

THE SCALAR-PLUS-COMPACT PROPERTY IN SPACES WITHOUT REFLEXIVE SUBSPACES

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ABSTRACT. A hereditarily indecomposable Banach space \mathfrak{X}_{nr} is constructed that is the first known example of a \mathcal{L}_∞ -space not containing c_0 , ℓ_1 , or reflexive subspaces, and it answers a question posed by J. Bourgain. Moreover, the space \mathfrak{X}_{nr} satisfies the “scalar-plus-compact” property and is the first known space without reflexive subspaces having this property. It is constructed using the Bourgain–Delbaen method in combination with a recent version of saturation under constraints in a mixed-Tsirelson setting. As a result, the space \mathfrak{X}_{nr} has a shrinking finite-dimensional decomposition and does not contain a boundedly complete sequence.

INTRODUCTION

The class of \mathcal{L}_∞ hereditarily indecomposable (HI) spaces is perhaps the most interesting class of nonclassical Banach spaces. This happens since in such a space \mathfrak{X} conditional and unconditional structures strongly coexist. More precisely, in \mathfrak{X} there is no unconditional basic sequence, and on the other hand $\mathfrak{X} = \overline{\bigcup_n F_n}$ with $(F_n)_n$ an increasing sequence of finite-dimensional subspaces, each one C -isomorphic to $\ell_\infty^{\dim F_n}$. The latter yields that \mathfrak{X} admits the Gordon–Lewis LUST [GL]. It is an important open problem whether there exists a reflexive HI space with LUST. As a consequence of the above described peculiar structure, the scalar-plus-compact property is satisfied by several \mathcal{L}_∞ -spaces (e.g., [AH],[A et al.]). In this paper we present a new \mathcal{L}_∞ HI space denoted \mathfrak{X}_{nr} . This is the first example of a \mathcal{L}_∞ HI space not containing a reflexive subspace. Moreover, every $T \in \mathcal{L}(\mathfrak{X}_{nr})$ is of the form $\lambda I + K$ with K a compact operator, and thus, this is the first example of a space without reflexive subspaces satisfying the scalar-plus-compact property.

In 1981 J. Bourgain [B, Problem 4, page 46] suggested the class of \mathcal{L}_∞ -spaces as a possible subclass of Banach spaces where the problem “ ℓ_1 , c_0 , or reflexive subspace” could have a positive answer. This would be in line with a multitude of results hinting that \mathcal{L}_∞ -spaces exhibit a highly canonical structure. For example, such spaces have the aforementioned Gordon–Lewis LUST, and the dual of a separable \mathcal{L}_∞ -space is either isomorphic to ℓ_1 or isomorphic to $\mathcal{M}[0, 1]$ [St]. Furthermore, as it was proved by H. P. Rosenthal in [R], whenever a \mathcal{L}_∞ -space embeds in a space with an unconditional basis, then it is necessarily isomorphic to c_0 . It follows from the work of D. Lewis–C. Stegall [LS], [St], and A. Pełczyński [P] that if the dual

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of a separable \mathcal{L}_∞ -space \mathfrak{X} is nonseparable, then ℓ_1 is isomorphic to a subspace of \mathfrak{X} . If in addition \mathfrak{X}^* is separable (i.e., $\mathfrak{X}^* \simeq \ell_1$) and \mathfrak{X} does not contain a reflexive subspace, then c_0 appears as a strong candidate to be a subspace of \mathfrak{X} .

The aforementioned problem, in the general setting, was answered in 1994 by W. T. Gowers. More precisely, in [G1], the Gowers Tree space is presented, the first example of a Banach space not containing ℓ_1 , c_0 , or a reflexive subspace. A systematic study of this type of spaces has appeared in [AAT]. The Gowers Tree space and the spaces in [AAT] satisfy a stronger property, namely every subspace has a nonseparable dual. Actually, in every subspace there exists a tree-basis similar to the basis of the classical James Tree space [J]. This is an obstacle to any attempt to combine Gowers' norming set with the Bourgain–Delbaen techniques to obtain a \mathcal{L}_∞ HI space with no reflexive subspace. Indeed, Gowers' norming set would enforce the dual of the space to be nonseparable and, since the space is \mathcal{L}_∞ , as we have mentioned before, the space ℓ_1 would be isomorphic to a subspace of the space.

Motivated by the above, we recently introduced a new method of defining norming sets that, among others, yields HI Banach spaces with a separable dual containing no reflexive subspaces. A Tsirelson version of this method and its consequences in a classical setting appeared in [AM2]. It is worth pointing out that the new method leads to a unified approach for constructing HI spaces that are either reflexive or do not contain a reflexive subspace. Moreover, this is rather simpler than the initial method for constructing Gowers Tree spaces [G1], [AAT].

All known nonclassical separable \mathcal{L}_∞ -spaces are Bourgain–Delbaen \mathcal{L}_∞ -spaces (BD- \mathcal{L}_∞ -spaces). This class of spaces was introduced by J. Bourgain and F. Delbaen [BD] and they are defined as follows. A BD- \mathcal{L}_∞ -space is a subspace \mathfrak{X} of $\ell_\infty(\Gamma)$, with Γ a countable set. It is determined by a sequence $(\Gamma_q, i_q)_q$ where $(\Gamma_q)_q$ is an increasing sequence of finite subsets of Γ with $\bigcup_q \Gamma_q = \Gamma$ and $i_q : \ell_\infty(\Gamma_q) \rightarrow \ell_\infty(\Gamma)$, $q \in \mathbb{N}$, are uniformly bounded extension operators (i.e., $i_q(x)|_{\Gamma_q} = x$) that are in addition compatible. This last property means that for $q < p$ and $x \in \ell_\infty(\Gamma_q)$ we have $i_q(x) = i_p(i_q(x)|_{\Gamma_p})$. For $q \in \mathbb{N}$ we set $\Delta_q = \Gamma_q \setminus \Gamma_{q-1}$, and for $\gamma \in \Delta_q$ we set $d_\gamma = i_q(e_\gamma)$. Then, the BD- \mathcal{L}_∞ -space is defined to be $\mathfrak{X} = \overline{\langle \{d_\gamma : \gamma \in \Gamma\} \rangle}$, as a subspace of $\ell_\infty(\Gamma)$. The sequence $(d_\gamma)_{\gamma \in \Gamma}$ forms a Schauder basis for \mathfrak{X} , however it is usually more convenient to consider the finite-dimensional decomposition (FDD) $(M_q)_q$ with $M_q = \langle \{d_\gamma : \gamma \in \Delta_q\} \rangle$. For an interval E of \mathbb{N} , P_E denotes the natural projection onto E associated to the FDD $(M_q)_q$. As we mentioned above, this class of \mathcal{L}_∞ -spaces appeared for the first time in [BD] as a specific class of \mathcal{L}_∞ -spaces. Recently, in [AGM] it was shown that every separable \mathcal{L}_∞ -space is isomorphic to a BD- \mathcal{L}_∞ -space. Thus, BD- \mathcal{L}_∞ -spaces are the generic ones.

A second component in the Bourgain–Delbaen invention, which is equally important to the definition of the spaces, concerns the method of constructing the sequence $(i_q)_q$. It is defined inductively in a way to preserve the uniform bound of the norms of the i_q 's. Moreover, by analyzing the initial spaces defined in [BD] one can observe that the saturation of the structure is an inevitable ingredient. This is more transparent in the alternative definition of ℓ_p -saturated \mathcal{L}_∞ -spaces in [GPZ]. This explains why it is possible to combine BD- \mathcal{L}_∞ structure with saturated norms resulting in \mathcal{L}_∞ HI spaces. The relation of BD- \mathcal{L}_∞ -spaces with saturated norms was established for the first time by R. Haydon in [H].

Let us pass to the description of some features of the space \mathfrak{X}_{nr} . As we have already mentioned, we will use a new method of defining HI spaces, which we will combine with the Bourgain–Delbaen techniques in order to obtain a \mathcal{L}_∞ HI space without reflexive subspaces. The new method requires a preexisting space that in the classical setting would be either the Tsirelson space [T] or a mixed-Tsirelson space $T[(\mathcal{A}_{n_j}, m_j^{-1})_j]$. The norming sets of the new spaces are defined to be subsets of the corresponding ones of the initial spaces. A typical and known example is the Schlumprecht space $S[(\mathcal{A}_n, 1/\sqrt{\log(n+1)})_n]$ [Sch] that serves as a preexisting reflexive space with an unconditional basis for the Gowers–Maurey space [GM]. Furthermore, in [AM2] the norming set W is a subset of W_T , the norming set of the Tsirelson space.

Attempting to adapt the above scheme to a \mathcal{L}_∞ setting, we have to use an initial \mathcal{L}_∞ -space in which the obvious norming set is the set $\{e_\gamma^* : \gamma \in \Gamma\}$. We then have to carefully select a subset of Γ that will define the space \mathfrak{X}_{nr} . This is the motivation behind introducing the self-determined subsets of a set Γ , which are defined as follows.

Definition. Let \mathfrak{X} be a BD- \mathcal{L}_∞ -subspace of $\ell_\infty(\Gamma)$. A subset Γ' of Γ is self-determined if $\langle\{d_\gamma^* : \gamma \in \Gamma'\}\rangle = \langle\{e_\gamma^* : \gamma \in \Gamma'\}\rangle$, where $(d_\gamma^*)_{\gamma \in \Gamma}$ denotes the sequence biorthogonal to the basis $(d_\gamma)_{\gamma \in \Gamma}$ and for $\gamma \in \Gamma$, e_γ^* denotes the element e_γ of $\ell_1(\Gamma)$ restricted on \mathfrak{X} .

The following holds.

Proposition 1. *Let \mathfrak{X} be a BD- \mathcal{L}_∞ -subspace of $\ell_\infty(\Gamma)$, and let Γ' be a self-determined subset of Γ .*

- (i) *The space $Y = \overline{\langle\{d_\gamma : \gamma \in \Gamma \setminus \Gamma'\}\rangle}$ is a \mathcal{L}_∞ -space.*
- (ii) *The quotient \mathfrak{X}/Y is a \mathcal{L}_∞ -space.*

The above proposition and a result from [KL] yield the following.

Theorem. *There is a continuum of \mathcal{L}_∞ -subspaces $\{Y_\alpha : \alpha \in \mathfrak{c}\}$ of \mathfrak{X}_{AH} , satisfying the following:*

- (i) *Each space Y_α has the scalar-plus-compact property and for every $\alpha \neq \beta$ every bounded linear operator $T : Y_\alpha \rightarrow Y_\beta$ is compact.*
- (ii) *Each $X_\alpha = \mathfrak{X}_{\text{AH}}/Y_\alpha$ is a hereditarily indecomposable space with the scalar-plus-compact property and for every $\alpha \neq \beta$ every bounded linear operator $T : X_\alpha \rightarrow X_\beta$ is compact.*

The space \mathfrak{X}_{AH} above is the \mathcal{L}_∞ HI space from [AH]. Another consequence of self-determined sets is an intriguing result that displays the complete divergence between the structure of a \mathcal{L}_∞ -space and its quotients. This contrasts corresponding results concerning classical \mathcal{L}_∞ -spaces [JZ].

Theorem. *There exist \mathcal{L}_∞ Banach spaces X_1, X_2, X_3 with a separable dual so that X_2 is a quotient of X_1 and X_3 is a quotient of X_2 and, moreover, X_1 and X_3 are reflexive saturated, whereas X_2 contains no reflexive subspaces.*

In the present construction, the preexisting space is the space \mathfrak{B}_{mT} from [AH]. This is a BD- \mathcal{L}_∞ -space, saturated by reflexive subspaces that have an unconditional basis. The norming set of the space \mathfrak{X}_{nr} will be a self-determined subset Γ' of the set Γ defining the space \mathfrak{B}_{mT} as a subspace of $\ell_\infty(\Gamma)$. As in all Bourgain–Delbaen constructions, to each $\gamma \in \Gamma$ with $\gamma \in \Delta_{q+1}$ we associate a linear functional

$c_\gamma^* : \ell_\infty(\Gamma_q) \rightarrow \mathbb{R}$ such that $e_\gamma^* = c_\gamma^* + d_\gamma^*$. In the case of \mathfrak{X}_{nr} , the functional c_γ^* is defined as

$$c_\gamma^* = \frac{1}{m_j} b^* \quad \text{or} \quad c_\gamma^* = e_\xi^* + \frac{1}{m_j} b^*,$$

where $\xi \in \Delta_p \cap \Gamma'$, $p < q$, and

$$b^* = \frac{1}{n} (\varepsilon_1 e_{\zeta_1}^* \circ P_{E_1} + \cdots + \varepsilon_n e_{\zeta_n}^* \circ P_{E_n})$$

with $\zeta_1, \dots, \zeta_n \in \Gamma' \cap (\Gamma_q \setminus \Gamma_p)$, $p < E_1 < \cdots < E_n \leq q$, and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$. If ξ does not exist, then $\zeta_1, \dots, \zeta_n \in \Gamma' \cap \Gamma_q$.

The functional b^* is a special type of average, called an α_c -average. The latter are defined by a countable tree \mathcal{U} with elements $\{(\gamma_k, x_k)\}_{k=1}^n$ so that $\gamma_k \in \Gamma$ and x_k is a block vector in \mathfrak{B}_{mT} with respect to its basis $(d_\gamma)_{\gamma \in \Gamma}$. The use of the tree \mathcal{U} in the definition of α_c -averages explains the necessity of the preexisting space \mathfrak{B}_{mT} . Furthermore, since Γ' is a self-determined subset of Γ , the space \mathfrak{X}_{nr} is a quotient of \mathfrak{B}_{mT} .

Note that in the definition of Γ' we use saturation under constraints, which has occurred in earlier papers (e.g., [AM1], [ABM]). The version appearing here is similar to the one used in [AM2]. The difference from [AM2] is that here we deal with families $(\mathcal{A}_{n_j})_j$ instead of $(\mathcal{S}_n)_n$, which makes the definitions and the proofs easier. It is also worth pointing out that, as in [AM2], the conditional structure of the space \mathfrak{X}_{nr} is imposed by certain α_c -averages and not by special sequences $(\gamma_k)_{k=1}^n$.

The space \mathfrak{X}_{nr} satisfies the scalar-plus-compact property. In the space \mathfrak{X}_{AH} from [AH] the same result is proved using LUST. In the case of \mathfrak{X}_{nr} the proof is more involved. This is due to the fact that we apply saturation under constraints. Actually, we combine the LUST of \mathfrak{X}_{nr} and the fact that $\mathfrak{X}_{nr}^* \simeq \ell_1$. Another property of \mathfrak{X}_{nr} is that every subspace fails the point of continuity property (PCP). Thus, \mathfrak{X}_{nr} answers in a strong sense a problem posed by J. Bourgain [B, Problem 3, p. 46]. We mention that a \mathcal{L}_∞ -space without PCP and not containing c_0 could also be obtained by the results in [FOS].

We close the introduction by mentioning a problem attributed to H. P. Rosenthal. The problem in question is the following.

Problem. Let \mathfrak{X} be a \mathcal{L}_∞ saturated Banach space. Does \mathfrak{X} contain c_0 isomorphically?

Note that if \mathfrak{X} is \mathcal{L}_∞ saturated and does not contain c_0 , then it does not contain an unconditional basic sequence. This follows from James' classical characterization of reflexivity for spaces with an unconditional basis. Indeed, let us assume that \mathfrak{X} contains a subspace Y with an unconditional basis. Then Y is \mathcal{L}_∞ saturated, i.e., it is not reflexive. For the same reason Y cannot contain ℓ_1 and by assumption it does not contain c_0 , which implies Y is reflexive, which is absurd. Therefore, by Gowers' dichotomy [G2] any \mathcal{L}_∞ saturated space \mathfrak{X} not containing c_0 is HI saturated. If a space \mathfrak{X} answers Rosenthal's problem negatively, then it has to be saturated with HI spaces with LUST and probably with the scalar-plus-compact property. We show that \mathfrak{X}_{nr} is not \mathcal{L}_∞ saturated. However, we believe that the techniques deployed in the present paper are a step towards the solution of Rosenthal's problem.

1. THE SELF-DETERMINED SETS

In this preparatory section we introduce the self-determined subsets of the norming set Γ of a BD- \mathcal{L}_∞ -space \mathfrak{X} . It is shown that the self-determined sets are able to provide \mathcal{L}_∞ subspaces and quotients of a given BD- \mathcal{L}_∞ -space. Also, they are a key ingredient for the definition of the space \mathfrak{X}_{nr} . We start by recalling the definition of the BD- \mathcal{L}_∞ -spaces given in [AGM]. We recall that every separable \mathcal{L}_∞ space is isomorphic to a BD- \mathcal{L}_∞ -space ([AGM, Theorem 3.6]).

Notation 1.1. For Γ_1, Γ sets with $\Gamma_1 \subseteq \Gamma$, we denote by $r : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma_1)$ the natural restriction operator. An operator $i : \ell^\infty(\Gamma_1) \rightarrow \ell^\infty(\Gamma)$ is an extension operator if $r \circ i$ is the identity operator of $\ell^\infty(\Gamma_1)$. Also, if $(\Gamma_q)_q$ is a strictly increasing sequence of nonempty sets and $\Gamma = \bigcup_q \Gamma_q$, then a sequence of extension operators $(i_q)_q$, with $i_q : \ell^\infty(\Gamma_q) \rightarrow \ell^\infty(\Gamma)$ for all $q \in \mathbb{N}$, will be called compatible, if for every $p, q \in \mathbb{N}$ with $p < q$, $i_p = i_q \circ r_q \circ i_p$, where r_q denotes the restriction onto Γ_q .

Definition 1.2. Let $(\Gamma_q)_{q=1}^\infty$ be a strictly increasing sequence of nonempty finite sets, let $\Gamma = \bigcup_q \Gamma_q$, and let compatible extension operators $(i_q)_{q=1}^\infty$, with $i_q : \ell^\infty(\Gamma_q) \rightarrow \ell^\infty(\Gamma)$ for all $q \in \mathbb{N}$, such that $C = \sup_q \|i_q\|$ be finite. Define $\Delta_1 = \Gamma_1$, $\Delta_{q+1} = \Gamma_{q+1} \setminus \Gamma_q$ for $q \in \mathbb{N}$, and for every $\gamma \in \Gamma$ we define d_γ , a vector in $\ell^\infty(\Gamma)$, as follows: if $\gamma \in \Delta_q$ for some $q \in \mathbb{N}$, then $d_\gamma = i_q(e_\gamma)$. The closed linear span of the set $\{d_\gamma : \gamma \in \Gamma\}$ will be denoted by $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ and called a Bourgain–Delbaen space.

If for all $q \in \mathbb{N}$ we define $M_q = \langle \{d_\gamma : \gamma \in \Delta_q\} \rangle$, then $(M_q)_q$ forms a Finite Dimensional Decomposition (FDD) for the space $\mathfrak{X}_{(\Gamma_q, i_q)_q}$, and for every interval E of \mathbb{N} we denote by P_E the projection associated to this FDD and E . For every $\gamma \in \Gamma$ we denote by $e_\gamma^* : \mathfrak{X}_{(\Gamma_q, i_q)_q} \rightarrow \mathbb{R}$ the evaluation functional on the γ th coordinate, defined on $\ell^\infty(\Gamma)$ and then restricted to the subspace $\mathfrak{X}_{(\Gamma_q, i_q)_q}$. Moreover, for every $\gamma \in \Gamma$ we define two specific linear functionals c_γ^* and d_γ^* so that $e_\gamma^* = c_\gamma^* + d_\gamma^*$. For the precise definition see [AGM, Definition 2.14]. We summarize some properties of these functionals. Their proofs can be found in [AGM, Lemma 2.16 and Proposition 2.17].

