ (because $0 < \theta \le 1$). Thus, by Corollary 2.5, there exists $F \subset \{1, \dots, N\}$ with $\#F = n$ so that for $i \ne j \in F$ we have $|\langle a_i, a_j\rangle| < \varepsilon$.

Consider now the matrices $L_{(A,F)}$ and $R_{(A,F)}$ given by Definition 2.7. By Proposition 2.9 and (1) we deduce

$$\|R_{(A,F)}\| \le \theta^{-1}, \quad \|L_{(A,F))}\| \le \tfrac{4}{3}, \quad \text{and} \quad \|L_{(A,F)}AR_{(A,F)} - I_n\| \le \tfrac{1}{4}. \tag{2}$$

Set $R = R_{(A,F)}$. To define L, recall that if S is an $n \times n$ matrix with $\|S - I_n\| = c < 1$ then S^{-1} exists and $\|S^{-1}\| \le 1/(1-c)$. One way to see this is to observe that $S^{-1} = \sum_{k=0}^{\infty}(I-S)^k$. Therefore, the matrix $(L_{(A,F)}AR_{(A,F)})^{-1}$ is well-defined and has norm at most $1/\left(1 - \frac{1}{4}\right) = \frac{4}{3}$. Finally, set $L = (L_{(A,F)}AR_{(A,F)})^{-1}L_{(A,F)}$ and observe that $LAR = I_n$, $\|R\| \le 1/\theta$, and $\|L\| \le \frac{16}{9} \le 2$. □

Remark 2.11. The theorem above may also be stated for an $N \times N$ matrix A without restrictions on $\|A\|$ as follows: if $\theta = \min_{1 \le i \le N} \|a_i\| > 0$ then for every $1 \le n \le \frac{1}{5}(\theta/\|A\|)^{4/3} N^{1/3}$ there exist $n \times N$ and $N \times n$ matrices L and R respectively so that $LAR = I_n$ and $\|L\|\|R\| \le 2\|A\|/\theta$. This estimate can be compared to [Bourgain and Tzafriri 1987, Theorem 1.2], which yields a similar result: there exist universal constants $c, C > 0$ so that if N, A, and θ are as above then for every $1 \le n \le c(\theta/\|A\|)^2 N$ there exist $n \times N$ and $N \times n$ matrices L and R respectively so that $LAR = I_n$ and $\|L\|\|R\| \le C\|A\|/\theta$. We observe that the result from [Bourgain and Tzafriri 1987] gives a better relation between the dimension n and N, whereas our result gives a better relation between n and the quantity $\theta/\|A\|$.

Remark 2.12. In Theorem 2.10 whenever $n \ge 2$, the quantity $\|L\|\|R\|$ cannot be demanded to be below $1/\theta$. To see this fix $0 < \theta \le 1$ and consider the $N \times N$ diagonal matrix A with first diagonal entry 1 and all other diagonal entries θ. If $n \ge 2$ and we assume that L, R are matrices with $LAR = I_n$, consider the subspace X of $\mathbb{R}^n$ of all vectors orthogonal to $R^\top e_1$. Then X has codimension at most 1 and in particular it is nontrivial; i.e., we may pick $x \in X$ with $\|x\| = 1$. Then,

$$Rx = \sum_{i=1}^{n} \langle e_i, Rx \rangle e_i = \sum_{i=2}^{n} \langle e_i, Rx \rangle e_i$$

and thus we can compute that $ARx = \sum_{i=2}^{n} \theta \langle e_i, Rx \rangle e_i = \theta Rx$. By assumption, $LAR = I_n$ and so $\|x\| = \|LARx\| = \theta \|LRx\| \le \theta \|L\|\|R\|\|x\|$. We conclude $\|L\|\|R\| \ge 1/\theta$.

3. The continuous case

In this section we present the main result of our paper. We demonstrate how the estimates from the previous section can be utilized to continuously factor the identity matrix through a continuous matrix function $A = A(t)$ with large diagonal entries. The idea behind the argument is to first obtain continuous factors $L(t)$, $R(t)$ on small intervals that cover the real line and then stitch the different solutions together in a continuous manner.

