Fall 2016, Math 409, Section 502
Further practice problems for the second Midterm Exam

(i) If $(x_n)_n$ is an unbounded real sequence, show that

$$\textbf{either} \limsup_n x_n = \infty \textbf{ or } \liminf_n x_n = -\infty.$$

(ii) Let $E \subseteq \mathbb{R}$, $E \neq \varnothing$, $f : E \to \mathbb{R}$ be a continuous function and $(x_n)_n$ be a bounded real sequence. Prove that

$$(*) \qquad f(\limsup_n x_n) \leqslant \limsup_n f(x_n) \text{ and } \liminf_n f(x_n) \leqslant f(\liminf_n x_n).$$

(iii) Let $E \subseteq \mathbb{R}$, $E \neq \varnothing$, and $f : E \to \mathbb{R}$ be a function and assume that for every bounded sequence $(x_n)_n$ in $\mathbb{R}$ $(*)$ holds. Show that f is continuous.

(iv) Let $E \subseteq \mathbb{R}$, $E \neq \varnothing$ and $f, g : E \to \mathbb{R}$ be continuous function. Prove that $f \vee g : E \to \mathbb{R}$, with $(f \vee g)(x) = \max\{f(x), g(x)\}$, is a continuous function. Give an example showing that the conclusion is false if continuity is replaced with differentiability.

(v) Let $f : [0,1] \to \mathbb{R}$ be a continuous function and assume that $f(x) > 0$ for all $x \in [0,1]$. Prove that there exists $c > 0$, so that $f(x) \geqslant c$ for all $x \in [0,1]$.

(vi) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) + \infty.$$

Prove that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = \inf\{f(x) : x \in \mathbb{R}\}$.

(vii) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that for all $x \in \mathbb{R}$ we have $f(x) \neq 0$. Prove that precisely one of the following holds:

(a) $f(x) > 0$ for all $x \in \mathbb{R}$, or
(b) $f(x) < 0$ for all $x \in \mathbb{R}$.

(viii) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that $f^{-1}[\mathbb{R} \setminus \mathbb{Q}] \subseteq \mathbb{Q}$. Prove that f is constant, i.e. there exists c in $\mathbb{R}$ with $f(x) = c$ for all $x \in \mathbb{R}$. (*Hint:* use an appropriate continuity theorem and a cardinality argument)

(ix) Let $f : [0, 1) \to \mathbb{R}$ be a bounded and continuous function that is **not** uniformly continuous. Show that there exists a Cauchy sequence $(x_n)_n$ in $[0, 1)$ so that

$$\liminf_n f(x_n) < \limsup_n f(x_n).$$

(x) Let $f : [0, +\infty) \to \mathbb{R}$ be a uniformly continuous function. Prove that there exist positive constants C and M, so that for all $x \geqslant 0$ we have $|f(x)| \leqslant Cx + M$. (*Hint*: use the definition of uniform continuity and for each $x > 0$ appropriately choose $0 = x_0 < \cdots < x_n = x$ and write $f(x) = f(0) + \sum_{k=1}^{n}(f(x_k) - f(x_{k-1}))$)

(xi) Let $f : [0, +\infty) \to \mathbb{R}$ be a uniformly continuous function. Prove that

$$\lim_{x \to +\infty} (x^2 - f(x)) = +\infty.$$

Hint: use (x).

(xii) Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period $T \in \mathbb{R}$ (i.e. $f(x) = f(x + T)$ for all $x \in \mathbb{R}$). If f is differentiable, show that f' is periodic with period T.

(xiii) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f' : \mathbb{R} \to \mathbb{R}$ is bounded, show that f is uniformly continuous.

(xiv) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function that is differentiable on $(0, \infty)$. Assume that for all $x > 0$, $|f'(x)| \leqslant 1/x^2$. Prove that f is bounded.
Hint: For each $m \geqslant 2$, $f(m) = f(1) + \sum_{k=2}^{m}(f(k) - f(k-1)))$. You also need the mean value theorem and the fact that $\sum_n 1/n^2 < \infty$.

(xv) Let $f : [a, b] \to \mathbb{R}$ be a decreasing function. Prove the following:

(a) For all $c \in [a, b)$ we have $f(c) \leqslant \lim_{x \to c^+} f(x) = \inf\{f(x) : c < x \leqslant b\}$ and

(b) For all $c \in (a, b]$ we have $f(c) \geqslant \lim_{x \to c^-} f(x) = \sup\{f(x) : a \leqslant x < c\}$.

(xvi) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that $f'[\mathbb{R}]$, i.e. the image of $\mathbb{R}$ under f', is an interval.

(xvii) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f' : \mathbb{R} \to \mathbb{R}$ is continuous, show that for every open interval I, there exists an open interval $J \subseteq I$ so that f is either constant or strictly monotone on J.