Proposition 1.3. *Let $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ be a Bourgain–Delbaen space. The following hold:*

- (i) *The sequence $(e_\gamma^*)_{\gamma \in \Gamma}$ is equivalent to the unit vector basis of $\ell_1(\Gamma)$.*
- (ii) *The functionals $(d_\gamma^*)_{\gamma \in \Gamma}$ are biorthogonal to the vectors $(d_\gamma)_{\gamma \in \Gamma}$.*
- (iii) *For $q \in \mathbb{N} \cup \{0\}$, $\{c_\gamma^* : \gamma \in \Delta_{q+1}\} \subset \langle \{e_\gamma^* : \gamma \in \Gamma_q\} \rangle = \langle \{d_\gamma^* : \gamma \in \Gamma_q\} \rangle$.*
- (iv) *If the FDD $(M_q)_q$ is shrinking, the closed linear span of the functionals $(d_\gamma^*)_{\gamma \in \Gamma}$ is $\mathfrak{X}_{(\Gamma_q, i_q)_q}^*$, and hence $\mathfrak{X}_{(\Gamma_q, i_q)_q}^*$ is isomorphic to ℓ_1 .*

1.1. Self-determined subsets of Γ . We define and study self-determined subsets Γ' of Γ , which define quotients of $\mathfrak{X}_{(\Gamma_q, i_q)_q}$, which are Bourgain–Delbaen spaces as well.

Definition 1.4. Let $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ be a Bourgain–Delbaen space. An infinite subset Γ' of Γ will be called self-determined if for every $\gamma \in \Gamma'$ the functional d_γ^* is in the linear span of $\{e_\gamma^* : \gamma \in \Gamma'\}$.

Proposition 1.5. *Let $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ be a Bourgain–Delbaen space, let Γ' be an infinite subset of Γ , and for all q set $\Gamma'_q = \Gamma' \cap \Gamma_q$, $\Delta'_q = \Gamma' \cap \Delta_q$, and $q_0 = \min\{q : \Gamma'_q \neq \emptyset\}$. The following assertions are equivalent:*

- (a) *The set Γ' is self-determined.*
- (b) *For all $q \geq q_0$, $\langle \{d_\gamma^* : \gamma \in \Gamma'_q\} \rangle = \langle \{e_\gamma^* : \gamma \in \Gamma'_q\} \rangle$.*
- (c) *We have $\langle \{d_\gamma^* : \gamma \in \Gamma'_{q_0}\} \rangle = \langle \{e_\gamma^* : \gamma \in \Gamma'_{q_0}\} \rangle$, and for all $q \geq q_0$, $\langle \{c_\gamma^* : \gamma \in \Delta'_{q+1}\} \rangle \subset \langle \{e_\gamma^* \circ P_E : \gamma \in \Gamma'_q, E \subset \mathbb{N} \cup \{0\}\} \rangle$.*
- (d) *For all $\gamma \in \Gamma \setminus \Gamma'$ and $\eta \in \Gamma'$, $e_\eta^*(d_\gamma) = 0$.*
- (e) *For all $\gamma \in \Gamma \setminus \Gamma'$ and $\eta \in \Gamma'$, $c_\eta^*(d_\gamma) = 0$.*

Proof. We add an auxiliary assertion in order to obtain the equivalence:

- (c') *We have $\langle \{d_\gamma^* : \gamma \in \Gamma'_{q_0}\} \rangle = \langle \{e_\gamma^* : \gamma \in \Gamma'_{q_0}\} \rangle$, and for all $q \geq q_0$, $\langle \{c_\gamma^* : \gamma \in \Delta'_{q+1}\} \rangle \subset \langle \{d_\gamma^* : \gamma \in \Gamma'_q\} \rangle$.*

The main facts we shall use are (ii) and (iii) from Proposition 1.3, as well as $e_\gamma^* = c_\gamma^* + d_\gamma^*$ for all $\gamma \in \Gamma$. The assertions (a) \Leftrightarrow (b), (b) \Rightarrow (c'), (c') \Rightarrow (e) and (d) \Leftrightarrow (e) are very easy to prove. To see (c') \Rightarrow (c) recall that from [AGM, Remark 2.15], for each $p \in \mathbb{N}$ and $\gamma \in \Delta_p$, $d_\gamma^* = e_\gamma^* \circ P_{\{p\}}$, while (c) \Rightarrow (b) is proved by induction on q . An argument involving kernels of linear functionals yields (e) \Rightarrow (c'). Drawing a diagram will convince the reader that the proof is complete. \square

Notation 1.6. Given a Bourgain–Delbaen space $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ as well as a self-determined subset Γ' of Γ , we denote by

- (i) R the restriction onto Γ' ,
- (ii) $\Gamma'_q = \Gamma' \cap \Gamma_q$ and $\Delta'_q = \Gamma' \cap \Delta_q$ for all $q \in \mathbb{N}$,
- (ii') $\Gamma''_q = \Gamma_q \setminus \Gamma'_q$ and $\Delta''_q = \Delta_q \setminus \Delta'_q$ for all $q \in \mathbb{N}$,
- (iii) $S = \{q \in \mathbb{N} \cup \{0\} : \Delta'_q \neq \emptyset\} = \{q_0 < q_1 < \dots < q_s \dots\}$,
- (iv) for all $s \in \mathbb{N} \cup \{0\}$, r'_{q_s} the restriction onto Γ'_{q_s} , and
- (v) for $s \in \mathbb{N} \cup \{0\}$, $i'_{q_s} : \ell_\infty(\Gamma'_{q_s}) \rightarrow \ell_\infty(\Gamma')$ with $i'_{q_s}(x) = R(i_{q_s}(x))$, where we naturally identify x with a vector in $\ell_\infty(\Gamma_{q_s})$.

Observe that $(\Gamma'_{q_s})_{s=1}^\infty$ is a strictly increasing sequence of finite sets whose union is Γ' and that $(i'_{q_s})_{q=1}^\infty$ is a uniformly bounded sequence of extension operators, in particular, $\sup_s \|i'_{q_s}\| \leq \sup_q \|i_q\|$. We will see next (Proposition 1.9) that the extensions $(i'_{q_s})_s$ are compatible, hence the sequence $(\Gamma'_{q_s}, i'_{q_s})$ defines a Bourgain–Delbaen space.

For the rest of this section we follow the above notation.

Proposition 1.7. *Let $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ be a Bourgain–Delbaen space, and let Γ' be a self-determined subset of Γ . Then for every $q \in \mathbb{N}$, we have that $i_q[\ell_\infty(\Gamma''_q)] = \langle \{d_\gamma : \gamma \in \Gamma''_q\} \rangle$, where we naturally identify $\ell_\infty(\Gamma''_q)$ with a subspace of $\ell_\infty(\Gamma_q)$. In particular, if we denote by Y the closed linear span of $\{d_\gamma : \gamma \in \Gamma \setminus \Gamma'\}$, then Y is a \mathcal{L}_∞ -space.*

Proof. Fix $q \in \mathbb{N}$. If $\Delta''_q = \emptyset$, then there is nothing to prove. If this is not the case, we will show that $d_\gamma \in i_q[\ell_\infty(\Gamma''_q)]$ for all $\gamma \in \Gamma''_q$ which, due to dimensional reasons, yields the desired result. The compatibility property of the operators implies $d_\gamma = i_q(\sum_{\eta \in \Gamma_q} e_\eta^*(d_\gamma)e_\eta)$, whereas Proposition 1.5 (d) yields $d_\gamma = i_q(\sum_{\eta \in \Gamma''_q} e_\eta^*(d_\gamma)e_\eta)$. The second part of this proposition follows from the fact that for all q , i_q is a C -isomorphism, where $C = \sup_q \|i_q\|$. \square

Lemma 1.8. *Let $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ be a Bourgain–Delbaen space, and let Γ' be a self-determined subset of Γ . Then for every $s \in \mathbb{N}$ and $x \in \ell_\infty(\Gamma_{q_s})$ we have $R(i_{q_s}(x)) = i'_{q_s}(r'_{q_s}(x))$.*

Proof. Note that $x - r'_{q_s}(x) = \sum_{\eta \in \Gamma''_{q_s}} e_\eta^*(x)e_\eta$ and hence $i'_{q_s}(r'_{q_s}(x)) = R(i_{q_s}(r'_{q_s}(x))) = R(i_{q_s}((x))) - R(i_{q_s}(\sum_{\eta \in \Gamma''_{q_s}} e_\eta^*(x)e_\eta))$. We will show that $R(i_{q_s}(e_\eta)) = 0$ for all $\eta \in \Gamma''_{q_s}$, which clearly yields the desired result. By Proposition 1.7 $i_{q_s}(e_\eta)$ is in $\langle \{d_\gamma : \gamma \in \Gamma''_q\} \rangle$, and by Proposition 1.5(d) the conclusion follows. \square

Proposition 1.9. *Let $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ be a Bourgain–Delbaen space, and let Γ' be a self-determined subset of Γ . Then the sequence $(i'_{q_s})_{s=1}^\infty$ is compatible, hence it defines a Bourgain–Delbaen space $\mathfrak{X}_{(\Gamma'_{q_s}, i'_{q_s})_s}$.*

Proof. Fix s, t in \mathbb{N} with $s < t$ and $x \in \ell_\infty(\Gamma'_{q_s})$; we will show that $i'_{q_s}(x) = i'_{q_t}(r'_{q_t}(i'_{q_s}(x)))$. Define $y = r'_{q_t}(i'_{q_s}(x))$ and observe that $r'_{q_t} \circ R = r'_{q_t} \circ r_{q_t}$, which in conjunction with $i'_{q_s} = R \circ i_{q_s}$ yields $y = r'_{q_t}(r_{q_t}(i_{q_s}(x)))$. Applying Lemma 1.8 and using the compatibility of $(i_q)_q$ we obtain

$$\begin{aligned} i'_{q_t}(r'_{q_t}(i'_{q_s}(x))) &= i'_{q_t}(r'_{q_t}(r_{q_t}(i_{q_s}(x)))) = R(i_{q_t}(r_{q_t}(i_{q_s}(x)))) \\ &= R(i_{q_s}(x)) = i'_{q_s}(x). \end{aligned}$$

Since $\sup_s \|i'_{q_s}\| \leq \sup_q \|i_q\|$, the proof is complete. \square

Notation 1.10. Given a Bourgain–Delbaen space $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ as well as a self-determined subset Γ' of Γ , we denote by $(d'_\gamma)_{\gamma \in \Gamma'}$ the vectors that span the space $\mathfrak{X}_{(\Gamma'_{q_s}, i'_{q_s})_s}$. Moreover, for $\gamma \in \Gamma'$ we denote by d'^*_γ and c'_γ the corresponding functionals from [AGM, Definition 2.14], whereas for the evaluation functionals on the γ th coordinate we retain the symbol e_γ . We also denote by $(M'_s)_{s=0}^\infty$ the FDD of $\mathfrak{X}_{(\Gamma'_{q_s}, i'_{q_s})_s}$ as defined in [AGM, Proposition 2.8] and by P'_E the corresponding projections.

Remark 1.11. Note that Lemma 1.8 easily implies that for every $\gamma \in \Gamma$, $R(d_\gamma)$ is in $\mathfrak{X}_{(\Gamma'_{q_s}, i'_{q_s})_s}$. Hence, $R : \mathfrak{X}_{(\Gamma_q, i_q)_q} \rightarrow \mathfrak{X}_{(\Gamma'_{q_s}, i'_{q_s})_s}$ is a well-defined linear operator of norm at most one. Moreover, the following hold:

- (i) for all $\gamma \in \Gamma'$, $R(d_\gamma) = d'_\gamma$,
- (ii) for all $\gamma \in \Gamma \setminus \Gamma'$, $R(d_\gamma) = 0$,
- (iii) for all $\gamma \in \Gamma'$, $R^*(e_\gamma) = e'_\gamma$,
- (iv) for all $\gamma \in \Gamma'$, $R^*(d'^*_\gamma) = d^*_\gamma$, and
- (v) for all $\gamma \in \Gamma'$, $R^*(c'^*_\gamma) = c^*_\gamma$.

The first assertion is easily implied by Lemma 1.8 while the second one clearly follows from Proposition 1.5(d). The third assertion is an easy consequence of the definition of R and the fourth follows from the first two, while the last one follows from (iii) and (iv).

Proposition 1.12. *Let $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ be a Bourgain–Delbaen space, let Γ' be a self-determined subset of Γ , and let Y be the closed linear span of $\{d_\gamma : \gamma \in \Gamma \setminus \Gamma'\}$. Then R is onto $\mathfrak{X}_{(\Gamma'_{q_s}, i'_{q_s})_s}$ and its kernel is the space Y . Hence, $\mathfrak{X}_{(\Gamma_q, i_q)_q}/Y$ is isomorphic to $\mathfrak{X}_{(\Gamma'_{q_s}, i'_{q_s})_s}$.*

Proof. To conclude that R is onto, it suffices to show that the closure of $R[A]$ contains B , where $A = \{x \in \mathfrak{X}_{(\Gamma_q, i_q)_q} : \|x\| \leq \sup_q \|i_q\|\}$ and B is the unit ball

of $\mathfrak{X}_{(\Gamma'_{q_s}, i'_{q_s})_s}$. To this end, let x be in the linear span of $\{d'_\gamma : \gamma \in \Gamma'\}$, with $\|x\| \leq 1$. Then there is $s \in \mathbb{N} \cup \{0\}$ and $y \in \ell_\infty(\Gamma'_{q_s})$ so that $x = i'_{q_s}(y)$. If we define $z = i_{q_s}(y)$, then $\|z\| \leq \|i_{q_s}\|$, and by Lemma 1.8, $R(z) = x$. To conclude the proof, observe that Remark 1.11, in conjunction with $((d_\gamma)_{\gamma \in \Delta_q})_q$ and $((d'_\gamma)_{\gamma \in \Delta'_{q_s}})_s$ being Schauder bases for $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ and $\mathfrak{X}_{(\Gamma'_{q_s}, i'_{q_s})_s}$, respectively, yields that $\ker R = Y$. \square

The proposition below and the remark following it state that if $\gamma \in \Gamma'$ and the action of the extension function c_γ^* is understood, then the action of $c_\gamma'^*$ is understood as well. This is useful for determining the evaluation analysis of the coordinate γ .

Proposition 1.13. *Let $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ be a Bourgain–Delbaen space, and let Γ' be a self-determined subset of Γ . Let, moreover, γ be in Γ' and assume that there are a finite subset F of Γ' , scalars $(\lambda_\eta)_{\eta \in F}$, and intervals $(E_\eta)_{\eta \in F}$ of \mathbb{N} so that $c_\gamma^* = \sum_{\eta \in F} \lambda_\eta e_\eta^* \circ P_{E_\eta}$. Then $c_\gamma'^* = \sum_{\eta \in F} \lambda_\eta e_\eta^* \circ P'_{E'_\eta}$, where for $\eta \in F$, $E'_\eta = \{s \in \mathbb{N} \cup \{0\} : q_s \in E_\eta\}$.*

Proof. We will show that for all ξ in Γ' , $c_\gamma'^*(d'_\xi) = \sum_{\eta \in F} \lambda_\eta e_\eta^* \circ P'_{E'_\eta}(d'_\xi)$. Fix $\xi \in \Gamma'$ with $\xi \in \Delta'_{q_s}$. Remark 1.11(i) and (v) yield $c_\gamma'^*(d'_\xi) = c_\gamma^*(d_\xi)$, and hence, setting $F' = \{\eta \in F : q_s \in E_\eta\}$, we obtain

$$\begin{aligned} c_\gamma'^*(d'_\xi) &= \sum_{\eta \in F} \lambda_\eta e_\eta^* \circ P_{E_\eta}(d_\xi) = \sum_{\eta \in F'} \lambda_\eta e_\eta^*(d_\xi) \\ &= \sum_{\eta \in F'} \lambda_\eta e_\eta^*(d'_\xi) = \sum_{\eta \in F'} \lambda_\eta e_\eta^* \circ P'_{E'_\eta}(d'_\xi). \end{aligned}$$

\square

Remark 1.14. The above argument actually yields that if γ, ξ are in Γ' and E is an interval of \mathbb{N} , then $e_\gamma^* \circ P_E(d_\xi) = e_\gamma^* \circ P'_{E'}(d'_\xi)$.

Remark 1.15. In [FOS] the authors introduce a method of embedding a separable Banach space X into a Bourgain–Delbaen space Z satisfying certain properties; in particular, if X has separable dual, then Z^* is isomorphic to ℓ_1 . This method is comprised of two steps. In the first one, X is embedded into a Bourgain–Delbaen space $Y = \mathfrak{X}_{(\Gamma_q, i_q)_q}$, whereas in the second one, the space $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ is “augmented” to obtain a space $Z = \mathfrak{X}_{(\bar{\Gamma}_q, \bar{i}_q)}$, with $\bar{\Gamma} = \bigcup_q \bar{\Gamma}_q$ a suitable superset of $\Gamma = \bigcup_q \Gamma_q$ so that a copy of X is naturally preserved in Z . As it is stated in that paper, the restriction operator R onto Γ maps elements of Z to elements of Y . We observe that Γ is a self-determined subset of $\bar{\Gamma}$, and hence $R : Z \rightarrow Y$ is a quotient map which moreover preserves a copy of X . In [A et al.] it is shown that if X is superreflexive, Z can be chosen to satisfy the scalar-plus-compact property. As we will also comment later, the Argyros–Haydon space from [AH] is a quotient of the mixed-Tsirelson Bourgain–Delbaen space \mathfrak{B}_{mT} defined in that paper.

2. THE DEFINITION OF THE SPACE \mathfrak{X}_{nr}

In this section we define the space \mathfrak{X}_{nr} , combining the method from [AH] with that from [AM2]. We start by recalling the definition of the Bourgain–Delbaen mixed-Tsirelson space \mathfrak{B}_{mT} from [AH], in fact a slight variation of it, and then we define the space \mathfrak{X}_{nr} as a quotient of \mathfrak{B}_{mT} by selecting an appropriate self-determined subset of the set $\bar{\Gamma}$ associated to \mathfrak{B}_{mT} . We follow the notation from

both papers [AH] and [AM2], and when they come in conflict, we shall use the one from [AH].

2.1. The space \mathfrak{B}_{mT} . In [AH] a Bourgain–Delbaen space \mathfrak{B}_{mT} is presented which is based on a mixed-Tsirelson space. We slightly modify this space but still denote it by \mathfrak{B}_{mT} . This construction can, by now, be considered standard, and therefore we do not include all of the details. We shall use notation such as $\bar{\Gamma}$, $\bar{\Gamma}_q$, \bar{i}_q , \bar{c}_γ^* , \bar{d}_γ^* , \bar{d}_γ , $(\bar{M}_q)_q$, and \bar{P}_E to refer to the corresponding components of the space \mathfrak{B}_{mT} , as we reserve the casual notation (i.e., Γ_q , c_γ^* , etc.) for the space \mathfrak{X}_{nt} that we will define in the sequel. We start with a sequence of pairs of natural numbers $(m_j, n_j)_{j=1}^\infty$ satisfying [AH, Assumption 2.3, p. 5]. We shall use, without mentioning it, properties of this sequence such as $\sum_j 1/m_j < 1/3$ and $\sum_{j>i} 1/m_j < 1/m_i$; however, we make the additional assumption that $m_1 \geq 8$, which is not needed in [AH].

Proposition 2.1. *There exists a Bourgain–Delbaen space $\mathfrak{B}_{mT} = \mathfrak{X}_{(\bar{\Gamma}_q, \bar{i}_q)}$, with $\sup_q \|\bar{i}_q\| \leq 2$, so that $\bar{\Delta}_1 = \{0\}$ and for $q \in \mathbb{N}$,*

$$\begin{aligned} \bar{\Delta}_{q+1} = & \bigcup_{j=0}^{q+1} \{(q+1, m_j^{-1}, b^*) : b^* \in B_{0,n}\} \\ & \cup \bigcup_{p=0}^{q-1} \bigcup_{j=0}^p \{(q+1, \xi, m_j^{-1}, b^*) : \xi \in \bar{\Delta}_p, \text{weight}(\xi) = m_j^{-1}, \text{age}(\xi) < n_j, \\ & \qquad \qquad \qquad b^* \in B_{p,q}\}, \end{aligned}$$

where for each $0 \leq p < q$, $B_{p,q}$ is the set of all linear combinations

$$(1) \qquad b^* = \sum_{\eta \in \bar{\Gamma}_q \setminus \bar{\Gamma}_p} \lambda_\eta e_\eta^* \circ \bar{P}_{E_\eta},$$

where $\sum_{\eta \in \bar{\Gamma}_q \setminus \bar{\Gamma}_p} |\lambda_\eta| \leq 1$, each λ_η is a rational number with denominator dividing the quantity $\mathcal{N}_{q+1} = (2^q \#\bar{\Gamma}_q)!$, and each E_η is an interval of $(p, q]$. For each $\gamma \in \bar{\Delta}_{q+1}$, $\text{rank}(\gamma) = q+1$, and if $\gamma = (q+1, m_j^{-1}, b^*)$, then $\text{age}(\gamma) = 1$, $\text{weight}(\gamma) = m_j^{-1}$, and

$$(2a) \qquad \bar{c}_\gamma^* = \frac{1}{m_j} b^*,$$

whereas if $\gamma = (q+1, \xi, m_j^{-1}, b^*)$, then $\text{age}(\gamma) = \text{age}(\xi) + 1$, $\text{weight}(\gamma) = m_j^{-1}$, and

$$(2b) \qquad \bar{c}_\gamma^* = e_\xi^* + \frac{1}{m_j} b^*.$$

Remark 2.2. Although the definition of the space \mathfrak{B}_{mT} is formulated slightly differently than in [AH], the only actual difference lies in the sets $B_{p,q}$, namely the E_η 's that appear in (1) are only allowed to be the interval $(p, q]$ in [AH, Section 4].