Let us recall the notion of a matrix function. We denote by $M_{m \times n}(\mathbb{R})$ the set consisting of all $m \times n$ matrices with real entries. We will write $M_N(\mathbb{R})$ instead of $M_{N \times N}(\mathbb{R})$. A matrix function A is a function with some domain D and range in some $M_{m \times n}(\mathbb{R})$; i.e., it maps every $t \in D$ to some $m \times n$ matrix $A(t) = (a_{i,j}(t))$. Whenever the domain D is equipped with a topology (e.g., when D is a subset of $\mathbb{R}$ with the usual distance) then we say that a matrix function A is continuous whenever all its entries $a_{i,j}$, viewed as scalar functions with domain D, are continuous. It is straightforward that for continuous matrix functions A, B with appropriate dimensions and common domain D the product AB is a continuous matrix function.

The first proposition of this section infers that to prove the main result it is enough to find continuous factors $L(t)$, $R(t)$ so that $L(t)A(t)R(t)$ is sufficiently close to the identity matrix for all t. We begin with two well-known lemmas, which we prove for the sake of completeness.

Lemma 3.1. *Let I be an interval of $\mathbb{R}$, $m, n \in \mathbb{N}$, and $A : I \to M_{m\times n}(\mathbb{R})$ be a matrix function. For any t_0 in I the matrix function A is continuous at t_0 if and only if* $\lim_{t\to t_0} \|A(t) - A(t_0)\| = 0$.

Proof. Note that for any $m \times n$ matrix $B = (b_{i,j})$ and any $1 \leq i_0 \leq m,\ 1 \leq j_0 \leq n$ we have $|b_{i_0,j_0}(t)| = |\langle e_{i_0}, Be_{j_0}\rangle| \leq \|B\|$. By Corollary 2.2 we also have $\|B\| \leq m^{1/2}n^{1/2} \max_{i,j} |b_{i,j}|$. For $t \in I$ we apply our observation to the matrix $B = A(t) - A(t_0)$ to obtain that for any $1 \leq i_0 \leq m,\ 1 \leq j_0 \leq n$ we have

$$|a_{i_0,j_0}(t) - a_{i_0,j_0}(t_0)| \leq \|A(t) - A(t_0)\| \leq m^{1/2}n^{1/2} \max_{i,j} |a_{i,j}(t) - a_{i,j}(t_0)|.$$

The desired conclusion immediately follows. □

Lemma 3.2. *Let $N \in \mathbb{N}$, I be an interval of $\mathbb{R}$, and $A : I \to M_N(\mathbb{R})$ be a continuous matrix function such that $A(t)$ is invertible for all $t \in I$. Then $A^{-1} : I \to M_N(\mathbb{R})$ is a continuous matrix function.*

Proof. We fix t_0 in I and estimate $\|A^{-1}(t) - A^{-1}(t_0)\|$ for t close to t_0. Observe that $A^{-1}(t) - A^{-1}(t_0) = A^{-1}(t)(A(t_0) - A(t))A^{-1}(t_0)$. We deduce

$$\|A^{-1}(t) - A^{-1}(t_0)\| \leq \|A^{-1}(t)\|\|A(t_0) - A(t)\|\|A^{-1}(t_0)\| \tag{3}$$

and

$$\|A^{-1}(t)\| \leq \|A^{-1}(t_0)\| + \|A^{-1}(t)\|\|A(t_0) - A(t)\|\|A^{-1}(t_0)\|,$$

which, solving for $\|A^{-1}(t)\|$, yields

$$\|A^{-1}(t)\| \leq \frac{\|A^{-1}(t_0)\|}{1 - \|A(t_0) - A(t)\|\|A^{-1}(t_0)\|}. \tag{4}$$

The quantity on the right-hand side of the inequality above is well-defined for t sufficiently close to t_0. We plug (4) into (3) to get rid of the term $\|A^{-1}(t)\|$:

$$\|A^{-1}(t) - A^{-1}(t_0)\| \leq \frac{\|A^{-1}(t_0)\|^2\|A(t_0) - A(t)\|}{(1 - \|A(t_0) - A(t)\|\|A^{-1}(t_0)\|)}.$$

This estimate, in conjunction with Lemma 3.1, yields that the continuity of $A : I \to M_N(\mathbb{R})$ at t_0 implies the continuity of $A^{-1} : I \to M_N(\mathbb{R})$ at t_0. □

Proposition 3.3. *Let $n \leq N \in \mathbb{N}$, I be an interval of $\mathbb{R}$, and $A : I \to M_N(\mathbb{R})$ be a continuous matrix function. Assume that $0 < C < 1$, $\Delta \geq 0$, and $L : I \to M_{n\times N}(\mathbb{R})$, $R : I \to M_{N\times n}(\mathbb{R})$ are continuous matrix functions so that for all $t \in I$ we have $\|L(t)A(t)R(t) - I_n\| \leq C$ and $\|L(t)\|\|R(t)\| \leq \Delta$. Then there exist continuous* *matrix functions* $\widetilde{L} : I \to M_{n\times N}(\mathbb{R})$, $\widetilde{R} : I \to M_{N\times n}(\mathbb{R})$ *so that for all* $t \in I$ *we have* $\widetilde{L}(t)A(t)\widetilde{R}(t) = I_n$ *and* $\|\widetilde{L}\|\|\widetilde{R}\| \leq \Delta/(1-C)$.

Proof. For each $t \in I$, because we have that $\|L(t)A(t)R(t) - I_n\| \leq C$, the matrix $L(t)A(t)R(t)$ is invertible, and in particular $\|(L(t)A(t)R(t))^{-1}\| \leq 1/(1-C)$. By Lemma 3.2 the matrix function $(LAR)^{-1} : I \to M_n(\mathbb{R})$ is continuous. We define $\widetilde{L} : I \to M_{n\times N}(\mathbb{R})$ as $\widetilde{L}(t) = (L(t)A(t)R(t))^{-1}L(t)$ and just set $\widetilde{R} = R$. Both $\widetilde{L}$ and $\widetilde{R}$ are continuous and clearly for all $t \in I$ we have $\widetilde{L}(t)A(t)\widetilde{R}(t) = I_n$. Additionally, for $t \in I$ we have $\|\widetilde{L}(t)\|\|\widetilde{R}\| \leq \|(L(t)A(t)R(t))^{-1}\|\|L\|\|R\| \leq \Delta/(1-C)$. □

Recall the matrices $L_{(A,F)}$ and $R_{(A,F)}$ from Definition 2.7. In the sequel we will start with two versions of pairs $L_{(A,F_1)}$, $R_{(A,F_1)}$, $L_{(A,F_2)}$ and $R_{(A,F_2)}$, and a scalar $0 \leq \lambda \leq 1$. We will combine them into a new pair $L^{\lambda}_{(A,F_1,F_2)}$ and $R^{\lambda}_{(A,F_1,F_2)}$.

Definition 3.4. Let $n \leq N \in \mathbb{N}$, let $A = [a_1 \cdots a_N]$ be an $N \times N$ matrix, let $F_1 = \{i_1 < \cdots < i_n\}$, $F_2 = \{j_1 < \cdots < j_n\}$ be disjoint subsets of $\{1, \ldots, N\}$, and let $0 \leq \lambda \leq 1$. We assume that $\|a_i\| > 0$ for $i \in F_1 \cup F_2$. Define the $N \times n$ and $n \times N$ matrices

$$R^{\lambda}_{(A,F_1,F_2)} = \lambda^{1/2}R_{(A,F_1)} + (1-\lambda)^{1/2}R_{(A,F_2)},$$
$$L^{\lambda}_{(A,F_1,F_2)} = \lambda^{1/2}L_{(A,F_1)} + (1-\lambda)^{1/2}L_{(A,F_2)}.$$

Remark 3.5. The matrices $R^{\lambda}_{(A,F_1,F_2)}$, $L^{\lambda}_{(A,F_1,F_2)}$ lie "between" $R_{(A,F_1)}$, $R_{(A,F_2)}$ and $L_{(A,F_1)}$, $L_{(A,F_2)}$ respectively. Clearly, if $\lambda = 1$ then

$$R^{1}_{(A,F_1,F_2)} = R_{(A,F_1)}, \quad L^{1}_{(A,F_1,F_2)} = L_{(A,F_1)}$$

and if $\lambda = 0$ then

$$R^{0}_{(A,F_1,F_2)} = R_{(A,F_2)}, \quad L^{0}_{(A,F_1,F_2)} = L_{(A,F_2)}.$$

Remark 3.6. Recall that for $k = 1, \ldots, n$, we have $R_{(A,F_1)}e_k = e_{i_k}/\|a_{i_k}\|$ and $R_{(A,F_2)}e_k = e_{j_k}/\|a_{j_k}\|$, which means that

$$R^{\lambda}_{(A,F_1,F_2)}e_k = \lambda^{1/2}e_{i_k}/\|a_{i_k}\| + (1-\lambda)^{1/2}e_{j_k}/\|a_{j_k}\|.$$