2.2. Constraints in the setting of the Bourgain–Delbaen construction method. We adapt some notation used in papers such as [ABM], [AM2] to the setting of our construction. We recall that such constraints have also been used in [AGM].

- (i) Let Ξ be a subset of $\bar{\Gamma}$. A functional b^* in $\mathfrak{B}_{m\Gamma}^*$ is called an α -average of Ξ of size $s(b^*) = n$ if there exist $1 \leq d \leq n$, successive intervals $(E_i)_{i=1}^d$ of \mathbb{N} , signs $(\varepsilon_i)_{i=1}^d$ in $\{-1, 1\}$, and $(\gamma_i)_{i=1}^d$ in Ξ so that

$$b^* = \frac{1}{n} \sum_{i=1}^d \varepsilon_i e_{\gamma_i}^* \circ \bar{P}_{E_i}.$$

Observe that if $p < \min E_1$, $\max E_d \leq q$, and $n \leq (2^q \# \bar{\Gamma}_q)!$, then $b^* \in B_{p,q}$.

- (ii) A (finite or infinite) sequence of α -averages $(b_k^*)_k$ of Ξ is called very fast growing if there are nonnegative integers $0 \leq p_1 < q_1 < p_2 < q_2 < \dots$ so that $b_k^* \in B_{p_k, q_k}$ for $k = 1, 2, \dots$ and $s(b_k^*) \geq \mathcal{N}_{q_{k-1}+1}$ for $k > 1$.

For the definition of \mathcal{N}_q see the text below (1). Note that a subsequence of a very fast growing sequence is itself very fast growing.

Remark 2.3. The above definition implies that if $(b_k^*)_k$ is very fast growing, then $s(b_k^*) < s(b_{k+1}^*)$ for all k .

2.3. The tree of special sequences. We denote by \mathcal{Q} the set of all finite sequences of pairs $\{(\gamma_1, x_1), \dots, (\gamma_k, x_k)\}$ satisfying the following:

- (i) $\gamma_i \in \bar{\Gamma}$ with $\text{rank}(\gamma_i) \geq \min \text{ran } x_i$ for $i = 1, \dots, k$ and
- (ii) the x_1, \dots, x_k are finite linear combinations of $(\bar{d}_\gamma)_{\gamma \in \bar{\Gamma}}$ with rational coefficients, which are successive with respect to the FDD $(\bar{M}_q)_q$.

We choose a one-to-one function $\sigma : \mathcal{Q} \rightarrow \mathbb{N}$, called the coding function, so that for every $\{(\gamma_1, x_1), \dots, (\gamma_k, x_k)\} \in \mathcal{Q}$,

$$(3) \quad \sigma(\{(\gamma_1, x_1), \dots, (\gamma_k, x_k)\}) > \text{weight}(\gamma_k)^{-1} \max \text{supp } x_k,$$

where the support x_k is considered with respect to the FDD $(\bar{M}_q)_q$.

A finite sequence $\{(\gamma_k, x_k)\}_{k=1}^d \in \mathcal{Q}$ is called a special sequence if:

- (i) $\text{weight}(\gamma_1) = m_1^{-1}$ and
- (ii) if $d \geq 2$, then $\text{weight}(\gamma_k) = m_{\sigma((\gamma_1, x_1), \dots, (\gamma_{k-1}, x_{k-1}))}^{-1}$ for $k = 2, \dots, d$.

We note by \mathcal{U} the tree of all special sequences, endowed with the natural ordering “ \sqsubseteq ” of initial segments.

Remark 2.4. Note that if $\{(\gamma_k, x_k)\}_{k=1}^d$ is a special sequence, then by (3) $\text{weight}(\gamma_1) > \dots > \text{weight}(\gamma_d)$.

Definition 2.5. We say that two distinct natural numbers $i, j \geq 2$ are incomparable if one of the following holds:

- (i) neither i nor j is in $\sigma(\mathcal{Q})$ or
- (ii) both i and j are in $\sigma(\mathcal{Q})$ and $\sigma^{-1}(i), \sigma^{-1}(j)$ are incomparable in the ordering of \mathcal{U} .

2.4. The α_c -averages. In a similar manner as in [AM2], we define specific types of averages, based on the tree \mathcal{U} and the notion of comparability of natural numbers from Definition 2.5.

Definition 2.6. Let Ξ be a subset of $\bar{\Gamma}$, $d \in \mathbb{N}$, let $E_1 < \dots < E_d$ be intervals of \mathbb{N} , and $\gamma_i \in \Xi$ with $\text{rank}(\gamma_i) \geq \min E_i$ for $i = 1, \dots, d$ and $(\text{weight}(\gamma_i))_{i=1}^d$ is strictly decreasing.

- (i) The sequence of pairs $(\gamma_i, E_i)_{i=1}^d$ is called incomparable if choosing j_i so that $\text{weight}(\gamma_i) = m_{j_i}^{-1}$, then the natural numbers $j_i, i = 1, \dots, d$, are pairwise incomparable, in the sense of Definition 2.5. In this case, if $n \in \mathbb{N}$ with $d \leq n$ and $(\varepsilon_i)_{i=1}^d$ are any signs in $\{-1, 1\}$, we call the average

$$b^* = \frac{1}{n} \sum_{i=1}^d \varepsilon_i e_{\gamma_i}^* \circ \bar{P}_{E_i}$$

an \mathcal{IC} -average of Ξ .

- (ii) The sequence of pairs $(\gamma_i, E_i)_{i=1}^d$ is called comparable if there exist $m \in \mathbb{N}$ with $d \leq m, \{(\eta_1, x_1), \dots, (\eta_m, x_m)\} \in \mathcal{U}$, and $1 \leq k_1 < \dots < k_d \leq m$ so that the following are satisfied:
 - (a) $\text{weight}(\eta_{k_i}) = \text{weight}(\gamma_i)$,
 - (b) if $d \geq 4$, then $|e_{\gamma_i}^* \circ \bar{P}_{E_i}(x_{k_i}) - e_{\gamma_j}^* \circ \bar{P}_{E_j}(x_{k_j})| < 1/2^i$ for $2 \leq i < j \leq d - 1$.

In this case, if $n \in \mathbb{N}$ with $d \leq n$ and $(\varepsilon_i)_{i=1}^d$ is a sequence of alternating signs in $\{-1, 1\}$, we call the average

$$b^* = \frac{1}{n} \sum_{i=1}^d \varepsilon_i e_{\gamma_i}^* \circ \bar{P}_{E_i}$$

a \mathcal{CO} -average of Ξ .

- (iii) The sequence of pairs $(\gamma_i, E_i)_{i=1}^d$ is called irrelevant if there exist $m \in \mathbb{N}$ with $d \leq m, \{(\eta_1, x_1), \dots, (\eta_m, x_m)\} \in \mathcal{U}$, and $1 \leq k_1 < \dots < k_d \leq m$ so that the following are satisfied:
 - (a) $\text{weight}(\eta_{k_i}) = \text{weight}(\gamma_i)$ and
 - (b) if $d \geq 3$, then $|e_{\gamma_i}^* \circ \bar{P}_{E_i}(x_{k_i})| > 16000$ for $2 = 1, \dots, d - 1$.

In this case, if $n \in \mathbb{N}$ with $d \leq n$ and $(\varepsilon_i)_{i=1}^d$ are any signs in $\{-1, 1\}$, we call the average

$$b^* = \frac{1}{n} \sum_{i=1}^d \varepsilon_i e_{\gamma_i}^* \circ \bar{P}_{E_i}$$

an \mathcal{IR} -average of Ξ .

- (iv) Moreover, we call a basic average of Ξ any average of the form

$$b^* = \frac{1}{n} \sum_{i=1}^d \varepsilon_i \bar{d}_{\gamma_i}^*$$

where $d \leq n, \gamma_i \in \Xi$ with $\text{rank}(\gamma_1) < \dots < \text{rank}(\gamma_d)$, and $(\varepsilon_i)_{i=1}^d$ are any signs in $\{-1, 1\}$. In this case we do not impose any restrictions on the weights of the γ_i 's. Note that $\bar{d}_{\gamma_i}^* = e_{\gamma_i}^* \circ \bar{P}_{\{\text{rank}(\gamma_i)\}}$, hence basic averages are α -averages.

Any average which is of one of the forms defined above shall be called an α_c -average of Ξ .

Remark 2.7. A sequence of pairs $(\gamma_i, E_i)_{i=1}^d$ can be of none or of more than one of the types described in Definition 2.6. If it is of any of the first three types, then any of its subsequences is of the same type as well. Moreover, if b^* is an α_c -average of Ξ and E is an interval of \mathbb{N} so that $e_{\gamma}^* \circ \bar{P}_E \neq 0$, then $e_{\gamma}^* \circ \bar{P}_E$ is also an α_c -average of Ξ .

Proposition 2.8. *Let Ξ be a subset of $\bar{\Gamma}$, let $(\gamma_i)_i$ be a sequence in Ξ , and let $(E_i)_i$ be a sequence of successive intervals of \mathbb{N} so that $\text{rank}(\gamma_i) \geq \min E_i$ for all $i \in \mathbb{N}$ and the set $\{(\text{weight}(\gamma_i))^{-1} : i \in \mathbb{N}\}$ is unbounded. Then there exists an infinite subset L of \mathbb{N} so that for every $d \in \mathbb{N}$ and $i_1 < \dots < i_d$, the sequence $(\gamma_{i_j}, E_{i_j})_{j=1}^d$ satisfies either (i), (ii), or (iii) of Definition 2.6.*

Proof. Choose j_i so that $\text{weight}(\gamma_{j_i}) = m_{j_i}^{-1}$. A Ramsey argument yields that in passing to a subsequence either the j_i 's are pairwise incomparable in the sense of Definition 2.5 or the sequence $(\sigma^{-1}(i_j))_j$ is a chain of elements of \mathcal{U} . In the first case it easily follows that (i) is satisfied. Otherwise, we conclude that there is a sequence of pairs $\{(\eta_k, x_k)\}_{k=1}^\infty$ and a strictly increasing sequence $(d_i)_i$ of \mathbb{N} so that for all $i \in \mathbb{N}$, $\sigma^{-1}(j_i) = \{(\eta_k, x_k)\}_{k=1}^{d_i}$. If, passing to a subsequence, for all $i \in \mathbb{N}$ $|e_{\gamma_i}^* \circ \bar{P}_{E_i}(x_{d_i+1})| > 16000$ we conclude that (ii) is satisfied. Otherwise, a compactness arguments yields that passing to a further subsequence (iii) is satisfied. \square

2.5. The space \mathfrak{X}_{nr} . We recursively choose subsets Δ_q of $\bar{\Delta}_q$ as follows: we set $\Delta_1 = \bar{\Delta}_1$, and if for $q \in \mathbb{N}$ we have chosen the sets $\Delta_1, \dots, \Delta_q$, set $\Gamma_q = \bigcup_{p=1}^q \Delta_p$ and

$$\begin{aligned} \Delta_{q+1} = & \{(q+1, m_j^{-1}, b^*) \in \bar{\Delta}_{q+1} : b^* \text{ is an } \alpha_c\text{-average of } \Gamma_q\} \\ & \cup \{(q+1, \xi, m_j^{-1}, b^*) \in \bar{\Delta}_{q+1} : \xi \in \Gamma_q, b^* \text{ is an } \alpha_c\text{-average of } \Gamma_q \\ & \text{with size } s(b^*) \geq \mathcal{N}_{\text{rank}(\xi)}\}. \end{aligned}$$

For the definition of $\mathcal{N}_{\text{rank}(\xi)}$ see the text below (1). Note that for all q the set Δ_q is nonempty; as for $q > 1$, $(q, m_1^{-1}, \bar{d}_0^*) \in \Delta_q$ (recall $\Delta_1 = \bar{\Delta}_1 = \{0\}$). We define $\Gamma = \bigcup_q \Gamma_q$.

Proposition 2.9. *The set Γ is a self-determined subset of $\bar{\Gamma}$, hence it defines a Bourgain–Delbaen space $\mathfrak{X}_{(\Gamma_q, i_q)_q} = \mathfrak{X}_{\text{nr}}$ so that the restriction from $\bar{\Gamma}$ to Γ defines a quotient operator $R : \mathfrak{B}_{\text{nr}\Gamma} \rightarrow \mathfrak{X}_{\text{nr}}$.*

Proof. We will use Proposition 1.5(c). As it clearly follows from Proposition 2.1 and the definition of the set Γ , for every $q \in \mathbb{N}$, if $\gamma \in \Delta_{q+1}$, then there is b^* in $\langle \{e_\eta^* \circ \bar{P}_E : \eta \in \Gamma_q, E \subset \mathbb{N}\} \rangle$ and $j \in \mathbb{N}$ so that either $\bar{c}_\gamma^* = (1/m_j)b^*$ or $\bar{c}_\gamma^* = e_\eta^* + (1/m_j)b^*$ for some $\eta \in \Gamma_q$. We conclude that condition (c) of Proposition 1.5 is indeed satisfied. \square

2.6. Some remarks on the space \mathfrak{X}_{nr} . As we have mentioned earlier, for the space \mathfrak{X}_{nr} we shall use the standard notation $d_\gamma, c_\gamma^*, P_E$, etc. Henceforth, whenever we say α_c -average, we shall mean an α_c -average of Γ . Moreover, in the light of Proposition 1.13, it makes sense to identify any such average either with a basic average of Γ , i.e., $b^* = (1/n) \sum_{i=1}^d \varepsilon_i d_{\gamma_i}^*$, or with a functional $b^* = (1/n) \sum_{i=1}^d \varepsilon_i e_{\gamma_i}^* \circ P_{E_i}$ so that $d, n, (E_i)_{i=1}^d, (\varepsilon_i)_{i=1}^d$, and $(\gamma_i)_{i=1}^d \in \Gamma_q$ satisfy one of (i), (ii), or (iii) of Definition 2.6. We remark that this is not independent of the set $\bar{\Gamma}$. Nevertheless, if $\gamma \in \Gamma$ with $\gamma = (q+1, m_j^{-1}, b^*)$ or $\gamma = (q+1, \xi, m_j^{-1}, b^*)$, we can assume that b^* is an α_c -average as it was just described. Furthermore, if $\gamma = (q+1, m_j^{-1}, b^*)$, then by (2a)

$$(4a) \quad c_\gamma^* = \frac{1}{m_j} \frac{1}{n} \sum_{i=1}^d \varepsilon_i e_{\gamma_i}^* \circ P_{E_i},$$

whereas if $\gamma = (q + 1, \xi, m_j^{-1}, b^*)$, then moreover $\text{rank}(\xi) < \min E_1$, $n > 2^{\text{rank}(\xi)}$, and by (2b)

$$(4b) \quad c_\gamma^* = e_\xi^* + \frac{1}{m_j} \frac{1}{n} \sum_{i=1}^d \varepsilon_i e_{\gamma_i}^* \circ P_{E_i}.$$

We also note that $\|i_q\| \leq 2$ for all q , hence $\|P_E\| \leq 4$ for all intervals E of \mathbb{N} which implies that for every α_c -average b^* , $\|b^*\| \leq 4$.

The following is a restatement of [AH, Proposition 4.5] in the present setting.

Proposition 2.10. *Let $q \in \mathbb{N}$ and $\gamma \in \Delta_{q+1}$ with $\text{weight}(\gamma) = m_j$ and $\text{age}(\gamma) = a \leq n_j$. Then there exist natural numbers $0 = p_0 < p_1 < \dots < p_a = q + 1$, elements $\xi_1, \dots, \xi_a = \gamma$ of weight m_j with $\xi_r \in \Delta_{p_r}$ for $r = 1, \dots, a$, and a very fast growing sequence of α_c -averages $(b_r^*)_{r=1}^a$ with $b_r^* \in B_{p_{r-1}, p_r-1}$ for $r = 1, \dots, a$ such that*

$$(5) \quad e_\gamma^* = \sum_{r=1}^a d_{\xi_r}^* + \frac{1}{m_j} \sum_{r=1}^a b_r^* = \sum_{r=1}^a d_{\xi_r}^* + \frac{1}{m_j} \sum_{r=1}^a b_r^* \circ P_{(p_{r-1}, p_r)}.$$

Moreover, if $1 \leq t < a$, then

$$(6) \quad e_\gamma^* = e_{\xi_t}^* + \sum_{r=t+1}^a d_{\xi_r}^* + \frac{1}{m_j} \sum_{r=t+1}^a b_r^*.$$

The form (5) of e_γ^* is called the evaluation analysis of γ .

A finite inductive argument also yields the following.

Proposition 2.11. *Let $j \in \mathbb{N}$, let $1 \leq a \leq n_j$, let $0 \leq p_0 < p_1 < \dots < p_a = q + 1$ with $j \leq p_1$, and let $(b_r^*)_{r=1}^a$ be a very fast growing sequence of α_c -averages with $b_r^* \in B_{p_{r-1}, p_r-1}$ for $r = 1, \dots, a$. Then there are $\gamma \in \Delta_{q+1}$ and $\xi_1, \dots, \xi_a = \gamma$, all of weight m_j , with $\xi_r \in \Delta_{p_r}$ for $r = 1, \dots, a$ so that γ has an evaluation analysis*

$$e_\gamma^* = \sum_{r=1}^a d_{\xi_r}^* + \frac{1}{m_j} \sum_{r=1}^a b_r^*.$$

2.7. Subspaces and quotients of \mathfrak{X}_{AH} defined by self-determined subsets.

We mention some results that can be derived by considering subspaces and quotients of the Argyros–Haydon space that are defined by self-determined sets.

Remark 2.12. The Argyros–Haydon space \mathfrak{X}_{AH} from [AH] can also be obtained by finding an appropriate self-determined subset Γ^{AH} of $\bar{\Gamma}$, hence the space \mathfrak{X}_{AH} is a quotient of \mathfrak{B}_{MT} as well.

Remark 2.13. In [KL] T. Kania and J. N. Laustsen choose a \mathcal{L}_∞ -subspace Y of \mathfrak{X}_{AH} so that every bounded linear operator $T : Y \rightarrow \mathfrak{X}_{\text{AH}}$ is a scalar multiple of the inclusion plus a compact operator. Following their notation, the Bourgain–Delbaen space \mathfrak{X}_{AH} is defined by a set Γ^{AH} and the space Y is the closed linear span of a subsequence $(d_\gamma)_{\gamma \in \Gamma'}$ of the basis, where Γ' is an appropriately chosen subset of Γ^{AH} . They prove that this set Γ' has the property that whenever $\gamma \in \Gamma'$, then $d_\gamma|_{\Gamma^{\text{AH}} \setminus \Gamma'} = 0$ ([KL, Lemma 2.5]), which by Proposition 1.5(d) is equivalent to $\Gamma^{\text{AH}} \setminus \Gamma'$ being self-determined. Actually, this is the only property of Γ' that they use to prove the properties of the space Y . Hence, they have proved the result below.

Proposition 2.14. *Let $\mathfrak{X}_{(\Gamma_q, i_q)_q} = \mathfrak{X}_{\text{AH}}$ be the \mathcal{L}_∞ -space with the scalar-plus-compact property from [AH]. Also let Γ' be a self-determined subset of $\Gamma^{\text{AH}} = \bigcup_q \Gamma_q$ and $Y = \overline{\langle d_\gamma : \gamma \in \Gamma^{\text{AH}} \setminus \Gamma' \rangle}$. Then, every bounded linear operator $T : Y \rightarrow \mathfrak{X}_{\text{AH}}$ is a multiple of the inclusion plus a compact operator.*

Lemma 2.15. *Let X be a Banach space with a basis $(e_i)_i$ and assume that A is a subset of \mathbb{N} so that every bounded linear operator $T : Y = \overline{\langle \{e_i : i \in A\} \rangle} \rightarrow X$ is a multiple of the inclusion plus a compact operator. If B is a subset of \mathbb{N} so that Y isomorphically embeds into $Z = \overline{\langle \{e_i : i \in B\} \rangle}$, then the set $A \setminus B$ is finite. In particular, if the set $A \setminus B$ is infinite, then every bounded linear operator $T : Y \rightarrow Z$ is compact.*

Proof. We may clearly assume that the basis is seminormalized. If the set $A \setminus B$ is infinite, it contains an infinite sequence $(n_k)_k$. Let $T : Y \rightarrow Z$ be a bounded linear operator; then $T = \lambda I_{Y,X} + K$ with K a compact operator. As $n_k \notin B$, we obtain $e_{n_k}^*(Te_{n_k}) = 0$ for all $k \in \mathbb{N}$. By the compactness of K and passing to a subsequence, there is x_0 in X so that $(Te_{n_k} - \lambda e_{n_k})_k$ converges to x_0 in norm, which yields $0 = \lim_k e_{n_k}^*(Te_{n_k}) = \lambda + \lim_k e_{n_k}^*(x_0) = \lambda$, therefore $T = K$. \square

Proposition 2.14 and Lemma 2.15 immediately yield the following.

Corollary 2.16. *Let Γ_1, Γ_2 be two self-determined subsets of Γ^{AH} so that $\Gamma_2 \setminus \Gamma_1$ is infinite. If $Y = \overline{\langle d_\gamma : \gamma \in \Gamma^{\text{AH}} \setminus \Gamma_1 \rangle}$ and $Z = \overline{\langle d_\gamma : \gamma \in \Gamma^{\text{AH}} \setminus \Gamma_2 \rangle}$, then every bounded linear operator $T : Y \rightarrow Z$ is compact.*

It is not very difficult to find a continuum of self-determined subsets of Γ^{AH} that pairwise satisfy the assumptions of Corollary 2.16. We choose these sets in such a way that the corresponding quotients have similar properties as well. Recall that the set Γ^{AH} is built using a sequence of weights $(m_j, n_j)_j$, where the even weights are used freely to define new coordinates, whereas some restrictions are applied to the odd weights. For each infinite subset L of \mathbb{N} , one can define a self-determined subset Γ_L of Γ^{AH} by only using the weights $(m_{2j}, n_{2j})_{j \in L}$. This is done so that the weights $(m_{4j}, n_{4j})_{j \in L}$ are used unconditionally, whereas the weights $(m_{4j-2}, n_{4j-2})_{j \in L}$ are used conditionally, i.e., they assume the role of the odd weights in the construction from [AH]. For this last part, a coding function specific to the subset Γ_L needs to be used.

We observe that the subset Γ_L of Γ^{AH} induces a Bourgain–Delbaen space which is qualitatively identical to the space $\mathfrak{X}_{\text{AH}}[(\mathcal{A}_{n_j}, 1/m_j)_{j \in 2L}]$ defined in [AH, Subsection 10.2]. Hence, if we choose a continuum $\{L_\alpha : \alpha \in \mathfrak{c}\}$ of infinite subsets of \mathbb{N} , with pairwise finite intersections, set $\Gamma_\alpha = \Gamma_{L_\alpha}$, and define the spaces $Y_\alpha = \overline{\langle d_\gamma : \gamma \in \Gamma \setminus \Gamma_\alpha \rangle}$, then the spaces $\{Y_\alpha : \alpha \in \mathfrak{c}\}$ satisfy the assumptions of Corollary 2.16 and the spaces $X_\alpha = \mathfrak{X}_{\text{AH}}/Y_\alpha$, $\alpha \in \mathfrak{c}$ satisfy the conclusion of [AH, Theorem 10.4], i.e., the following holds.

Theorem 2.17. *There is a continuum of \mathcal{L}_∞ -subspaces $\{Y_\alpha : \alpha \in \mathfrak{c}\}$ of \mathfrak{X}_{AH} satisfying the following:*

- (i) *Each space Y_α has the scalar-plus-compact property, and for every $\alpha \neq \beta$ every bounded linear operator $T : Y_\alpha \rightarrow Y_\beta$ is compact.*
- (ii) *Each $X_\alpha = \mathfrak{X}_{\text{AH}}/Y_\alpha$ is a hereditarily indecomposable space with the scalar-plus-compact property, and for every $\alpha \neq \beta$ every bounded linear operator $T : X_\alpha \rightarrow X_\beta$ is compact.*

Observe that for fixed α , all three spaces \mathfrak{X}_{AH} , Y_α , and $X_\alpha = \mathfrak{X}_{\text{AH}}/Y_\alpha$ are hereditarily indecomposable \mathcal{L}_∞ -spaces with the scalar-plus-compact property.

Remark 2.18. A version $\tilde{\mathfrak{X}}_{\text{nr}}$ of the space \mathfrak{X}_{nr} can be obtained as a quotient of a version $\tilde{\mathfrak{X}}_{\text{AH}}$ of the space \mathfrak{X}_{AH} (the difference being similar to the one stated in Remark 2.2). This is achieved by defining a self-determined subset $\tilde{\Gamma}$ of Γ^{AH} defined only on coordinates with even weight. This construction also satisfies that if Y is the kernel of the quotient operator $R : \tilde{\mathfrak{X}}_{\text{AH}} \rightarrow \tilde{\mathfrak{X}}_{\text{nr}}$, then $\tilde{\mathfrak{X}}_{\text{AH}}$, $\tilde{\mathfrak{X}}_{\text{nr}}$, and Y all have the scalar-plus-compact property.

Remark 2.19. A self-determined subset Γ of Γ^{AH} can be chosen so that the corresponding quotient is isomorphic to c_0 . This set can be chosen starting with a random point γ and then only allowing operations that result in new coordinates with age at most one. One can then choose a self-determined subset Γ' of Γ^{AH} , almost disjoint to Γ , with the same properties. We set $Y = \langle \{d_\gamma : \gamma \notin \Gamma\} \rangle$ and $Y' = \langle \{d_\gamma : \gamma \notin \Gamma'\} \rangle$. Then Y, Y' are subspaces of \mathfrak{X}_{AH} with the scalar-plus-compact property so that every operator from one to the other is compact (by Corollary 2.16). However, both spaces $\mathfrak{X}_{\text{AH}}/Y$ and $\mathfrak{X}_{\text{AH}}/Y'$ are isomorphic to c_0 .

3. THE α -INDEX

A tool that has been used in recent constructions involving saturation under constraints is the α -index of a block sequence ([ABM], [AM1], [AM2], and more). This index helps characterize what spreading models a given block sequence admits. However, due to the Bourgain–Delbaen construction and the mixed-Tsirelson setting, in the space \mathfrak{X}_{nr} the index does not fully determine spreading models. Nevertheless, it remains an integral part of the study of spaces constructed with the method of saturation under constraints.

Definition 3.1. Let $(x_k)_k$ be a block sequence in \mathfrak{X}_{nr} so that for every very fast growing sequence of α_c -averages $(b_j^*)_j$ and every subsequence $(x_{k_j})_j$ of $(x_k)_k$,

$$\lim_j |b_j^*(x_{k_j})| = 0.$$

Then we say that the α -index of $(x_k)_k$ is zero and write $\alpha((x_k)_k) = 0$. Otherwise, we write $\alpha((x_k)_k) > 0$.

Proposition 3.2. Let $(x_k)_k$ be a block sequence in \mathfrak{X}_{nr} . The following assertions are equivalent:

- (i) The α -index of $(x_k)_k$ is zero.
- (ii) For every $\varepsilon > 0$ there exist k_0 and $j_0 \in \mathbb{N}$ so that for every $k \geq k_0$, interval E of \mathbb{N} , and α_c -average b^* with $s(b^*) \geq j_0$, $|b^*(P_E x_k)| < \varepsilon$.

Remark 3.3. Using the above characterization and that for every α_c -average b^* , $\|b^*\| \leq 4$, it easily follows that if $(x_k)_k$ is a block sequence in \mathfrak{X}_{nr} with $\alpha((x_k)_k) = 0$, then for every $a \in \mathbb{N}$ and $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ so that for all $k \geq k_0$ and a very fast growing sequence of α_c -averages $(b_r^*)_{r=1}^a$, $\sum_{r=1}^a |b_r^*(x_k)| < 4\|x_k\| + \varepsilon$.

The proof of the next result is an easy consequence of Definition 3.1 and Proposition 2.11.

Proposition 3.4. *Let $(x_k)_k$ be a seminormalized block sequence in \mathfrak{X}_{nr} with $\alpha((x_k)_k) > 0$. Then there exist $\theta > 0$ and a subsequence of $(x_k)_k$, again denoted by $(x_k)_k$, so that for all natural numbers $j \leq k_1 < \dots < k_{n_j}$ and scalars $(\lambda_i)_{i=1}^{n_j}$,*

$$\left\| \sum_{i=1}^{n_j} \lambda_i x_{k_i} \right\| \geq \theta \frac{1}{m_j} \sum_{i=1}^{n_j} |\lambda_i|.$$

Proposition 3.5. *Let $(x_k)_k$ be a normalized block sequence in \mathfrak{X}_{nr} with $\alpha((x_k)_k) = 0$ and $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$. Then $(x_k)_k$ has a subsequence, which we also denote by $(x_k)_k$, that generates a spreading model isometric to c_0 . Moreover, there exists a strictly increasing sequence of natural numbers $(j_k)_k$ so that for every natural number $n \leq k_1 < \dots < k_n$, scalars $(\lambda_i)_{i=1}^n$, $\gamma \in \Gamma$ with $\text{weight}(\gamma) = m_j^{-1} > m_{j_n}^{-1}$, and interval E of \mathbb{N} ,*

$$(7) \quad \left| e_\gamma^* \circ P_E \left(\sum_{i=1}^n \lambda_i x_{k_i} \right) \right| \leq \frac{C}{m_j} \max_{1 \leq i \leq n} |\lambda_i|,$$

where $C = 8$.

Proof. Using Proposition 3.2 and $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$, we pass to a subsequence of $(x_k)_k$, again denoted by $(x_k)_k$, and choose a strictly increasing sequence of natural numbers (j_k) so that the following are satisfied:

- (i) for every $k \in \mathbb{N}$, $j_{k+1} > \max \text{supp } x_k$,
- (ii) for every $k \in \mathbb{N}$, $\sum_{m \geq k} \sup_{\gamma \in \Gamma} |d_\gamma^*(x_m)| < 1/(2km_{j_k}n_{j_k})$, and
- (iii) for every $k_0, k \in \mathbb{N}$ with $k \geq k_0$, every interval E of \mathbb{N} , and every α_c -average b^* with $s(b^*) \geq \min \text{supp } x_k$, $|b^*(P_E x_k)| < 1/(2k_0n_{j_{k_0}})$.

We claim that $(x_k)_k$ satisfies the conclusion. By induction on q we shall prove the following: for every $\gamma \in \Gamma_q$, interval E of \mathbb{N} , natural numbers $n \leq k_1 < \dots < k_n$, and scalars $\lambda_1, \dots, \lambda_n$ in $[-1, 1]$:

$$(8) \quad \left| e_\gamma^* \left(\sum_{i=1}^n \lambda_i x_{k_i} \right) \right| < 1 + \frac{15}{m_{j_n}} \quad \text{and} \quad \left| e_\gamma^* \circ P_E \left(\sum_{i=1}^n \lambda_i x_{k_i} \right) \right| \leq 7.$$

If moreover $\text{weight}(\gamma) = m_j^{-1}$ with $j < j_n$, then

$$(9) \quad \left| e_\gamma^* \circ P_E \left(\sum_{i=1}^n \lambda_i x_{k_i} \right) \right| < \frac{8}{m_j}.$$

The desired conclusion clearly follows from the above.

The case $q = 1$ is an easy consequence of the definition of Δ_1 . Assume now that q is such that the conclusion holds for every $\gamma \in \Gamma_q$, and let $\gamma \in \Gamma_{q+1}$ with $\text{weight}(\gamma) = m_j^{-1}$ and let E be an interval of \mathbb{N} . Let $e_\gamma^* = \sum_{t=1}^a d_{\xi_t}^* + (1/m_j) \sum_{t=1}^a b_t^*$ be the evaluation analysis of γ , according to Proposition 2.10. Then $1 \leq a \leq n_j$ and $(b_t^*)_{t=1}^a$ is a very fast growing sequence of α_c -averages of Γ_q . Assuming that $\text{rank}(\gamma) \geq \min \text{supp } x_{k_1}$ (otherwise the estimates appearing in (8) and (9) are all zero), set $t_0 = \min\{t : \max \text{supp } b_q^* \geq \min \text{supp } x_{k_1}\}$. The inductive assumption easily implies that for $1 \leq i_0 \leq n$,

$$(10) \quad \left| b_{t_0}^* \circ P_E \left(\sum_{i=1}^{i_0} \lambda_i x_{k_i} \right) \right| \leq 7.$$

We shall distinguish three cases concerning the weight of γ .

Case 1: $j < j_{k_1}$. Since the sequence $(b_q^*)_{q=1}^d$ is very fast growing, for $t > t_0$ we have $s(b_t^*) > \max \text{supp } b_{t_0}^* \geq \min \text{supp } x_{k_1}$. Also $a \leq n_j < n_{j_{k_1}}$ and $n \leq k_1$, and hence by (iii) we conclude:

$$(11) \quad \sum_{t=t_0+1}^a \left| b_t^* \circ P_E \left(\sum_{i=1}^n \lambda_i x_{k_i} \right) \right| < a \frac{n}{2k_1 n_{j_{k_1}}} \leq \frac{1}{2}.$$

By (ii) we obtain

$$(12) \quad \left| \sum_{t=1}^a d_{\xi_t}^* \circ P_E \left(\sum_{i=1}^n \lambda_i x_{k_i} \right) \right| < a \frac{n}{2k_1 m_{j_{k_1}} n_{j_{k_1}}} \leq \frac{1}{2m_{j_{k_1}}}.$$

Combining (10) with (11) and (12):

$$(13) \quad \left| e_\gamma^* \circ P_E \left(\sum_{i=1}^n \lambda_i x_{k_i} \right) \right| < \frac{8}{m_j}.$$

This concludes the proof of the first case and also (9) of the inductive assumption (the first part of (8) follows if we set $E = \mathbb{N}$ and use $m_1 \geq 8$).

Case 2: there is $1 \leq i_0 < n$ so that $j_{k_{i_0}} \leq j < j_{k_{i_0+1}}$. Arguing in a similar manner as in the previous case, we obtain

$$(14) \quad \left| e_\gamma^* \circ P_E \left(\sum_{i>i_0} \lambda_i x_{k_i} \right) \right| < \frac{8}{m_{j_{k_{i_0}}}}.$$

Note that (i) and $\text{weight}(d_{\xi_t}) = m_j^{-1}$ imply $\text{rank}(\xi_t) > \max \text{supp } x_{i_0-1}$, which yields $\sum_{t=1}^a d_{\xi_t}^* \circ P_E(\sum_{i<i_0} \lambda_i x_{k_i}) = 0$ and $\sum_{t=t_0+1}^a b_t^* \circ P_E(\sum_{i<i_0} \lambda_i x_{k_i}) = 0$. Combining this with (10):

$$(15) \quad \left| e_\gamma^* \circ P_E \left(\sum_{i<i_0} \lambda_i x_{k_i} \right) \right| = \frac{1}{m_j} \left| b_{t_0}^* \circ P_E \left(\sum_{i<i_0} \lambda_i x_{k_i} \right) \right| < \frac{7}{m_{j_{k_{i_0}}}}.$$

As $\|e_\gamma^* \circ P_E\| \leq 4$ we obtain that $|e_\gamma^* \circ P_E(x_{k_{i_0}})| \leq 4$, which in conjunction with (14) and (15) yields

$$(16) \quad \left| e_\gamma^* \circ P_E \left(\sum_{i=1}^n \lambda_i x_{k_i} \right) \right| \leq 4 + \frac{15}{m_{j_{k_{i_0}}}} \leq 4 + \frac{15}{m_1} \leq 7.$$

Similarly, for $E = \mathbb{N}$ and using $|e_\gamma^*(x_{k_{i_0}})| \leq 1$ we obtain

$$(17) \quad \left| e_\gamma^* \left(\sum_{i=1}^n \lambda_i x_{k_i} \right) \right| < 1 + \frac{15}{m_{j_{k_{i_0}}}} \leq 1 + \frac{1}{m_{j_n}}.$$

This concludes the proof of the second case. The third case, in which $j \geq j_{k_n}$, is treated in a similar manner as the second one. \square

Remark 3.6. We point out that the space without reflexive subspaces constructed in [AM2] admits precisely three spreading models in every subspace, namely the unit vector basis of ℓ_1 , the unit vector basis of c_0 , and the summing basis of c_0 . This is no longer true for the space \mathfrak{X}_{nr} presented in this paper, as this space admits a large variety of spreading models. This is due to the \mathcal{L}_∞ structure and mainly due to the mixed-Tsirelson frame used to define the norm, as opposed to the Tsirelson frame used in [AM2]. We also point out that in [AM2] the α -index alone is sufficient

to fully describe the spreading models admitted by a block sequence. Here, this is no longer the case, and the condition $\alpha((x_k)_k) = 0$ is not sufficient for a sequence to generate a c_0 spreading model and $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$ is necessary as well. As it was shown in the proof [AH, Proposition 10.1], the sequence $(y_q)_q$, with $y_q = \sum_{\gamma \in \Delta_q} d_\gamma$, generates an ℓ_1 spreading model. The same sequence in \mathfrak{X}_{nr} generates an ℓ_1 spreading model as well, however it can be shown that $\alpha((y_q)_q) = 0$. In the special case when $(x_k)_k$ is a subsequence of the basis $(d_\gamma)_{\gamma \in \Gamma}$, we have $\alpha((x_k)_k) = 0$ and $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| \neq 0$. However, $(x_k)_k$ has a subsequence generating a c_0 spreading model. This is proved by replacing the condition $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$ with the conclusion of the following lemma.

Lemma 3.7. *Let $\{\gamma_k : k \in \mathbb{N}\}$ be an infinite subset of Γ . Then there exists an infinite subset L of \mathbb{N} satisfying the following: for every $\gamma \in \Gamma$, if $e_\gamma^* = \sum_{r=1}^a d_{\xi_r}^* + (1/m_j) \sum_{r=1}^a b_r^*$ is the evaluation analysis of γ , then the set $\{\xi_r : r = 1, \dots, a\} \cap \{\gamma_k : k \in L\}$ is at most a singleton.*

Proof. For $\eta, \gamma \in \Gamma$ with $\text{rank}(\eta) < \text{rank}(\gamma)$, we shall say that η is in the analysis of γ if the following happens. If $(d_{\xi_r}^*)_{r=1}^a$ is the sequence appearing in the evaluation analysis of γ as in (5), then there is some $1 \leq r \leq a$ so that $\xi_r = \eta$. Note that (6) implies that this property is transitive and it also easily follows that there exists no infinite chain with this property.

By passing to an infinite subset we may assume that $(\text{rank}(\gamma_k))_k$ is strictly increasing, and using an easy Ramsey argument we may also assume that for $k < m$, γ_k is not in the analysis of γ_m , which implies the desired result. □

An application of the above lemma and arguments similar to those used in the proof of Proposition 3.5 yield the next result.

Proposition 3.8. *Let $(d_{\gamma_k})_k$ be a subsequence of the basis $(d_\gamma)_{\gamma \in \Gamma}$ of \mathfrak{X}_{nr} . Then it admits a subsequence generating an isometric c_0 spreading model. Furthermore, there exists a constant $C > 0$ so that for every natural number $n \leq k_1 < \dots < k_n$, scalars $(\lambda_i)_{i=1}^n$, $\gamma \in \Gamma$ with $\text{weight}(\gamma) = m_j^{-1} > m_{j_n}^{-1}$, and interval E of \mathbb{N} ,*

$$(18) \quad \left| e_\gamma^* \circ P_E \left(\sum_{i=1}^n \lambda_i d_{\gamma_{k_i}} \right) \right| \leq \frac{C}{m_j} \max_{1 \leq i \leq n} |\lambda_i|.$$

If the set $\{(\text{weight}(\gamma_k))^{-1} : k \in \mathbb{N}\}$ is unbounded, then $C = 8$. Otherwise there is $j_0 \in \mathbb{N}$ with $C = 2 + m_{j_0}$.