Therefore

$$AR^{\lambda}_{(A,F_1,F_2)} = \left[\left(\lambda^{1/2}\frac{a_{i_1}}{\|a_{i_1}\|} + (1-\lambda)^{1/2}\frac{a_{j_1}}{\|a_{j_1}\|}\right) \cdots \left(\lambda^{1/2}\frac{a_{i_n}}{\|a_{i_n}\|} + (1-\lambda)^{1/2}\frac{a_{j_n}}{\|a_{j_n}\|}\right)\right].$$

Remark 3.7. It will be important to note for the sequel the following: if $n \leq N \in \mathbb{N}$, I is an interval of $\mathbb{R}$, $\lambda : I \to [0, 1]$ is a continuous scalar function, $A = [a_1 \cdots a_N] : I \to M_N(\mathbb{R})$ is a continuous matrix function, and F_1, F_2 are disjoint subsets of $\{1, \ldots, N\}$ with $\#F_1 = \#F_2 = n$ so that $\|a_i(t)\| > 0$ for all $i \in F_1 \cup F_2$ and $t \in I$, then the matrix functions $R^{\lambda(t)}_{(F_1,F_2,A(t))} : I \to M_{N\times n}(\mathbb{R})$, $L^{\lambda(t)}_{(F_1,F_2,A(t))} : I \to M_{n\times N}(\mathbb{R})$ are both continuous.

The following proposition basically states that if we have appropriately picked $L_{(A,F_1)}$, $R_{(A,F_1)}$, $L_{(A,F_2)}$ and $R_{(A,F_2)}$ then for any scalar $0 \le \lambda \le 1$ the new pair $L^{\lambda}_{(A,F_1,F_2)}$, $R^{\lambda}_{(A,F_1,F_2)}$ satisfies a conclusion similar to that of Proposition 2.9.

Proposition 3.8. *Let* $n \le N \in \mathbb{N}$, *let* $A = [a_1 \cdots a_N]$ *be an* $N \times N$ *matrix, let* $F_1 = \{i_1 < \cdots < i_n\}$, $F_2 = \{j_1 < \cdots < j_n\}$ *be disjoint subsets of* $\{1, \ldots, N\}$ *and let* $0 \le \lambda \le 1$. *Set*

$$\theta = \min_{i \in F_1 \cup F_2} \|a_i\| \quad \textit{and} \quad \varepsilon = \max_{i \neq j \in F_1 \cup F_2} |\langle a_i, a_j \rangle|.$$

If $\theta > 0$ *then we have*

$$\|R^{\lambda}_{(A,F_1,F_2)}\| \le \theta^{-1}, \quad \|L^{\lambda}_{(A,F_1,F_2)}\| \le 1 + \frac{(2n)^{1/2}\varepsilon^{1/2}}{\theta},$$

and

$$\|L^{\lambda}_{(A,F_1,F_2)} A R^{\lambda}_{(A,F_1,F_2)} - I_n\| \le \frac{2n\varepsilon}{\theta^2}.$$

Proof. This proof is very similar in spirit to that of Proposition 2.9. We examine for $1 \le k \le n$ column k of $R^{\lambda}_{(A,F_1,F_2)}$, i.e., the vector $R^{\lambda}_{(A,F_1,F_2)} e_k$:

$$\|R^{\lambda}_{(A,F_1,F_2)} e_k\|^2 = \frac{\lambda}{\|a_{i_k}\|^2} + \frac{(1-\lambda)}{\|a_{j_k}\|^2} \le \frac{1}{\theta^2}.$$

It is also easy to see that for $k_1 \neq k_2$ the columns of $R^{\lambda}_{(A,F_1,F_2)}$ are orthogonal. Therefore, by Proposition 2.1 we have $\|R^{\lambda}_{(A,F_1,F_2)}\| \le 1/\theta$.