Lemma 3.9. *Let $(x_k)_k$ be a block sequence in \mathfrak{X}_{nr} generating a c_0 spreading model, and let $(\gamma_k)_k$ be a sequence in Γ so that $|e_{\gamma_k}^*(x_k)| > (3/4)\|x_k\|$ for all $k \in \mathbb{N}$. If the set $\{(\text{weight}(\gamma_k))^{-1} : k \in \mathbb{N}\}$ is bounded, then there exist $\varepsilon > 0$, an infinite subset L of \mathbb{N} , and a sequence $(\eta_k)_{k \in L}$ of Γ so that $|d_{\eta_k}^*(x_k)| > \varepsilon$ for all $k \in L$.*

Proof. Passing to a subsequence, there are $j \in \mathbb{N}$ and $1 \leq a \leq n_j$ so that each γ_k has an evaluation analysis $e_{\gamma_k}^* = \sum_{r=1}^a d_{\xi_r^k}^* + (1/m_j) \sum_{r=1}^a b_{k,r}^*$. Since $(x_k)_k$ generates a c_0 spreading model, by Proposition 3.4 we conclude that $\alpha((x_k)_k) = 0$. By Remark 3.3 we can assume that $|(1/m_j) \sum_{r=1}^a b_{k,r}^*(x_k)| < (5/m_j)\|x_k\| \leq (5/8)\|x_k\|$, which yields $|\sum_{r=1}^a d_{\xi_r^k}^*(x_k)| > 1/8\|x_k\|$, for all $k \in \mathbb{N}$. Setting $\varepsilon = \inf_k \|x_k\|/(8a)$, the result easily follows. □

Lemma 3.10. *Let $(x_k)_k$ be a block sequence in \mathfrak{X}_{nr} generating a c_0 spreading model. Then there are $\varepsilon > 0$ and a subsequence of $(x_k)_k$, again denoted by $(x_k)_k$, so that for every natural number $n \leq k_1 < \dots < k_n$ and sequence of alternating signs $(\varepsilon_i)_{i=1}^n$, if $y = \sum_{i=1}^n \varepsilon_i x_i$, there is an α_c -average b^* of size $s(b^*) = n$ so that $b^*(y) > \varepsilon$.*

Proof. Choose a sequence $(\gamma_k)_k$ in Γ so that $|e_{\gamma_k}^*(x_k)| > (3/4)\|x_k\|$ for all $k \in \mathbb{N}$. Passing to a subsequence, and perhaps considering the sequence $(-x_k)_k$, we may assume that $e_{\gamma_k}^*(x_k) > (3/4)\|x_k\|$ for all $k \in \mathbb{N}$. If the set $\{(\text{weight}(\gamma_k))^{-1} : k \in \mathbb{N}\}$ is unbounded, set $E_k = \text{ran } x_k$ and pass to a subsequence satisfying the conclusion of Proposition 2.8. Setting $\varepsilon = \inf_k \|x_k\|/4$, it easily follows that for $n \leq k_1 < \dots < k_n$ and alternating signs $(\varepsilon_i)_{i=1}^n$, $b^* = (1/n) \sum_{i=1}^n \varepsilon_i e_{\gamma_{k_i}}^* \circ P_{\text{ran } x_{k_i}}$ is the desired α_c -average. Otherwise, we apply Lemma 3.9 and argue in a similar manner. \square

Remark 3.11. The proof of Lemma 3.10 actually yields that in the case $(x_k)_k$ where generates a c_0 spreading model and $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$, if $(\gamma_k)_k$ satisfies $e_{\gamma_k}^*(x_k) > (3/4)\|x_k\|$ for all k , we can choose a subsequence of $(x_k)_k$, again denoted by $(x_k)_k$, so that for any natural numbers $n \leq k_1 < \dots < k_n$ and sequence of alternating signs $(\varepsilon_i)_{i=1}^n$, if $y = \sum_{i=1}^n (\varepsilon_i/e_{\gamma_i}^*(x_i))x_i$, then $b^* = (1/n) \sum_{i=1}^n \varepsilon_i e_{\gamma_i}^* P_{\text{ran } x_{k_i}}$ is an α_c -average of size $s(b^*) = n$ so that $b^*(y) = 1$.

It immediately follows that if $(x_k)_k$ is a sequence generating a c_0 spreading model, then it has a further block sequence $(y_k)_k$ with $\alpha((y_k)_k) > 0$. Hence by Proposition 3.4 we deduce the following.

Corollary 3.12. *The space \mathfrak{X}_{nr} does not contain c_0 .*

4. EXACT PAIRS AND DEPENDENT SEQUENCES

In this section we define exact pairs and dependent sequences and we also show that they can be found in every block subspace. They are important tools used in what follows to deduce all the properties of the space. The definition of a dependent sequence is based on that from [AH] and has been slightly modified in order to obtain a stronger result.

4.1. Rapidly increasing sequences and ℓ_1^n -averages. We recall the definition of a rapidly increasing sequence (RIS), state the basic inequality, for which we do not include a proof (for details see [AH, Section 5]), and also recall the notion of normalized ℓ_1^n -averages (see also [AH, Section 8]). The auxiliary space used for the basic inequality is $T[(\mathcal{A}_{3n_j}, m_j^{-1})_j]$ (see [AH, Section 2.4]).

Definition 4.1. A (finite or infinite) block sequence $(x_k)_k$ is called a C -rapidly increasing sequence, or a C -RIS, where $C \geq 1$, if there is a strictly increasing sequence of natural numbers $(j_k)_k$ so that the following hold:

- (i) $\|x_k\| \leq C$,
- (ii) $j_{k+1} > \max \text{supp } x_k$, and
- (iii) $|e_\gamma^*(x_k)| < C/m_j$ whenever $\text{weight}(\gamma) = m_j^{-1}$ and $j < j_k$, for all k .

Remark 4.2. Note that if an infinite block sequence satisfies (i) and (iii) of Definition 4.1, for some C and a strictly increasing sequence $(j_k)_k$, then it has a subsequence which is a C -RIS.

The following proposition has been essentially proven in [AH]. It is a consequence of the basic inequality and it follows by combining [AH, Corollary 5.5] with [AH, Proposition 5.6]. Statement (21) in particular follows by applying [AH, Lemma 5.3] to [AH, Proposition 5.6 (1)].

Proposition 4.3. *If $(x_k)_k$ is a C -RIS, then for any scalars $(\lambda_k)_k$ we have*

$$(19) \quad \left\| \sum_k \lambda_k x_k \right\| \leq 10C \left\| \sum_k \lambda_k e_k \right\|_{T[(\mathcal{A}_{3n_j}, m_j^{-1})_j]},$$

where the right-hand norm is taken in $T[(\mathcal{A}_{3n_j}, m_j^{-1})_j]$. More precisely, if $j \in \mathbb{N}$ and $(x_k)_{k=1}^{n_j}$ is a C -RIS, then

$$(20) \quad \left\| \frac{m_j}{n_j} \sum_{k=1}^{n_j} x_k \right\| \leq 10C,$$

and if $\gamma \in \Gamma$ with $\text{weight}(\gamma) = m_i^{-1}$ and E is an interval of \mathbb{N} , then

$$(21) \quad \left| e_\gamma^* \circ P_E \left(\frac{m_j}{n_j} \sum_{k=1}^{n_j} x_k \right) \right| \leq \begin{cases} \frac{112C}{m_i} & \text{if } i < j \text{ and} \\ \frac{16Cm_j}{n_j} + \frac{24Cm_j}{m_i} + \frac{80C}{m_i} & \text{if } i \geq j. \end{cases}$$

Definition 4.4. An element x of \mathfrak{X}_{nr} will be called a C - ℓ_1^n -average if there exists a block sequence $(x_k)_{k=1}^n$ in \mathfrak{X}_{nr} such that $x = (1/n) \sum_{k=1}^n x_k$ and $x_k \leq C$ for all k . We say that x is a normalized C - ℓ_1^n average if, in addition, $\|x\| = 1$.

Proposition 3.4 implies that a sequence $(x_k)_k$ with positive α -index supports normalized C - ℓ_1^n -averages. This can be deduced using an argument similar to that in the proof of [AT, Lemma II.22, p. 33].

Lemma 4.5. *Let $(x_k)_k$ be a seminormalized block sequence in \mathfrak{X}_{nr} with $\alpha((x_k)_k) > 0$. Then for every $C > 1$ and $n \in \mathbb{N}$ there exist further normalized block vectors $(y_k)_{k=1}^n$ of $(x_k)_k$ so that $y = (1/n) \|\sum_{k=1}^n y_k\| \geq 1/C$. In particular, the vector $(1/\|y\|)y$ is a normalized C - ℓ_1^n -average.*

A standard argument yields the following result (for a proof see, e.g., [ABM, Lemma 3.3]).

Lemma 4.6. *Let y be a normalized C - ℓ_1^n -average, and let b^* be an α_c -average. Then $|b^*(y)| < 4C/s(b^*) + 8C/n$. In particular, if $(y_k)_k$ is a block sequence in \mathfrak{X}_{nr} so that each y_k is a C - $\ell_1^{r_k}$ -average with $(r_k)_k$ strictly increasing, then $\alpha((y_k)_k) = 0$ and $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(y_k)| = 0$.*

Remark 4.7. As it is shown in [AH, Lemma 8.4], a sequence of C - ℓ_1^n -averages with strictly increasing n 's has a subsequence which is a $2C$ -RIS.

A standard argument, using either Lemma 4.6 and Proposition 3.5 or Remark 4.7 and (20), yields the following.

Proposition 4.8. *The FDD of \mathfrak{X}_{nr} is shrinking. In particular, \mathfrak{X}_{nr}^* is isomorphic to ℓ_1 .*

Proposition 4.9. *Every block subspace X of \mathfrak{X}_{nr} contains a normalized block sequence $(y_k)_k$ with $\alpha((y_k)_k) = 0$ and $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(y_k)| = 0$. In particular, every subspace of \mathfrak{X}_{nr} admits a c_0 spreading model.*

Proof. As the sequence $(d_\gamma^*)_{\gamma \in \Gamma}$ is weak-star null, Corollary 3.12 implies that there is a further block subspace Z of X so that for every bounded block sequence $(z_k)_k$ in Z , $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(z_k)| = 0$. We fix any normalized block sequence $(z_k)_k$ in Z . If $\alpha((z_k)_k) = 0$, then this is the desired sequence. Otherwise, $\alpha((z_k)_k) > 0$ and Lemmas 4.5 and 4.6 yield that there is a further block sequence $(y_k)_k$ of $(z_k)_k$ satisfying the conclusion. The second part follows from Proposition 3.5. \square

Proposition 4.10. *The space \mathfrak{X}_{nr} is not \mathcal{L}_∞ -saturated. More precisely, if X is generated by a skipped-block sequence of \mathfrak{X}_{nr} , then X does not contain a \mathcal{L}_∞ -subspace.*

Proof. Assume that there is X as in the statement that contains a \mathcal{L}_∞ -subspace Y . Proposition 4.8 and [LS, Corollary, p. 182] yield that Y^* is isomorphic to ℓ_1 . By Proposition 4.9 we may then find a normalized block sequence in Y , which is a perturbation of a skipped block sequence $(x_i)_i$, and generates a c_0 spreading model. By Lemma 3.10 there are a normalized block sequence $(y_i)_i$ of $(x_i)_i$, $\varepsilon > 0$, and a very fast growing sequence of α_c -averages $(b_i^*)_i$ so that $b_i^*(y_i) > \varepsilon$ for all $i \in \mathbb{N}$. It follows that if f_i is the restriction of b_i^* onto Y , then $(f_i)_i$ has a subsequence equivalent to the unit vector basis of ℓ_1 . By Proposition 2.11, for every $j \in \mathbb{N}$ we can find $i_1 < \dots < i_{n_j}$ and $(d_{\xi_r}^*)_r$ in the annihilator of X , and hence also of Y , so that $\|\sum_{r=1}^{n_j} d_{\xi_r}^* + (1/m_j) \sum_{r=1}^{n_j} b_{i_r}^*\| \leq 1$, which implies that $\|\sum_{r=1}^{n_j} f_{i_r}^*\| \leq m_j$. We conclude that $(f_i)_i$ cannot be equivalent to the basis of ℓ_1 , a contradiction. \square

4.2. Exact vectors and exact pairs. An exact pair is a pair of the form (γ, x) , where $\gamma \in \Gamma$ and $x \in \mathfrak{X}_{nr}$, which satisfies certain properties. The terms of dependent sequences, which we study in the next subsection, are such pairs. The coordinate x of an exact pair is an exact vector. Its definition below is worth comparing to [AH, Definition 6.1], as the constraints appear in the present case.

Definition 4.11. Let $C \geq 1$, and let $j \in \mathbb{N}$. A finitely supported vector x in \mathfrak{X}_{nr} is called a (C, j) -exact vector if

- (i) $\sup_{\eta \in \Gamma} |d_\eta^*(x)| \leq Cm_j/n_j$,
- (ii) $\|x\| \leq C$,
- (iii) for every $i > j$, $\eta \in \Gamma$ with $\text{weight}(\eta) = m_i^{-1}$ and interval F of N ,

$$|e_\eta^* \circ P_E(x)| < \frac{C}{m_j},$$

- (iv) for every $i < j$, $a \leq n_i$ and very fast growing sequence of α_c -averages $(b_r^*)_{r=1}$,

$$\sum_{r=1}^a |b_r^*(x)| < \frac{C}{s(b_1^*)} + \frac{Cm_i}{m_j},$$

and

- (v) $\min \text{supp } x \geq m_j$.

Definition 4.12. Let $C \geq 1$, let $\theta > 0$, and let $j \in \mathbb{N}$. A pair (γ, x) is called a (C, j, θ) -exact pair if $\gamma \in \Gamma$ with $\text{weight}(\gamma) = m_j^{-1}$, x is a (C, j) -exact vector, and $e_\gamma^*(x) = \theta$.

Remark 4.13. If $(x_k)_k$ is a block sequence so that each x_k is a (C, j_k) -exact vector with $(j_k)_k$ strictly increasing, then (i) and (iv) of Definition 4.11 easily imply that $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$ and $\alpha((x_k)_k) = 0$.

The following estimate is similar to [AM1, Proposition 2.5] in a mixed-Tsirelson setting.

Lemma 4.14. *Let $j \in \mathbb{N}$, and let $k \leq n_j$. Then*

$$(22) \quad \left\| \frac{m_j}{n_j} \sum_{i=1}^k e_i \right\|_{T[(\mathcal{A}_{4n_j}, m_j^{-1})_j]} \leq \frac{k}{n_j} + \frac{1}{m_j},$$

where the the norm is taken in $T[(\mathcal{A}_{4n_j}, m_j^{-1})_j]$.

Proof. Let $x = (m_j/n_j) \sum_{i=1}^k e_i$, and let f be a functional in the norming set of $W[(\mathcal{A}_{4n_i}, m_i^{-1})_i]$. Define $E_1 = \{i : |f(e_i)| \leq 1/m_j\}$, $f_1 = E_1 f$, and $f_2 = f - f_1$. Clearly, $|f_1(x)| \leq k/n_j$. One can also verify that $f_2 \in W[(\mathcal{A}_{4n_i}, m_i^{-1})_{i \neq j}]$. By the last statement of [AH, Proposition 2.5] we obtain $|f_2(x)| \leq 1/m_j$. \square

The following estimate is a refinement of Lemma 4.6 for rapidly increasing sequences. It is based on (19) and Lemma 4.14. Its proof is very similar to that of [AM1, Lemma 3.7], however we include it for completeness.

Lemma 4.15. *Let $j \in \mathbb{N}$, let $(x_k)_{k=1}^{n_j}$ be a C-RIS, let $x = (m_j/n_j) \sum_{k=1}^{n_j} x_k$, and let b^* be a α_c -average. If $K = \#\{k : \text{ran } b^* \cap \text{ran } x_k \neq \emptyset\}$, then*

$$|b^*(x)| < \min \left\{ m_j \frac{CK/n_j}{s(b^*)}, \frac{10CK/n_j}{s(b^*)} + \frac{10C}{m_j} \right\} + 8C \frac{m_j}{n_j}.$$

Proof. If $b^* = (1/p) \sum_{i=1}^d \varepsilon_i e_{\gamma_i}^* \circ P_{E_i}$ with $1 \leq d \leq p$, define $G = \{k : \text{ran } b^* \cap \text{ran } x_k \neq \emptyset\}$,

$$A_1 = \{k \in G : \text{there exists } 1 \leq i \leq d \text{ with } \text{ran } x_k \subset E_i\},$$

$$A_2 = G \setminus A_1, \text{ and for each } k \in A_2 \text{ set}$$

$$J_k = \{1 \leq i \leq d : E_i \cap \text{ran } x_k \neq \emptyset\}.$$

It is easy to see that $\sum_{k \in A_2} (\#J_k) \leq 2\#(\bigcup_{k \in A_2} J_k) \leq 2p$, hence we obtain

$$(23) \quad \begin{aligned} \left| b^* \left(\frac{m_j}{n_j} \sum_{k \in A_2} x_k \right) \right| &\leq \frac{m_j}{pn_j} \sum_{k \in A_2} \sum_{i \in J_k} |e_{\gamma_i}^* \circ P_{E_i}(x_k)| \\ &\leq \frac{m_j}{pn_j} \sum_{k \in A_2} \sum_{i \in J_k} \|P_{E_k}\| \|x_k\| \leq \frac{4Cm_j}{pn_j} \sum_{k \in A_2} (\#J_k) \\ &\leq \frac{8Cm_j}{n_j}. \end{aligned}$$

We used that $\sup\{\|P_E\| : E \text{ is an interval of } \mathbb{N}\} \leq 2$.

We now estimate the action of b^* on A_1 . Define $S = \{1 \leq i \leq d : \text{there exists } k \in A_1 \text{ with } \text{ran } x_k \subset E_i\}$, and for $i \in S$ define $S_i = \{k \in A_1 : \text{ran } x_k \subset E_i\}$. Note that $(S_i)_{i \in S}$ defines a partition of A_1 into disjoint sets. We evaluate

$$(24) \quad \begin{aligned} \left| b^* \left(\frac{m_j}{n_j} \sum_{k \in A_1} x_k \right) \right| &= \frac{1}{p} \left| \sum_{i \in S} e_{\gamma_i}^* \circ P_{E_i} \left(\frac{m_j}{n_j} \sum_{k \in S_i} x_k \right) \right| \\ &= \frac{1}{p} \left| \sum_{i \in S} e_{\gamma_i}^* \left(\frac{m_j}{n_j} \sum_{k \in S_i} x_k \right) \right| \leq \frac{1}{p} \sum_{i \in S} \left\| \left(\frac{m_j}{n_j} \sum_{k \in S_i} x_k \right) \right\|. \end{aligned}$$

We shall treat (24) in two different ways. We first just apply the triangle inequality to obtain

$$(25) \quad \left| b^* \left(\frac{m_j}{n_j} \sum_{k \in A_1} x_k \right) \right| \leq \frac{1}{p} \sum_{i \in S} (\#S_i) \frac{m_j}{n_j} C = \frac{Cm_j}{pn_j} \#A_1 \leq \frac{Cm_j}{pn_j} K \\ = m_j \frac{C(K/n_j)}{s(b^*)}.$$

The second way is to apply (19) to (24) and then combine the result with Lemma 4.14 as follows:

$$(26) \quad \left| b^* \left(\frac{m_j}{n_j} \sum_{k \in A_1} x_k \right) \right| \leq \frac{1}{p} \sum_{i \in S} 10C \left\| \frac{m_j}{n_j} \sum_{k \in S_i} e_k \right\|_{T[(\mathcal{A}_{n_j, m_j^{-1}})_j]} \\ \leq \frac{10C}{p} \sum_{i \in S} \left(\frac{\#S_i}{n_j} + \frac{1}{m_j} \right) \\ = \frac{10C}{s(b^*)} \frac{\sum_{i \in S} \#S_i}{n_j} + \frac{10C}{m_j} \frac{\#S}{p} \leq \frac{10C}{s(b^*)} \frac{\#A_1}{n_j} + \frac{10C}{m_j} \frac{d}{p} \\ \leq \frac{10C(K/n_j)}{s(b^*)} + \frac{10C}{m_j}.$$

We combine (23), (25), and (26) to obtain the desired conclusion. □

Lemma 4.16. *Let $j > i$ be natural numbers, let $(x_k)_{k=1}^{n_j}$ be a C -RIS with $\min \text{supp } x_k \geq j - 1$, and set $x = (m_j/n_j) \sum_{k=1}^{n_j} x_k$. Moreover, let $(b_r^*)_{r=1}^a$ be a very fast growing sequence of α_c -averages, with $a \leq n_i$, and assume that for every $1 \leq r \leq a$ there is at most one $1 \leq k \leq n_j$ so that $\text{ran } b_r^* \cap \text{ran } x_k \neq \emptyset$. Then*

$$\sum_{r=1}^a |b_r^*(x)| < \frac{24Cm_i}{m_j}.$$

Proof. Set $G = \{k : \text{there is } r \text{ with } \text{ran } b_r^* \cap \text{ran } x_k \neq \emptyset\}$ and $x' = (m_j/n_j) \sum_{k \in G} x_k$. Note that $\#G \leq a \leq n_i$. By changing the signs of the b_r^* and, perhaps, omitting some of the first few terms, we may assume that $\max \text{supp } b_1^* \geq \min \text{supp } x_1 \geq j$ and that $|b_r^*(x)| = b_r^*(x')$ for $r = 1, \dots, a$. Proposition 2.11 implies that there are γ and $(\xi_r)_{r=1}^a$ in Γ so that

$$(27) \quad \frac{1}{m_i} \sum_{r=1}^a |b_r^*(x)| = \frac{1}{m_i} \sum_{r=1}^a b_r^*(x') = e_\gamma^*(x') - \sum_{r=1}^a d_{\xi_r}^*(x').$$

Lemma 4.14 and (19) imply that $|e_\gamma^*(x')| \leq 10Cn_i/n_j + 10C/m_j$, while it easily follows that $\sum_{r=1}^a |d_{\xi_r}^*(x')| \leq 4Cn_i m_j/n_j$. We obtain $\sum_{r=1}^a |b_r^*(x)| < 10Cn_i m_i/n_j + 10Cm_i/m_j + 4Cn_i m_i m_j/n_j$. The choice of the sequences $(m_j)_j, (n_j)_j$ yields the desired estimate. □

The following lemma is proved using Lemmas 4.15 and 4.16 and arguments very similar to those used in the proof of [AM1, Lemma 3.8]. We include a proof for completeness.

Lemma 4.17. *Let $j > i$ be natural numbers, let $(x_k)_{k=1}^{n_j}$ be a C -RIS with $\min \text{supp } x_k \geq m_j n_j$, and set $x = (m_j/n_j) \sum_{k=1}^{n_j} x_k$. Moreover, let $(b_r^*)_{r=1}^a$ be a*

very fast growing sequence of α_c -averages, with $a \leq n_i$. Then

$$\sum_{r=1}^a |b_r^*(x)| < \frac{10C}{s(b_1^*)} + \frac{50Cm_i}{m_j}.$$

Proof. We define r_1 to be the minimum r for which $\text{ran } b_r^* \cap \text{ran } x \neq \emptyset$. Define the sets $R_1 = \{r > r_1 : \text{there is at most one } k \text{ with } \text{ran } b_r^* \cap \text{ran } x_k \neq \emptyset\}$ and $R_2 = \{r_1 + 1, \dots, a\} \setminus R_1$. By Lemma 4.16 we obtain

$$(28) \quad \sum_{r \in R_1} |b_r^*(x)| < \frac{24Cm_i}{m_j}.$$

On the other hand, for $r \in R_2$, by Lemma 4.15 we obtain the estimate $|b_r^*(x)| < m_j C / s(b_r^*) + 8Cm_j / n_j < m_j C / 2^{\min \text{supp } x} + 8Cm_j / n_j$, and hence

$$(29) \quad \sum_{r \in R_2} |b_r^*(x)| < n_i \frac{10Cm_j}{2^{m_j n_j}} + n_i \frac{8Cm_j}{n_j}.$$

For $r = 1$ by Lemma 4.15 we obtain

$$(30) \quad |b_{r_1}^*(x)| < \frac{10C}{s(b_{r_1}^*)} + \frac{10C}{m_j} + \frac{8Cm_j}{n_j}.$$

We obtain the result by combining (28), (29), and (30) and using the lacunarity properties of $(m_j, n_j)_j$. □

Proposition 4.18. *Let $j \in \mathbb{N}$, and let $(x_k)_{k=1}^{n_j}$ be a C -RIS with $\min \text{supp } x_1 \geq m_j n_j$. Then $x = (m_j / n_j) \sum_{k=1}^{n_j} x_k$ is a $(112C, j)$ -exact vector. If, moreover, $(x_k)_{k=1}^{n_j}$ is skipped, and there are $\theta > 0$ and a very fast growing sequence of α_c -averages $(b_k^*)_{k=1}^{n_j}$ so that $b_k^*(x_k)_k = \theta$ for $k = 1, \dots, n_j$, then there is $\gamma \in \Gamma$ so that (γ, x) is a $(112C, j, \theta)$ -exact pair.*

Proof. The first part follows from (20), (21), and Lemma 4.17, while the second part follows from Proposition 2.11. □

4.3. Dependent sequences. We finally define dependent sequences and describe how they can be found in every block subspace. Their definition is based on the tree \mathcal{U} of special sequences (see Subsection 2.3). Note that \mathcal{U} was defined using the space \mathfrak{B}_{mT} . Here, we naturally identify finitely supported vectors in \mathfrak{X}_{nr} with ones in \mathfrak{B}_{mT} .

Notation 4.19. For a finitely supported vector $x = \sum_{\gamma \in \Gamma} \lambda_\gamma d_\gamma$ in \mathfrak{X}_{nr} , we denote by \bar{x} the vector $\sum_{\gamma \in \Gamma} \lambda_\gamma \bar{d}_\gamma$ in \mathfrak{B}_{mT} .

Remark 4.20. Remark 1.14 yields that if $\gamma \in \Gamma$ and E is an interval of \mathbb{N} , then $e_\gamma^* \circ \bar{P}_E(\bar{x}) = e_\gamma^* \circ P_E(x)$.

Definition 4.21. Let $C \geq 0$, and let $\theta > 0$. A sequence of pairs $\{(\gamma_k, x_k)\}_{k=1}^\ell$, where $\gamma_k \in \Gamma$ and x_k is a finitely supported vector of $(d_\gamma)_{\gamma \in \Gamma}$ with rational coefficients for $k = 1, \dots, \ell$, is called a (C, θ) -dependent sequence if the following are satisfied:

- (i) (γ_k, x_k) is a (C, j_k, θ) -exact pair, where $\text{weight}(\gamma_k) = m_{j_k}^{-1}$,
- (ii) $\{(\gamma_k, \bar{x}_k)\}_{k=1}^\ell$ is a special sequence (i.e., it is in \mathcal{U}), and
- (iii) $\min \text{supp } x_{k+1} > \max\{\max_{i \leq k} \text{rank}(\gamma_i), \max \text{supp } x_k\}$ for $k < \ell$.

An infinite sequence of pairs $\{(\gamma_k, x_k)\}_{k=1}^\infty$, where for each $\ell \in \mathbb{N}$ the first ℓ -terms $\{(\gamma_k, x_k)\}_{k=1}^\ell$ define a (C, θ) -dependent sequence, will be called a (C, θ) -dependent sequence as well.

Remark 4.22. Let $0 < C \leq 16000$, let $\theta > 0$, and let $\{(\gamma_k, x_k)\}_{k=1}^\ell$ be a (C, θ) -dependent sequence. If $E_k = \text{ran } x_k$ for $k = 1, \dots, \ell$, then Remark 4.20 easily implies that $\{(\gamma_k, E_k)\}_{k=1}^\ell$ is comparable, in the sense of Definition 2.6.

Proposition 4.23. *Let $(x_k)_k$ be a normalized block sequence with rational coefficients in \mathfrak{X}_{nr} , which satisfies $\alpha((x_k)_k) = 0$ and $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$. Also, let $(\eta_k)_k$ be a sequence in Γ so that $|e_{\eta_k}^*(x_k)| > (3/4)\|x_k\|$ for all $k \in \mathbb{N}$. Then for every $j \in \mathbb{N}$ there exists a $(3584, j, 1)$ -exact pair (γ, y) so that*

$$(31) \quad \begin{aligned} y &= \frac{m_j}{n_j} \sum_{r=1}^{n_j} \sum_{i \in F_r} \lambda_i \varepsilon_i x_i \text{ and} \\ e_\gamma^* &= \frac{1}{m_j} \sum_{r=1}^{n_j} \frac{1}{\#F_r} \sum_{i \in F_r} \varepsilon_i e_{\eta_i}^* \circ P_{E_i} + \sum_{r=1}^{n_j} d_{\xi_r}^*, \end{aligned}$$

where $0 < |\lambda_i| < 4/3$, actually $\lambda_i = 1/e_{\gamma_i}^*(x_i)$, $(F_r)_r$ are successive subsets in \mathcal{S}_1 , $(\varepsilon_i)_{i \in F_r}$ are alternating signs for $r \in \mathbb{N}$, and $\text{rank}(d_{\xi_r}^*) \notin \text{ran } x_m$ for all r, m .

Proof. Pass to a subsequence satisfying the assumption of Remark 3.11 and define $\lambda_k = 1/e_{\gamma_k}^*(x_k)$ for all $k \in \mathbb{N}$. The conclusion follows from Proposition 3.5 and then applying Proposition 4.18. □

Remark 4.24. Proposition 4.9 and Proposition 4.23 immediately yield that, up to a perturbation, in every infinite-dimensional subspace X of \mathfrak{X}_{nr} there is a $(3584, 1)$ -dependent sequence $\{(\gamma_k, x_k)\}_{k=1}^\infty$ so that $x_k \in X$ for all $k \in \mathbb{N}$.

Using Proposition 3.8, the following result can be shown, where the sequence $(x_k)_k$ satisfying $\alpha((x_k)_k) = 0$ and $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$ can be replaced with a subsequence of the basis.

Proposition 4.25. *Let (d_{γ_i}) be a subsequence of the basis of \mathfrak{X}_{nr} . Then there exist $\theta > 0$ and a $(3584, \theta)$ -dependent sequence $\{(\gamma_k, y'_k)\}_{k=1}^\infty$ so that for each k ,*

$$(32) \quad \begin{aligned} y_k &= \frac{m_{j_k}}{n_{j_k}} \sum_{r=1}^{n_{j_k}} \theta \sum_{i \in F_{r,k}} \varepsilon_i d_{\gamma_i} \text{ and} \\ e_{\gamma'_k}^* &= \frac{1}{m_{j_k}} \sum_{r=1}^{n_{j_k}} \frac{1}{\#F_{r,k}} \sum_{i \in F_{r,k}} \varepsilon_i d_{\gamma_i}^* + \sum_{r=1}^{n_{j_k}} d_{\xi_{r,k}}^*, \end{aligned}$$

where $(F_{r,k})_{r,k}$ are successive subsets in \mathcal{S}_1 and $(\varepsilon_i)_{i \in F_r}$ are alternating signs for $r \in \mathbb{N}$.

5. ESTIMATIONS ON DEPENDENT SEQUENCES

In the previous section we defined dependent sequences and proved their existence in every block subspace. In this section we provide an estimate for the norm of finite sums of consecutive terms of such sequences, which yields that the space \mathfrak{X}_{nr} contains no boundedly complete sequence and that it is hereditarily indecomposable. We also observe that every subspace of \mathfrak{X}_{nr} fails the PCP and hence also the RNP.

Lemma 5.1. *Let $1 \leq C \leq 4000$, let $\theta > 0$, let $\{(\gamma_k, x_k)\}_{k=1}^\ell$ be a (C, θ) -dependent sequence, and let $1 \leq n \leq m \leq \ell$ be natural numbers. Also, let $(\eta_j)_{j=1}^d$ be a sequence in Γ , let $(E_j)_{j=1}^d$ be a sequence of intervals of \mathbb{N} , and let $(\varepsilon_j)_{j=1}^d$ be a sequence of signs so that one of the following is satisfied:*

- (i) *the sequence $(\eta_j, E_j)_{j=1}^d$ is comparable and the signs $(\varepsilon_j)_{j=1}^d$ are alternating or*
- (ii) *the sequence $(\eta_j, E_j)_{j=1}^d$ is either incomparable or irrelevant.*

If for $j = 1, \dots, d$ we define $D_j = \{n \leq k \leq m : \text{weight}(\gamma_k) < \text{weight}(\eta_j)\}$, then

$$\left| \sum_{j=1}^d \varepsilon_j e_{\eta_j}^* \circ P_{E_j} \left(\sum_{k=n}^m x_k \right) - \sum_{j=1}^d \varepsilon_j e_{\eta_j}^* \circ P_{E_j} \left(\sum_{k \in D_j} x_k \right) \right| \leq 9C + 2dC \text{weight}(\gamma_n).$$

Proof. The goal of this lemma is to show that the two sums in the absolute value of the statement are sufficiently close to each other. The concept behind the proof is to compare the weights of the γ_k 's and the weights of the η_j 's. It is first shown that the action of all $e_{\eta_j}^*$'s on all x_k 's that have different weights is negligible. This is done using standard techniques from HI constructions. On the other hand, to show that the action of the $e_{\eta_j}^*$'s with weights equal to some of those of the γ_k 's, it is necessary to use the notion of comparable sequences (Definition 2.6(ii)). Actually, the treatment of this case is the main reason for introducing this notion.

For $k = 1, \dots, \ell$ we define $A_k = \{j : \text{weight}(\eta_j) = \text{weight}(\gamma_k)\}$, $B_k = \{j : \text{weight}(\eta_j) > \text{weight}(\gamma_k)\}$, and $C_k = \{j : \text{weight}(\eta_j) < \text{weight}(\gamma_k)\}$. By Definition 4.11(iii), for $k = 1, \dots, m$ we obtain

$$(33) \quad \left| \sum_{j \in C_k} \varepsilon_j e_{\eta_j}^* \circ P_{E_j}(x_k) \right| < C \text{weight}(\gamma_k).$$

Observe that

$$\sum_{j=1}^d \varepsilon_j e_{\eta_j}^* \circ P_{E_j} \left(\sum_{k \in D_j} x_k \right) = \sum_{k=n}^m \sum_{j \in B_k} \varepsilon_j e_{\eta_j}^* \circ P_{E_j}(x_k).$$

Therefore

$$\begin{aligned} & \left| \sum_{j=1}^d \varepsilon_j e_{\eta_j}^* \circ P_{E_j} \left(\sum_{k=n}^m x_k \right) - \sum_{j=1}^d \varepsilon_j e_{\eta_j}^* \circ P_{E_j} \left(\sum_{k \in D_j} x_k \right) \right| \\ &= \left| \sum_{k=n}^m \sum_{j \in A_k} \varepsilon_j e_{\eta_j}^* \circ P_{E_j}(x_k) + \sum_{k=n}^m \sum_{j \in C_k} \varepsilon_j e_{\eta_j}^* \circ P_{E_j}(x_k) \right| \\ &\leq \left| \sum_{k=n}^m \sum_{j \in A_k} \varepsilon_j e_{\eta_j}^* \circ P_{E_j}(x_k) \right| + Cd \sum_{k=n}^m \text{weight}(\gamma_k), \end{aligned}$$

where the inequality follows (33). By the choice of the sequence $(m_j)_j$, we obtain $\sum_{k=n}^m \text{weight}(\gamma_k) \leq 2 \text{weight}(\gamma_n)$. Hence, all that remains to be shown is that $\left| \sum_{k=n}^m \sum_{j \in A_k} \varepsilon_j e_{\eta_j}^* \circ P_{E_j}(x_k) \right| \leq 9C$.

Observe that each set A_k is either empty or a singleton, and in particular, we note that $j \in A_k$ if and only if $\text{weight}(\eta_j) = \text{weight}(\gamma_k)$. If the sets A_k are all empty there is nothing to prove. Otherwise, let $k_1 < \dots < k_s$ be all the k 's in $\{1, \dots, \ell\}$ satisfying $A_{k_i} \neq \emptyset$. Also, let $1 \leq j_1 < \dots < j_s \leq d$ be so that for each i , j_i is the unique element of A_{k_i} , and hence $\text{weight}(\eta_{j_i}) = \text{weight}(\gamma_{k_i})$ for $i = 1, \dots, s$.

Is $s \leq 2$, then the desired estimate follows from $\|x_k\| \leq C$ and $\|e_{\eta_i}^* \circ P_{E_i}\| \leq 4$. Otherwise, $s \geq 3$, which implies the sequence $(\eta_j, E_j)_{j=1}^d$ (for a detailed argument see the proof of [AM2, Lemma 5.8]).

We conclude that the sequence $(\eta_j, E_j)_{j=1}^d$ is either comparable or irrelevant, and therefore there exists $m' \in \mathbb{N}$ with $d \leq m'$, natural numbers $1 \leq k'_1 < \dots < k'_d \leq m'$, and $\{(\xi_k, y_k)\}_{k=1}^{m'}$ in \mathcal{U} so that $\text{weight}(\eta_j) = \text{weight}(\xi_{k'_j})$ for $j = 1, \dots, d$. It follows that

- (a) $j_i = i$ for $i = 1, \dots, s$,
- (b) $k'_i = k_i$ for $i = 1, \dots, s$, and
- (c) $\xi_k = \gamma_k, y_k = \bar{x}_k$ for $k = 1, \dots, k_s - 1$.

For a detailed argument explaining the above, once again see [AM2, Lemma 5.8]. We observe that the sequence is not irrelevant. Indeed, the opposite would imply $16000 < |e_{\eta_2}^* \circ \bar{P}_{E_2}(\bar{x}_{k_2})| = |e_{\eta_2}^* \circ P_{E_2}(x_{k_2})| \leq 4C \leq 16000$, where the equality follows from Remark 4.20 and the first inequality from $\|x_k\| \leq C$ and $\|e_{\eta_2}^* \circ P_{E_2}\| \leq 4$.

We have therefore shown that the sequence $(\eta_j, E_j)_{j=1}^d$ is comparable. Define $J = \{i : k_i \in \{n, \dots, m\}\}$, observe that J is an interval of $\{1, \dots, s\}$, and choose successive two-point intervals J_1, \dots, J_p of $J \setminus \{\max J, \min J\}$ so that $J \setminus \bigcup_{i=1}^p J_i$ has at most three elements. The fact that the sequence $(\eta_j, E_j)_{j=1}^d$ is comparable yields

$$(34) \quad \sum_{i=1}^p \left| \sum_{j \in J_i} \varepsilon_j e_{\eta_j}^* \circ P_{E_j}(x_{k_j}) \right| \leq 1,$$

and hence

$$\left| \sum_{k=n}^m \sum_{j \in A_k} \varepsilon_j e_{\eta_j}^* \circ P_{E_j}(x_k) \right| = \left| \sum_{j \in J} \varepsilon_j e_{\eta_j}^* \circ P_{E_j}(x_{k_j}) \right| \leq 8C + 1 \leq 9C.$$

For a more detailed explanation of (34) see the proof of [AM2, Lemma 5.8]. □

The following result is the main estimate of this section.