For the second estimate, we define, for $1 \le k \le n$,

$$b_k = \frac{\lambda^{1/2} a_{i_k}}{\|a_{i_k}\|} + \frac{(1-\lambda)^{1/2} a_{j_k}}{\|a_{j_k}\|}.$$

By Remark 3.6 we have

$$(L^{\lambda}_{(A,F_1,F_2)})^T = A R^{\lambda}_{(A,F_1,F_2)} = [b_1 \cdots b_k].$$

We calculate, for $1 \le k \le n$, the norm of column k:

$$\begin{aligned}\|b_k\|^2 &= \left\langle \frac{\lambda^{1/2}}{\|a_{i_k}\|} a_{i_k} + \frac{(1-\lambda)^{1/2}}{\|a_{j_k}\|} a_{j_k}, \frac{\lambda^{1/2}}{\|a_{i_k}\|} a_{i_k} + \frac{(1-\lambda)^{1/2}}{\|a_{j_k}\|} a_{j_k} \right\rangle \\ &= \lambda + (1-\lambda) + 2\lambda^{1/2}(1-\lambda)^{1/2} \left\langle \frac{a_{i_k}}{\|a_{i_k}\|}, \frac{a_{j_k}}{\|a_{j_k}\|} \right\rangle;\end{aligned}$$

that is,

$$\big|\|b_k\|^2 - 1\big| \le 2\lambda^{1/2}(1-\lambda)^{1/2} \frac{\varepsilon}{\theta^2} \le \frac{\varepsilon}{\theta^2} \quad \text{for } 1 \le k \le n, \tag{5}$$

where we used $0 \le 2\lambda^{1/2}(1-\lambda)^{1/2} \le 1$ for $0 \le \lambda \le 1$. In particular, we have

$$\|b_k\| \le \left(1 + \frac{\varepsilon}{\theta^2}\right)^{1/2} \quad \text{for } 1 \le k \le n. \tag{6}$$

Next, we will show that

$$\text{for } 1 \le k_1 \ne k_2 \le n, \quad |\langle b_{k_1}, b_{k_2}\rangle| \le 2\frac{\varepsilon}{\theta^2}. \tag{7}$$

We have

$$\begin{aligned}|\langle b_{k_1}, b_{k_2}\rangle| \le \lambda\left|\left\langle \frac{a_{i_{k_1}}}{\|a_{i_{k_1}}\|}, \frac{a_{i_{k_2}}}{\|a_{i_{k_2}}\|}\right\rangle\right| + (1-\lambda)\left|\left\langle \frac{a_{j_{k_1}}}{\|a_{j_{k_1}}\|}, \frac{a_{j_{k_2}}}{\|a_{j_{k_2}}\|}\right\rangle\right| \\ + \lambda^{1/2}(1-\lambda)^{1/2}\left(\left|\left\langle \frac{a_{i_{k_1}}}{\|a_{i_{k_1}}\|}, \frac{a_{j_{k_2}}}{\|a_{j_{k_2}}\|}\right\rangle\right| + \left|\left\langle \frac{a_{j_{k_1}}}{\|a_{j_{k_1}}\|}, \frac{a_{i_{k_2}}}{\|a_{i_{k_2}}\|}\right\rangle\right|\right) \\ \le \frac{\varepsilon}{\theta^2} + 2\lambda^{1/2}(1-\lambda)^{1/2}\frac{\varepsilon}{\theta^2} \le 2\frac{\varepsilon}{\theta^2}.\end{aligned}$$

We now apply Proposition 2.1, which by (6) and (7), gives that

$$\begin{aligned}\|L^{\lambda}_{(A,F_1,F_2)}\| = \|AR^{\lambda}_{(A,F_1,F_2)}\| \le \left(1 + \frac{\varepsilon}{\theta^2} + (n-1)2\frac{\varepsilon}{\theta^2}\right)^{1/2} \\ \le 1 + (2n-1)^{1/2}\frac{\varepsilon^{1/2}}{\theta} \le 1 + (2n)^{1/2}\frac{\varepsilon^{1/2}}{\theta}.\end{aligned}$$

The final estimate follows from Corollary 2.3 directly applied to the matrix $AR^{\lambda}_{(A,F_1,F_2)} = [b_1 \cdots b_k]$ and (5), (7). □

We are finally ready to state and prove the main result of this paper.

Theorem 3.9. *Let $N \in \mathbb{N}$, let I be an interval of $\mathbb{R}$ and let $A = [a_1 \cdots a_N] : I \to M_N(\mathbb{R})$ be a continuous function so that the following hold*:

(i) *For $t \in I$ we have $\|A(t)\| \le 1$.*

(ii) $\theta = \inf_{t\in I} \min_{1\le i\le N} \|a_i(t)\| > 0$.