Proposition 5.2. *Let $1 \leq C \leq 4000$, let $\theta > 0$, and let $\{(\gamma_k, x_k)\}_{k=1}^\ell$ be a (C, θ) -dependent sequence.*

- (i) *Let $\gamma \in \Gamma$, and let E be an interval of \mathbb{N} . If for some natural numbers $1 \leq n \leq m \leq \ell$ we set $D = \{k \in [n, m] : \text{weight}(\gamma_k) < \text{weight}(\gamma)\}$, then*

$$\left| e_\gamma^* \circ P_E \left(\sum_{k \in D} x_k \right) \right| \leq 63C \text{weight}(\gamma).$$

- (ii) *For all natural numbers $1 \leq n \leq m \leq \ell$ we have*

$$\left\| \sum_{k=n}^m x_k \right\| \leq 10C.$$

Proof. We use a standard argument to prove the second statement, given the first one. Let $1 \leq n \leq m \leq \ell$, and let $\gamma \in \Gamma$. Define $k_0 = \min\{n \leq k \leq m : \text{weight}(\gamma_k) \leq \text{weight}(\gamma)\}$ (if k_0 is not well defined, then the result follows from a similar argument as the one we shall use in this case). We conclude that $|e_\gamma^*(\sum_{k=k_0+1}^m x_k)| \leq 63C \text{weight}(\gamma)$ and $|e_\gamma^*(x_{k_0})| \leq C$. To compute the action on the rest of the vectors we use the evaluation analysis of γ , $e_\gamma^* = \text{weight}(\gamma) \sum_{r=1}^a b_r^* + \sum_{r=1}^a d_{\xi_r}^*$. If $\text{weight}(\gamma) = m_{j_0}^{-1}$, then $\text{weight}(\xi_r) = m_{j_0}^{-1}$ as well, which yields $\text{rank}(\xi_r) \geq j_0$ for $r = 1, \dots, a$. Recall that $\{(\gamma_k, \bar{x}_k)\}_{k=1}^\ell$, which in conjunction with (3) implies that $j_0 > \max \text{supp } x_{k_0-1} \text{weight}(\gamma_{k_0-1})$. We obtain

$$\left| e_\gamma^* \left(\sum_{k=n}^{k_0-1} x_k \right) \right| = \text{weight}(\gamma) \left| b_1^* \left(\sum_{k=n}^{k_0-1} x_k \right) \right| \leq \text{weight}(\gamma) 4Ck_0 \leq C.$$

Combining the above the conclusion follows.

We now proceed to prove the first statement by induction on the rank of γ . The case $\text{rank}(\gamma) = 1$ is easy, so let $p \in \mathbb{N}$ such that for every $\gamma \in \Gamma$ with $\text{rank}(\gamma) \leq p$ and interval E of \mathbb{N} the conclusion is satisfied.

We remark the following: let b^* be an α_c -average of $B_{0,p}$, and let $n \leq m$; then

$$(35) \quad \left| b^* \left(\sum_{k=n}^m x_k \right) \right| \leq \frac{30C}{s(b^*)} + 2C \text{weight}(\gamma_n).$$

This follows from Definition 4.11(i) in case b^* is a basic average and from the inductive assumption and Lemma 5.1 and the choice of $(m_j)_j$ otherwise.

Now let $\gamma \in \Gamma$ with $\text{rank}(\gamma) = p + 1$, and let E be an interval of \mathbb{N} . Let $e_\gamma^* = (1/m_j) \sum_{r=1}^a b_r^* + \sum_{r=1}^a d_{\xi_r}^*$, with $a \leq n_j$, be the evaluation analysis of γ and note that $e_\gamma^* = (1/m_j) \sum_{r=1}^a \bar{b}_r^* + \sum_{\text{rank}(\xi_r) \in E} d_{\xi_r}^*$, where $\bar{b}_r^* = b_r^* \circ P_E$. We note that $(\bar{b}_r^*)_{r=1}^a$ is a very fast growing sequence of α_c -averages. Let $1 \leq n \leq m \leq \ell$, set $D = \{k \in [n, m] : \text{weight}(\gamma_k) < \text{weight}(\gamma)\}$, and for $k \in D$ set $M_k = \{r : \text{ran } \bar{b}_r^* \cap \text{ran } x_k \neq \emptyset\}$, $N_k = \{r \in M_k : s(\bar{b}_r^*) \geq (\text{weight}(\gamma_k))^{-1}\}$. By Definition 4.11(iv) we obtain

$$(36) \quad \left| \sum_{r=1}^a \bar{b}_r^* \left(\sum_{k \in D} x_k \right) \right| \leq \left| \sum_{k \in D} \sum_{q \in M_k \setminus N_k} \bar{b}_r^*(x_k) \right| + 2C \sum_{k \in D} \frac{\text{weight}(\gamma_k)}{\text{weight}(\gamma)}.$$

Define $A = \bigcup_{k \in D} M_k \setminus N_k$, for $r \in A$ set $D_r = \{k \in D : r \in M_k \setminus N_k\}$, and observe that

$$(37) \quad \left| \sum_{k \in D} \sum_{r \in M_k \setminus N_k} \bar{b}_r^*(x_k) \right| = \left| \sum_{r \in A} \bar{b}_r^* \left(\sum_{k \in D_r} x_k \right) \right|.$$

Following the arguments used in the proof of [AM2, Proposition 5.9] and using Definition 4.11(iv), we conclude that the D_r 's are disjoint intervals of $\{n, \dots, m\}$. We set $n_r = \min D_r$ and note that the n_r 's are strictly increasing. By (35) we obtain

$$(38) \quad \left| \sum_{r \in A} \bar{b}_r^* \left(\sum_{k \in D_r} x_k \right) \right| \leq \sum_{r \in A} \left(\frac{30C}{s(\bar{b}_r^*)} + 2C \text{weight}(\gamma_{n_r}) \right).$$

By using the fact that $(\bar{b}_r^*)_{r=1}^a$ is very fast growing, the choice of the sequence $(m_j)_j$, and (36), (37), and (38) we obtain

$$(39) \quad \left| \sum_{r=1}^a \bar{b}_r^* \left(\sum_{k \in D} x_k \right) \right| \leq 62C,$$

whereas Definition 4.11(i) and the choice of $(m_j)_j$ imply

$$(40) \quad \sum_{\text{rank}(\xi_r) \in E} d_{\xi_r}^* \left(\sum_{k \in D} x_k \right) \leq C \text{ weight}(\gamma).$$

Combining (39) and (40) we conclude $|e_\gamma^* \circ P_E(\sum_{k \in D} x_k)| \leq 63C \text{ weight}(\gamma)$. \square

Remark 4.24 and Proposition 5.2(ii) yield that \mathfrak{X}_{nr} contains no boundedly complete sequence, which yields the following.

Theorem 5.3. *The space \mathfrak{X}_{nr} contains no reflexive subspace.*

Proposition 5.4. *Let $1 \leq C \leq 4000$, let $\theta > 0$, and let $\{(\gamma_k, x_k)\}_{k=1}^\ell$ be a (C, θ) -dependent sequence. Then*

$$\sup_n \left\| \sum_{k=1}^n (-1)^k x_k \right\| = \infty.$$

Proof. Inductively choose successive a sequence $(E_r)_r$ of finite intervals of \mathbb{N} so that $\bigcup_r E_r = \mathbb{N}$, for each $r \in \mathbb{N}$

$$b_r^* = \frac{1}{\#E_n} \sum_{k \in E_n} (-1)^k e_{\gamma_k}^* \circ P_{\text{ran } x_k}$$

is an α_c -average, and the sequence $(b_r^*)_r$ is very fast growing. Note that for $n \geq \max E_r$ we have $b_r^*(\sum_{k=1}^n (-1)^k x_k) = \theta$. For $j \in \mathbb{N}$ we construct a $\gamma \in \Gamma$ with $e_\gamma^* = (1/m_{j_0}) \sum_{r=r_0+1}^{r_0+n_j} b_r^* + \sum_{r=r_0+1}^{r_0+n_j} d_{\xi_r}^*$, for $r_0 \in \mathbb{N} \cup \{0\}$ appropriately large so that $d_{\xi_r}^*(x_k) = 0$ for $r_0 < r \leq r_0 + n_j$ and $k \in \mathbb{N}$. Finally, observe that for $k \geq E_{r_0+n_j}$ we have $e_\gamma^*(\sum_{k=1}^n (-1)^k x_k) = (n_j/m_j)\theta$, which yields the desired result. \square

Remark 5.5. A very similar proof also yields that if $1 \leq C \leq 4000$, $\theta > 0$, $\{(\gamma_k, x_k)\}_{k=1}^\ell$ is a (C, θ) -dependent sequence, and L is an infinite subset of \mathbb{N} with $\mathbb{N} \setminus L$ infinite, then

$$\sup_n \left\| \sum_{\substack{k \in L \\ k \leq n}} x_k \right\| = \infty.$$

Theorem 5.6. *The space \mathfrak{X}_{nr} is hereditarily indecomposable.*

Proof. As is stated in Remark 4.24, every infinite-dimensional subspace contains a perturbation of a (3584, 1)-dependent sequence $\{(\gamma_k, x_k)\}_{k=1}^\infty$. Given two infinite-dimensional subspaces X and Y , the dependent sequence can be chosen so that for k even x_k is in X and for k odd x_k is in Y , at least up to a small enough perturbation. Set $u_n = \sum_{k=1}^n x_{2k}$ and $w_n = \sum_{k=1}^n x_{2k-1}$ for all $n \in \mathbb{N}$. Then $u_n \in X$, $w_n \in Y$ for all $n \in \mathbb{N}$, and by Proposition 5.2(ii) $\sup_n \|u_n + w_n\| < \infty$, whereas by Proposition 5.4 $\sup_n \|u_n - w_n\| = \infty$. This yields that \mathfrak{X}_{nr} is hereditarily indecomposable. \square

Remark 5.7. Bourgain posed the question of whether there exists a \mathcal{L}_∞ -space not containing c_0 that also fails the RNP [B, Problem 3, p. 46]. This was answered positively in [FOS], where it was proved that every Banach space X with a separable dual embeds into a \mathcal{L}_∞ -space Y with a separable dual as well. If X contains no copy of c_0 , Y can be chosen to contain no copy of c_0 either. The space in this paper also provides an answer to Bourgain’s question in a strong way. Indeed, by [EW, Theorem 4.1], every subspace of \mathfrak{X}_{nr} fails the PCP and hence also the RNP.

6. OPERATORS ON THE SPACE \mathfrak{X}_{nr}

The goal of this section is to prove that the space \mathfrak{X}_{nr} satisfies the scalar-plus-compact property. We initially characterize strictly singular operators with respect to their behavior on certain sequences generating c_0 spreading models. Then, we prove that the space has the scalar-plus-strictly singular property. Finally, we use the aforementioned characterization to deduce that strictly singular operators on \mathfrak{X}_{nr} are compact. Recall that in [AH], an operator $T : \mathfrak{X}_{AH} \rightarrow \mathfrak{X}_{AH}$ is compact if and only if it maps all rapidly increasing sequences to norm-null ones. In the present case, noncompact operators $T : \mathfrak{X}_{nr} \rightarrow \mathfrak{X}_{nr}$ always map some rapidly increasing sequence to a sequence with α -index positive.

Definition 4.11 (i) and the fact that the extension operators have norm at most two, yields the following result.

Remark 6.1. Let (γ, x) be a (C, j, θ) -exact pair, and let $\rho \in [0, \theta]$. Then there is an interval E of $\text{ran } x$ so that $|e_\gamma^* \circ P_E(x) - \rho| < 2Cm_j/n_j$.

Proposition 6.2. *Let X be an infinite-dimensional closed subspace of \mathfrak{X}_{nr} , and let $T : X \rightarrow \mathfrak{X}_{nr}$ be a bounded linear operator. The following assertions are equivalent:*

- (i) *The operator T is strictly singular.*
- (ii) *There is a normalized weakly null sequence $(x_k)_k$ in X so that $(Tx_k)_k$ is norm-null.*
- (iii) *For every normalized sequence $(x_k)_k$ in X generating a c_0 spreading model and $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$, $(Tx_k)_k$ is norm-null.*

Proof. The equivalence of (i) and (ii) is a general property of hereditarily indecomposable Banach spaces, whereas the implication (iii) \Rightarrow (i) is an immediate consequence of Proposition 4.9. It remains to prove that (ii) implies (iii), and towards a contradiction assume that this is not the case, i.e., there are a normalized weakly null sequence $(z_k)_k$ in X so that $(Tz_k)_k$ is norm-null and a normalized sequence $(x_k)_k$ in X generating a c_0 spreading model with $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$ so that $(Tx_k)_k$ is not norm-null. We note that $(Tx_k)_k$ only admits the unit vector basis of c_0 as a spreading model. We apply Proposition 4.9, and perturb and scale the operator T , perhaps defining it on a different subspace X' of \mathfrak{X}_{nr} , so that $(x_k)_k, (Tx_k)_k, (z_k)_k$ are all block sequences with rational coefficients, $\alpha((z_k)_k) = 0$ as well as $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(z_k)| = 0, \|Tx_k\| = 1$, and $Tz_k = 0$ for all $k \in \mathbb{N}$.

By perhaps changing the signs of some of the sequences or the operator T and passing to subsequences, we may choose three sequences $(\eta_k)_k, (\eta'_k)_k, (\theta_k)_k$ of Γ so that $\sum_k |e_{\eta_k}^*(x_k) - 1| < \infty, \sum_k |e_{\eta'_k}^*(z_k) - 1| < \infty$ and $\sum_k |e_{\theta_k}^*(Tx_k) - 1| < \infty$. We consider two cases, namely whether or not the set $\{(\text{weight}(\theta_k))^{-1} : k \in \mathbb{N}\}$ is bounded. We shall only treat the second case, as the first one follows by using Lemma 3.9 and the fact that basic averages are always α_c -averages. We therefore assume that the set $\{(\text{weight}(\theta_k))^{-1} : k \in \mathbb{N}\}$ is unbounded.

We define $E_k = \text{ran } x_k$, $E'_k = \text{ran } z_k$ and F_k to be the smallest interval containing E_k as well as $\text{ran } Tx_k$. By passing to a subsequence we may choose $\rho \in [0, 1]$ so that $\sum_k |e_{\theta_k}^* \circ P_{F_k}((1/e_{\eta_k}^*(x_k))x_k) - \rho| < \infty$.

Using Proposition 4.23 we carefully construct a $(3584, 1)$ -dependent sequence $\{(\gamma_k, y_k)\}$ so that for k even (γ_k, y_k) is built on $(\eta_i)_i, (x_i)_i$, whereas for k odd it is built on $(\eta'_i)_i, (z_i)_i$, as in (31). For k even, γ_k can be built in such a manner that there is $\zeta_k \in \Gamma$ (ζ_k is built on the θ_i 's) and an interval J_k of \mathbb{N} (J_k can be chosen to be the smallest interval containing the ranges of both y_k and Ty_k) with $e_{\zeta_k}^* \circ P_{J_k}(Ty_k) > 1/2$ with $\text{weight}(\zeta_k) = \text{weight}(\gamma_k)$ and $|e_{\zeta_k}^* \circ P_{J_k}(y_k) - \rho| < 1/2^{k+1}$. For k odd, by Remark 6.1, we can choose $\zeta_k \in \Gamma$ $\text{weight}(\zeta_k) = \text{weight}(\gamma_k)$ (actually $\zeta_k = \gamma_k$) and an interval J_k of \mathbb{N} so that $|e_{\zeta_k}^* \circ P_{J_k}(y_k) - \rho| < 7168m_{j_k}/n_{j_k}$ (where $\text{weight}(\gamma_k) = m_{j_k}^{-1}$), which, using (3), can be chosen to be below $1/2^{k+1}$. It is also important to note that for k odd, $Ty_k = 0$. We conclude that the sequence (ζ_k, J_k) is comparable, for k odd $e_{\zeta_k}^* \circ P_{J_k}(Ty_k) > 1/2$ and for k even $e_{\zeta_k}^* \circ P_{J_k}(Ty_k) = 0$. Hence, using an argument very similar to that in the proof of Proposition 5.4 we can find $n \in \mathbb{N}$ so that the norm of $\sum_{k=1}^n Ty_k$ is arbitrarily large, whereas by Proposition 5.2(ii) $\|\sum_{k=1}^n y_k\| \leq 35840$. This means that T is unbounded, which completes the proof. \square

Remark 6.3. We point out a fact that we will use to prove the next result. Recall that for each $n \in \mathbb{N}$, $(d_\gamma)_{\gamma \in \Delta_n}$ is 2-equivalent to the unit vector basis of ℓ_∞^n . This easily implies the following: if $(x_k)_k$ is a block sequence in \mathfrak{X}_{nr} , then $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(x_k)| = 0$ if and only if $\lim_k \sup_{n \in \mathbb{N}} \|P_{\{n\}}x_k\| = 0$, where $P_{\{n\}}$ denotes the Bourgain–Delbaen projection onto the n th coordinate of the FDD of \mathfrak{X}_{nr} .

Proposition 6.4. *For every bounded linear operator $T : \mathfrak{X}_{\text{nr}} \rightarrow \mathfrak{X}_{\text{nr}}$ there is a scalar λ so that $T - \lambda I$ is strictly singular.*

Proof. We choose an accumulation point λ of the sequence $(d_\gamma^*(Td_\gamma))_\gamma$ and we will show that the operator $S = T - \lambda I$ is strictly singular. Passing to a subsequence $(d_{\gamma_k})_k$ of the basis and adding a compact perturbation, we may assume that $d_{\gamma_k}^*(Sd_{\gamma_k}) = 0$ for all $k \in \mathbb{N}$. By Proposition 6.2, it suffices to show that $(Sd_{\gamma_k})_k$ converges to zero in norm. Towards a contradiction, we assume that this is not the case. We shall follow steps similar to those used in the proof of Proposition 6.2 to blow up the norm of T , in a slightly different way. More precisely, the goal is to use Proposition 4.25 to find a $(3584, \theta)$ -dependent sequence $\{(\gamma'_k, y_k)\}_{k=1}^\infty$ and a sequence $\{(\zeta_k, J_k)\}_k$ with $\text{weight}(\zeta_k) = \text{weight}(\gamma'_k)$, $e_{\zeta_k}^* \circ P_{J_k}((-1)^k Sy_k) > \varepsilon$, for some $\varepsilon > 0$, and $e_{\zeta_k}^* \circ P_{J_k}(y_k) = 0$ for all $k \in \mathbb{N}$. This last part in particular implies that $\{(\zeta_k, J_k)\}_k$ is comparable, and using the fact that $e_{\zeta_k}^* \circ P_{J_k}((-1)^k Sy_k) > \varepsilon$ and an argument similar to that used in the proof of Proposition 5.4 implies that one can find $n \in \mathbb{N}$ so that the norm of $\sum_{k=1}^n Sy_k$ is arbitrarily large, which in conjunction with Proposition 5.2(ii) implies that S is unbounded.

It remains to describe how to find $\{(\gamma'_k, y_k)\}_k$ and $\{(\zeta_k, J_k)\}_k$. By Proposition 3.8, $(d_{\gamma_k})_k$ admits only a c_0 spreading model, and hence the same is true for $(Sd_{\gamma_k})_k$, which we may assume is a normalized block sequence. We distinguish two cases, namely whether or not $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(Sd_{\gamma_k})|$ is zero. We treat the second case, i.e., on some subsequence there are $\varepsilon > 0$ and $(\eta_k)_k$ so that $d_{\eta_k}^* Sd_{\gamma_k} > \varepsilon$ for all $k \in \mathbb{N}$. As $d_{\gamma_k}^*(Sd_{\gamma_k}) = 0$, we obtain $\gamma_k \neq \eta_k$, and keeping this in mind we can apply Proposition 4.25 to find a $(3584, \theta)$ -dependent sequence $\{(\gamma'_k, y_k)\}_{k=1}^\infty$ as in

(32) and a sequence $\{(\zeta_k, J_k)\}_k$, where ζ_k is built on $(d_{\eta_k}^*)_k$ using appropriate signs, $J_k = \text{ran } Sd_{\gamma_k}$, with $\text{weight}(\zeta_k) = \text{weight}(\gamma'_k)$ and $e_{\zeta_k}^* \circ P_{J_k}((-1)^k Sy_k) > \varepsilon\theta$.