Then for every $1 \le n \le \frac{1}{12}\theta^{4/3}N^{1/3}$ there exist continuous functions $L : I \to M_{n\times N}(\mathbb{R})$ and $R : I \to M_{N\times n}(\mathbb{R})$ so that for all $t \in I$ we have $L(t)A(t)R(t) = I_n$ and $\|L(t)\|\|R(t)\| \le 2/\theta$.

Proof. By Proposition 3.3 it is sufficient to find continuous $L(t)$, $R(t)$ so that for all $t \in I$ we have $\|L(t)A(t)R(t) - I_n\| \le \frac{1}{4}$ and $\|L(t)\|\|R(t)\| \le 4/(3\theta)$.

The case $n = 1$ is treated easily by taking an arbitrary $1 \le i \le N$ and defining $R(t) = e_i/\|a_i(t)\|$ and $L(t) = a_i(t)/\|a_i(t)\|$; thus we assume that $2 \le n \le \frac{1}{12}\theta^{4/3}N^{1/3}$. Define $\varepsilon = \theta^2/(18n)$. This choice of ε is related to the estimates from Proposition 3.8 and also Corollaries 2.5 and 2.6. Let us note that we have

$$\frac{(2n)^{1/2}\varepsilon^{1/2}}{\theta} = \frac{1}{3} \quad \text{and} \quad \frac{2n\varepsilon}{\theta^2} \le \frac{1}{4} \tag{8}$$

and also

$$5\frac{n}{\varepsilon^2} = 5\frac{18^2 n^3}{\theta^4} \le 5\frac{18^2}{\theta^4}\frac{\theta^4 N}{12^3} \le N. \tag{9}$$

Let us assume henceforth that $I = [0, \infty)$. The case $I = \mathbb{R}$ is treated by performing the same argument on both sides of 0. Other cases are treated similarly. Otherwise they can be deduced from the previous two cases by using, e.g., that any open interval is homeomorphic to $\mathbb{R}$ and every half-open interval is homeomorphic to $[0, +\infty)$, and any continuous function on a closed bounded interval $[t_1, t_2]$ can be continuously extended to $\mathbb{R}$ by assigning the value $A(t_1)$ to each $t \le t_1$ and the value $A(t_2)$ to each $t \ge t_2$.

We start by finding a strictly increasing sequence $0 = t_0 < t_1 < t_2 < \cdots$ with $\lim_m t_m = \infty$ so that for all $m \in \mathbb{N}$ there exists $F_m \subset \{1, \ldots, N\}$ with

(a) $\#F_m = n$ and

(b) for all $i \neq j \in F_m$ and $t_{m-1} \le t \le t_m$ we have $|\langle a_i(t), a_j(t)\rangle| < \varepsilon$.

This is achieved as follows. For each $r \in [0, 1]$ we use Corollary 2.5 to find $F_r \subset \{1, \ldots, N\}$ so that for all $i \neq j \in F_r$ we have $|\langle a_i(t), a_j(t)\rangle| < \varepsilon$. Because A is continuous, we may find a small open interval I_r containing r (half-open if $r = 0$) so that for all $i \neq j \in F_r$ and $t \in I_r$ we still have $|\langle a_i(t), a_j(t)\rangle| < \varepsilon$. Because $[0, 1] \subset \bigcup_{r\in[0,1]} I_r$ and the interval $[0, 1]$ is compact there must exist $r_1 < \cdots < r_{m_1}$ so that $[0, 1] \subset \bigcup_{i=1}^{m_1} I_{r_i}$. By perhaps getting rid of a few intervals we may assume that none of them is contained in the union of the others. Then, by perhaps making some of the intervals a little shorter we may assume that $\sup(I_{r_i}) \le r_{i+1}$ for $1 \le i < m_1 - 1$ and $r_{i-1} \le \inf(I_{r_i})$ for $1 < i \le m_1$. In other words, for $i = 1, \ldots, m_1 - 1$ we have $\varnothing \neq I_{r_i} \cap I_{r_{i+1}} \subset (r_i, r_{i+1})$. Define $t_0 = 0,\ t_{m_1} = 1$ and for $1 \le i < m_1$ pick $t_i \in (r_i, r_{i+1})$. If we then set $F_i = F_{r_i}$ for $1 \le i \le m_1$ we obtain that (a) and (b) are satisfied up to $m = m_1$. For $k = 2, 3, \ldots$ repeat the same argument on $[k-1, k]$ to find $(t_i)_{i=m_{k-1}+1}^{m_k}$ and $(F_i)_{i=m_{k-1}+1}^{m_k}$ that satisfy (a) and (b).

The next step is to apply for each $m = 1, 2, \ldots$ Corollary 2.6 to the matrix $A(t_m)$ and the sets F_m, F_{m+1}. By doing so we find a set $G_m \subset \{1, \ldots, N\} \setminus (F_m \cup F_{m+1})$ with $\#G_m = n$ so that for all $i \neq j$ with $i \in G_m$ and $j \in G_m \cup F_m \cup F_{m+1}$ we have $|\langle a_i(t_m), a_j(t_m)\rangle| < \varepsilon$. We now use the continuity of A once more to find $s_m < t_m < u_m$ so that for all $t \in (s_m, u_m)$ the above hold as well. By perhaps moving s_m, u_m a bit closer to t_m we have the following situation:

(c) $0 = t_0 < s_1 < t_1 < u_1 < s_2 < t_2 < u_2 < s_3 < t_3 < u_3 < \cdots$.

(d) For $m = 1, 2, \ldots$ we have $G_m \subset \{1, \ldots, N\} \setminus (F_m \cup F_{m+1})$ with $\#G_m = n$ so that for all $t \in (s_m, u_m)$, $i \neq j$, with $i \in G_m$ and $j \in G_m \cup F_m \cup F_{m-1}$, we have $|\langle a_i(t), a_j(t)\rangle| < \varepsilon$.

We are finally ready to define $L(t)$ and $R(t)$. Set $0 = u_0$. For each $m = 1, 2, \ldots$ take a continuous $\lambda_m : [s_m, u_m] \to [0, 1]$ with $\lambda_m(s_m) = \lambda_m(u_m) = 1$ and $\lambda_m(t_m) = 0$:

(A) For $m = 0, 1, \ldots$ and $t \in [u_m, s_{m+1}]$ set $R(t) = R_{(A(t), F_{m+1})}$.

(B) For $m = 1, 2, \ldots$ and $t \in [s_m, t_m]$ define

$$R(t) = R^{\lambda_m(t)}_{A(t), F_m, G_m}.$$

We point out that, by Remark 3.5,

$$R(s_m) = R^1_{A(s_m), F_m, G_m} = R_{(A(s_m), F_m)} \quad \text{and} \quad R(t_m) = R^0_{A(t_m), F_m, G_m} = R_{A(t_m), G_m}.$$

(C) For $m = 1, 2, \ldots$ and $t \in [t_m, u_m]$ define

$$R(t) = R^{\lambda_m(t)}_{A(t), F_{m+1}, G_m}.$$

Once more, by Remark 3.5,

$$R(t_m) = R^0_{A(t_m), F_{m+1}, G_m} = R_{(A(t_m), G_m)} \quad \text{and} \quad R(u_m) = R^1_{A(u_m), F_{m+1}, G_m} = R_{A(u_m), F_{m+1}}.$$

By Remark 3.7, in each case (A), (B), and (C) the function R is continuous and the values at the endpoints of the corresponding intervals match. Thus R defines a continuous function on I and thus so does $L = (AR)^T$.

We next wish to show that for $t \geq 0$ we have $\|L(t)A(t)R(t) - I_n\| \leq \frac{1}{4}$ and $\|L(t)\|\|R(t)\| \leq 4/(3\theta)$ and the proof will be complete. If $t \in [u_m, s_{m+1}]$, for some $m \in \mathbb{N}$, then this follows from definition (A) above and (8) applied to Proposition 2.9. If $t \in [s_m, u_m]$ for some $m \in \mathbb{N}$ then this follows from definition (B) or (C), property (d), and (8) applied to Proposition 3.8. □

We conclude with some open questions regarding the topic of the paper.

Question 1. As was pointed out in Remark 2.11, [Bourgain and Tzafriri 1987] implies a version of Theorem 2.10 (in which $2/\theta$ is replaced by C/θ and C is a nonexplicit finite constant) with an estimate $n \gtrsim \theta^2 N$. This is better than our estimate $n \gtrsim \theta^{3/4} N^{1/3}$, provided that $N \gtrsim 1/\theta$. Can the probabilistic technique from [Bourgain and Tzafriri 1987] be used to obtain a similar version of the continuous Theorem 3.9 with an estimate $n \gtrsim \theta^2 N$?

Question 2. For the theorem in the continuous case, we considered $A : I \to M_N(\mathbb{R})$, where I is an interval of $\mathbb{R}$. We conjecture that a version of Theorem 3.9 is also true for a continuous matrix function $A : \mathbb{R}^d \to M_N(\mathbb{R})$. What is the relation between d, N, θ, and the dimension n in the conclusion of such a theorem?

For $1 \leq p \leq \infty$ and an $N \times N$ matrix A let $\|A\|_p$ denote the quantity $\max\{\|Ax\|_p : \|x\|_p \leq 1\}$. In particular, $\|A\| = \|A\|_2$.

Question 3. The methods used in this paper rely heavily on properties of the Euclidean norm. In the statement of Theorem 3.9 we may replace condition (i) with $\|A(t)\|_p \leq 1$. It would be interesting to prove a version of this theorem, as different methods might be necessary.

Acknowledgements

We would like to thank the anonymous referee for recommending the inclusion of Question 1.

This work was done in the research project "Continuous factorization of the identity matrix" at the Illinois Geometry Lab in Spring 2019. Dai, Jiao, and Lan participated as undergraduate scholars, Hore served as graduate student team leader, and Motakis as faculty mentor. The project was supported by the National Science Foundation under grant number DMS-1449269. Motakis was supported by the National Science Foundation under grant number DMS-1912897.

References

[Andrew 1979] A. D. Andrew, "Perturbations of Schauder bases in the spaces $C(K)$ and L^p, $p > 1$", *Studia Math.* **65**:3 (1979), 287–298. MR Zbl

[Bourgain and Tzafriri 1987] J. Bourgain and L. Tzafriri, "Invertibility of "large" submatrices with applications to the geometry of Banach spaces and harmonic analysis", *Israel J. Math.* **57**:2 (1987), 137–224. MR Zbl

[Laustsen et al. 2018] N. J. Laustsen, R. Lechner, and P. F. X. Müller, "Factorization of the identity through operators with large diagonal", *J. Funct. Anal.* **275**:11 (2018), 3169–3207. MR Zbl

[Lechner 2017] R. Lechner, "Direct sums of finite dimensional SL_n^∞ spaces", preprint, 2017. arXiv

[Lechner 2018a] R. Lechner, "Factorization in mixed norm Hardy and BMO spaces", *Studia Math.* **242**:3 (2018), 231–265. MR Zbl

[Lechner 2018b] R. Lechner, "Factorization in SL^∞", *Israel J. Math.* **226**:2 (2018), 957–991. MR Zbl

[Lechner 2019a] R. Lechner, "Dimension dependence of factorization problems: biparameter Hardy spaces", *Proc. Amer. Math. Soc.* **147**:4 (2019), 1639–1652. MR Zbl

[Lechner 2019b] R. Lechner, "Dimension dependence of factorization problems: Hardy spaces and SL_n^∞", *Israel J. Math.* **232**:2 (2019), 677–693. MR Zbl

[Lechner 2019c] R. Lechner, "Subsymmetric weak* Schauder bases and factorization of the identity", *Studia Math.* **248**:3 (2019), 295–319. MR Zbl

[Lechner et al. 2018] R. Lechner, P. Motakis, P. F. X. Müller, and T. Schlumprecht, "Strategically reproducible bases and the factorization property", preprint, 2018. arXiv

[Meckes and Meckes 2018] E. S. Meckes and M. W. Meckes, *Linear algebra*, Cambridge University Press, 2018. Zbl

Received: 2019-08-31 Revised: 2019-10-07 Accepted: 2019-10-14

yuyingd2@illinois.edu *Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, United States*

ahore2@illinois.edu *Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, United States*

sjiao2@illinois.edu *Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, United States* tl2971@columbia.edu *Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, United States*

pmotakis@illinois.edu *Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, United States*

mathematical sciences publishers

msp

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2020 is US $205/year for the electronic version, and $275/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2020 vol. 13 no. 1