Otherwise, $\lim_k \sup_{\gamma \in \Gamma} |d_{\gamma}^*(d_{\gamma_k})| = 0$. For each $k \in \mathbb{N}$, we define $D_k^- = \text{ran } Sd_{\gamma_k} \cap [1, \text{rank}(\gamma_k))$ and $D_k^+ = \text{ran } Sd_{\gamma_k} \cap (\text{rank}(\gamma_k), \infty)$. Remark 6.3 yields that either $\limsup_k \|P_{D_k^-} Sd_{\gamma_k}\| > 0$ or $\limsup_k \|P_{D_k^+} Sd_{\gamma_k}\| > 0$. We shall assume the first; set $D_k = D_k^-$ for all $k \in \mathbb{N}$ and note that $\text{rank}(d_{\gamma_k}) \notin D_k$ for all $k \in \mathbb{N}$. Clearly, $\lim_k \sup_{\gamma \in \Gamma} |d_{\gamma}^*(P_{D_k} Sd_{\gamma_k})| = 0$, and since $(Sd_{\gamma_k})_k$ only admits c_0 as a spreading model, we can deduce that $\alpha((P_{D_k} Sd_{\gamma_k})_k) = 0$. We pass to a subsequence, and perhaps consider $-S$, to choose a sequence $(\eta_k)_k$ with $e_{\eta_k}^*(P_{D_k} Sd_{\gamma_k}) > 3/4 \|P_{D_k} Sd_{\gamma_k}\|$ for all $k \in \mathbb{N}$. Since $e_{\eta_k}^* \circ P_{D_k} d_{\gamma_k} = 0$ for all k , we can use Remark 3.11 and Proposition 4.25 to proceed as in the previous case, that is, we can find a $(3584, \theta)$ -dependent sequence $\{(\gamma'_k, y_k)\}_{k=1}^\infty$ as in (32) and a sequence $\{(\zeta_k, J_k)\}_k$, where ζ_k is built on $(e_{\eta_k}^* \circ P_{D_k})_k$ using appropriate signs, with $\text{weight}(\zeta_k) = \text{weight}(\gamma'_k)$ and $e_{\zeta_k}^* \circ P_{J_k}((-1)^k Sy_k) > \varepsilon$. □

Remark 6.5. The same result can be proved, using similar arguments (see also [AM2, Lemma 8.8]), for operators $T : X \rightarrow X$, where X is a block subspace of \mathfrak{X}_{nr} generated by a block sequence $(x_k)_k$ which is either a subsequence of the basis or satisfies $\alpha((x_k)_k) = 0$ as well as $\lim_k \sup_{\gamma \in \Gamma} |d_{\gamma}^*(x_k)| = 0$.

Remark 6.6. The proof of Proposition 6.4 yields that if $(d_{\gamma_k})_k$ is a subsequence of the basis of \mathfrak{X}_{nr} and $Y = \overline{\langle \{d_{\gamma_k} : k \in \mathbb{N}\} \rangle}$, then every bounded linear operator $T : Y \rightarrow \mathfrak{X}_{\text{nr}}$ is a multiple of the inclusion plus a strictly singular operator.

Recall that a Banach space X is called an ℓ_1 -predual if X^* is isomorphic to ℓ_1 or, equivalently ([LS, Corollary, p. 182]), if X is a \mathcal{L}_∞ -space with separable dual.

Lemma 6.7. *Let X be an ℓ_1 -predual, let $T : X \rightarrow \mathfrak{X}_{\text{nr}}$ be a bounded linear operator, and assume that for every very fast growing sequence of α_c -averages $(b_k^*)_k$, $(T^* b_k^*)_k$ is norm-null. Then:*

- (i) *for every subsequence $(\gamma_k)_k$ of Γ , $(T^* d_{\gamma_k}^*)_k$ is norm-null,*
- (ii) *for every subsequence $(\gamma_k)_k$ of Γ with $\{(\text{weight}(\gamma_k))^{-1} : k \in \mathbb{N}\}$ unbounded and successive intervals $(E_k)_k$ of \mathbb{N} , $(T^*(e_{\gamma_k}^* \circ P_{E_k}))_k$ is norm-null,*
- (iii) *for every very fast growing sequence of α_c -averages $(b_k^*)_k$ and successive intervals $(E_k)_k$ of \mathbb{N} , $(T^*(b_k^* \circ P_{E_k}))_k$ is norm-null.*

Proof. Note that the third statement immediately follows from Remark 2.7, which yields that $(b_k^* \circ P_{E_k})_k$ is a very fast growing sequence of α_c -averages as well. To see the proof of first two statements, note that if for each k , $x_k^* = d_{\gamma_k}^*$ or $x_k^* = e_{\gamma_k}^* \circ P_{E_k} = P_{E_k}^* e_{\gamma_k}^*$, then $(x_k^*)_k$ is w^* -null and hence, so is $(T^* x_k^*)_k$. We conclude that if it is not norm-null, then it has a subsequence equivalent to the unit vector basis of ℓ_1 . In each case, either because elements of the basis always define α_c -averages or by using Proposition 2.8, one can find a very fast growing sequence of α_c -averages whose image under T^* is not norm-null, contradicting the initial assumption. □

Lemma 6.8. *Let X be a Banach space, and let $T : X \rightarrow \mathfrak{X}_{\text{nr}}$ be a bounded and noncompact linear operator. Then there are a subsequence $(\gamma_k)_k$ of Γ and a sequence of successive intervals $(E_k)_k$ of \mathbb{N} so that $\limsup_k \|T^*(e_{\gamma_k}^* \circ P_{E_k})\| > 0$.*

Proof. As T is not compact, there is a normalized sequence $(x_k)_k$ so that $(Tx_k)_k$ has no norm-convergent subsequence. Hence, passing if necessary to a subsequence, there are $\varepsilon > 0$ and a sequence of successive intervals $(E_k)_k$ of \mathbb{N} so that $\|P_{E_k}Tx_k\| > \varepsilon$ for all k . Choose a sequence $(\gamma_k)_k$ of Γ so that $e_{\gamma_k}^*(P_{E_k}Tx_k) > \varepsilon$ for all k and observe that it is the desired one. \square

Proposition 6.9. *Let X be an ℓ_1 -predual, and let $T : X \rightarrow \mathfrak{X}_{\text{nr}}$ be a bounded and noncompact linear operator. Then there exists a very fast growing sequence of α_c -averages $(b_k^*)_k$ so that $\limsup_k \|T^*b_k^*\| > 0$.*

Proof. Let us assume that the conclusion is false, i.e., T satisfies the assumptions of Lemma 6.7. We shall prove a statement, which in conjunction with Lemma 6.8 and a finite inductive argument yields that $\|T\|$ is arbitrarily large, a contradiction. The statement is the following: if $(\gamma_k)_k$ is a subsequence of Γ , $(E_k)_k$ is a sequence of successive intervals of \mathbb{N} , and $\varepsilon > 0$ so that $\|T^*(e_{\gamma_k}^* \circ P_{E_k})\| > \varepsilon$ for all k , then there are a subsequence $(\eta_k)_k$ of Γ and a sequence of successive intervals $(F_k)_k$ of \mathbb{N} so that $\|T^*(e_{\eta_k}^* \circ P_{F_k})\| > (m_1/2)\varepsilon$.

Let (γ_k) , $(E_k)_k$, and ε be as above. By Lemma 6.7(ii) and passing to a subsequence, there is j_0 so that $\text{weight}(\gamma_k) = m_{j_0}^{-1}$ for all $k \in \mathbb{N}$. Considering the evaluation analysis of each γ_k , and passing to a subsequence, there is $a \in \mathbb{N}$ so that $e_{\gamma_k}^* \circ P_{E_k} = \sum_{r=1}^a d_{\xi_{k,r}}^* \circ P_{E_k} + (1/m_{j_0}) \sum_{r=1}^a b_{k,r}^* \circ P_{E_k}$. As for each k and r , $d_{\xi_{k,r}}^* \circ P_{E_k}$ is either $d_{\xi_{k,r}}^*$ or zero, Lemma 6.7(i) yields that $(T^*(\sum_{r=1}^a d_{\xi_{k,r}}^* \circ P_{E_k}))_k$ is norm-null. Recall that for each k , the sequence $(b_{k,r}^*)_{r=1}^a$ is very fast growing. Hence Lemma 6.7(iii) implies that, passing to a subsequence, there is $\ell \in \mathbb{N}$ so that for each k there is r_k so that $s(b_{k,r_k}) = \ell$ and $\|T^*((1/m_{j_0})b_{k,r_k}^* \circ P_{E_k})\| > \varepsilon/2$. Moreover, Lemma 6.7(i) implies that $b_{k,r}^*$ cannot be a basic average; hence $b_{k,r}^* = (1/\ell) \sum_{i=1}^s \varepsilon_i e_{\eta_{k,i}}^* \circ P_{F_{k,i}}$ with $s \leq \ell$. We conclude that for each k there is i_k so that $\|T^*e_{\eta_{k,i_k}}^* \circ P_{F_{k,i_k} \cap E_k}\| > (m_{j_0}/2)\varepsilon$, which completes the proof. \square

Theorem 6.10. *A bounded linear operator $T : \mathfrak{X}_{\text{nr}} \rightarrow \mathfrak{X}_{\text{nr}}$ is strictly singular if and only if it is compact. Hence, \mathfrak{X}_{nr} satisfies the scalar-plus-compact property.*

Proof. Towards a contradiction, assume that there is a strictly singular operator $T : \mathfrak{X}_{\text{nr}} \rightarrow \mathfrak{X}_{\text{nr}}$ which is not compact. By Proposition 6.9 there is a very fast growing sequence $(b_k^*)_k$ of α_c -averages so that $\limsup_k \|T^*b_k^*\| > 0$. As the sequence $(b_k^*)_k$ is w^* -null, so is $(T^*b_k^*)_k$, and hence by a sliding hump argument we can pass to a subsequence and find a normalized block sequence $(x_k)_k$ in \mathfrak{X}_{nr} so that $\limsup_k T^*b_k^*(x_k) > 0$. An argument in which each vector x_k is split according to the weights of its local support ([AH, Definition 5.7]), yields that on some subsequence there are $\varepsilon > 0$, $C \geq 1$, and a C -RIS $(y_k)_k$ so that $b_k^*(Ty_k) = T^*b_k^*(y_k) > \varepsilon$ for all $k \in \mathbb{N}$. To be more precise, the sequence $(y_k)_k$ is chosen so that it either has bounded local weights or rapidly decreasing local weights ([AH, Definition 5.9]) which yields that it is an RIS ([AH, Proposition 5.10]). For a more detailed exposition of the argument see [AH, Proposition 5.11]. As $(y_k)_k$ is weakly null, we may assume that $(Ty_k)_k$ is a block sequence, i.e., we have found a C -RIS $(y_k)_k$ so that $(Ty_k)_k$ is a block sequence with α -index positive. Combining Proposition 3.4 and (20) of Proposition 4.3 we obtain that passing to a subsequence, there is $\delta > 0$, so

that for all $j \leq i_1 < \dots < i_{n_j}$,

$$(41) \quad \frac{\delta}{\|T\|} \leq \frac{1}{\|T\|} \left\| \frac{m_j}{n_j} \sum_{k=1}^{n_j} T y_{i_k} \right\| \leq \left\| \frac{m_j}{n_j} \sum_{k=1}^{n_j} y_{i_k} \right\| \leq 10C.$$

By the above and Proposition 4.18, we can construct a sequence $(u_k)_k$, with each u_k a $(112C, k)$ -exact vector, so that both $(u_k)_k$ and $(Tu_k)_k$ are seminormalized. By Remark 4.13 we obtain $\lim_k \sup_{\gamma \in \Gamma} |d_\gamma^*(u_k)| = 0$ as well as $\alpha((w_k)_k) = 0$. This contradicts Proposition 6.2(iii), since T was assumed to be strictly singular. \square

Remark 6.11. The above proof actually yields that if an operator $T : \mathfrak{X}_{\text{nr}} \rightarrow \mathfrak{X}_{\text{nr}}$ is noncompact, then it maps some RIS to a sequence with α -index positive (to be more precise, to a weakly null sequence that is a perturbation of a block sequences with α -index positive).

Remark 6.12. The space \mathfrak{X}_{nr} does not have the scalar-plus-compact property hereditarily, i.e., there exists a subspace Y of \mathfrak{X}_{nr} and a strictly singular operator $T : Y \rightarrow Y$ that is not compact. This is also true for the space \mathfrak{X}_{AH} constructed in [AH]. We repeat the argument for completeness. As it was explained in Remark 3.6, the sequence $(y_q)_q$ with $y_q = \sum_{\gamma \in \Delta_q} d_\gamma$ generates an ℓ_1 spreading model. If we set Y to be the closed linear span of $(y_q)_q$, then by Proposition 4.9 there is a sequence in Y generating a c_0 spreading model. By a theorem in [AOST] there exists a strictly singular operator $S : Y \rightarrow Y$ that is not compact.

7. QUOTIENTS OF \mathfrak{B}_{mT} WITH THE SCALAR-PLUS-COMPACT PROPERTY

Recall that the space \mathfrak{X}_{nr} is defined using the tree \mathcal{U} of all finite special sequences $\{(\gamma_k, x_k)\}_{k=1}^d$ (see Subsection 2.3). A defining property of the tree \mathcal{U} is that any of its maximal chains is infinite. This is precisely the reason why there are no boundedly complete sequences in the space \mathfrak{X}_{nr} .

In a way analogous to that in [AM2], for each ordinal number $2 \leq \xi < \omega_1$ we can consider the tree \mathcal{U}_ξ of all finite special sequences $\{(\gamma_k, x_k)\}_{k=1}^d$ so that $\{\text{rank}(\gamma_k) : k = 1, \dots, d\} \in \mathcal{S}_\xi$. Each such tree \mathcal{U}_ξ defines a different class of α_c -averages which induce a self-determined subset Γ_ξ of $\bar{\Gamma}$, resulting in a hereditarily indecomposable \mathcal{L}_∞ -space \mathfrak{X}_ξ with the scalar plus compact property, which is a quotient of \mathfrak{B}_{mT} . We note that $\xi \geq 2$ is necessary to be able to prove the aforementioned properties.

As the tree \mathcal{U}_ξ is well founded, it can be shown that the space \mathfrak{X}_ξ is reflexively saturated, in particular, its FDD is shrinking and every skipped block sequence in the space \mathfrak{X}_ξ is boundedly complete. Therefore, if for some $\lambda > 0$ and block subspace X of \mathfrak{X}_ξ we define the tree $\mathcal{N}\text{-}\mathcal{BC}_{\text{sk}}(X, \lambda)$ of all skipped block sequences $(x_k)_{k=1}^d$ in X satisfying $1 \leq \|\sum_{k=n}^m x_k\| \leq \lambda$ for $1 \leq n \leq m \leq d$, then $\mathcal{N}\text{-}\mathcal{BC}_{\text{sk}}(X, \lambda)$ is well founded. We conclude that there is an ordinal number ζ_ξ so that the order of the tree $\mathcal{N}\text{-}\mathcal{BC}_{\text{sk}}(X, \lambda)$ is at most ζ_ξ for every block subspace X of \mathfrak{X}_ξ and $\lambda > 0$. On the other hand, it can be deduced that for every block subspace X of \mathfrak{X}_ξ and $\eta < \xi$, the order of $\mathcal{N}\text{-}\mathcal{BC}_{\text{sk}}(X, \lambda)$ is at least η , for λ sufficiently large. This easily yields that for every $\zeta > \zeta_\xi + 1$, the spaces \mathfrak{X}_ξ and \mathfrak{X}_ζ are totally incomparable. By passing to a co-final subset of the countable ordinal numbers and relabeling, we can find an uncountable family of pairwise totally incomparable Banach spaces $\{\mathfrak{X}_\xi : \xi < \omega_1\}$ so that each space \mathfrak{X}_ξ is a hereditarily indecomposable and reflexively saturated \mathcal{L}_∞ -space with the “scalar-plus-compact” property which is moreover a quotient of \mathfrak{B}_{mT} .

A noteworthy fact is that the spaces in the family $\{\mathcal{Y}_\xi : \xi < \omega_1\}$ are defined using common weights and the same coding function, the difference between any two of them being that for each space a tree of special sequences with different complexity is used. It is also true that for $\zeta \neq \xi$ every bounded operator $T : \mathcal{Y}_\zeta \rightarrow \mathcal{Y}_\xi$ is compact. To see this, note that Proposition 6.9 also holds if $T : \mathcal{Y}_\zeta \rightarrow \mathcal{Y}_\xi$ is noncompact. Hence arguing as in the proof Theorem 6.10 one can find a sequence $(x_k)_k$ so that both it and $(Tx_k)_k$ are seminormalized and satisfy the assumptions of Proposition 3.5. Using the fact that T is necessarily strictly singular, a construction can be carried out, similar to the one from the proof of Proposition 6.2, however some extra cases may need to be treated.

Summarizing the above we obtain the following result.

Theorem 7.1. *There exists an uncountable family of pairwise totally incomparable Banach spaces $\{\mathcal{Y}_\xi : \xi < \omega_1\}$ satisfying the following:*

- (i) *Each space \mathcal{Y}_ξ is a hereditarily indecomposable and reflexively saturated \mathcal{L}_∞ -space with the “scalar-plus-compact” property.*
- (ii) *Each space \mathcal{Y}_ξ is a quotient of \mathfrak{B}_{mT} .*
- (iii) *For each $\xi \neq \zeta$, every bounded linear operator $T : \mathcal{Y}_\xi \rightarrow \mathcal{Y}_\zeta$ is compact.*

We recall that in [AH, Section 10.2] a continuum of pairwise incomparable spaces $\{X_a : a \in \mathfrak{c}\}$ is defined so that for $a \neq b$, every bounded operator $T : X_a \rightarrow X_b$ is compact. This is achieved by defining versions of the Argyros–Haydon space using almost disjoint families of weights and hence also different coding functions. All these spaces are actually quotients of \mathfrak{B}_{mT} as well, hence the class $\{X_a : a \in \mathfrak{c}\}$ satisfies the conclusion of Theorem 7.1.

Remark 7.2. As it was mentioned in Remark 2.12, a version $\tilde{\mathfrak{X}}_{nr}$ of \mathfrak{X}_{nr} can be obtained as a quotient of a version $\tilde{\mathfrak{X}}_{AH}$ of \mathfrak{X}_{AH} . Actually, all spaces in the classes $\{\mathcal{Y}_\xi : \xi < \omega_1\}$ and $\{X_a : a \in \mathfrak{c}\}$ can be constructed to be quotients of one same Argyros–Haydon space \mathfrak{X}_{AH} .

8. SUBSPACES AND QUOTIENTS DETERMINED BY SELF-DETERMINED SUBSETS IN \mathfrak{X}_{nr}

In this section we very briefly describe some results primarily concerning quotients of \mathfrak{X}_{nr} . It is of some interest that one may find Banach spaces X_1, X_2, X_3 , each one being a quotient of the previous one, so that X_1 and X_3 are reflexive saturated, whereas X_2 contains no reflexive subspaces.

The following is the analogue of Proposition 2.14 in the case of the space \mathfrak{X}_{nr} . The same argument used in that proof is necessary here as well, however some extra care needs to be taken.

Proposition 8.1. *Let Γ' be a self-determined subset of Γ , and also let $Y = \{\{d_\gamma : \gamma \in \Gamma \setminus \Gamma'\}\}$. Then every bounded linear operator $T : Y \rightarrow \mathfrak{X}_{nr}$ is a scalar multiple of the inclusion plus a compact operator.*

Proof. By Remark 6.6, it suffices to show that if T is strictly singular, then it is also compact. Recall that by Proposition 1.7, Y is a \mathcal{L}_∞ -space; hence if T is not compact, then the conclusion of Proposition 6.9 holds. As the set Γ' is self-determined, Proposition 1.5(d) yields that the argument used in the proof of Theorem 6.10 can be repeated, since when splitting a vector according to its local support, the components remain in the subspace Y . □

By combining Proposition 8.1 and Lemma 2.15 we observe that Corollary 2.16 also holds for the space \mathfrak{X}_{nr} . One can then perform a construction similar to that presented in Subsection 2.7, using, e.g., sizes instead of weights, to obtain a family $\{\Gamma_\alpha : \alpha < \mathfrak{c}\}$ of subsets of Γ so that for $\alpha \neq \beta$ the set $\Gamma_\alpha \setminus \Gamma_\beta$ is infinite. This is achieved by choosing a family $\{L_\alpha : \alpha < \mathfrak{c}\}$ of infinite subsets of \mathbb{N} with pairwise finite intersections. Each set Γ_α is chosen so that its elements are built by using only α_c -averages of sizes from the set L_α . Corollary 2.16 (which as we explained holds for \mathfrak{X}_{nr}) yields the following.

Proposition 8.2. *There exists a continuum of pairwise nonisomorphic \mathcal{L}_∞ -subspaces $\{Y_a : a \in \mathfrak{c}\}$ of \mathfrak{X}_{nr} , each one of which has the scalar-plus-compact property. Moreover, for every $\alpha \neq \beta$ every bounded linear operator $T : Y_\alpha \rightarrow Y_\beta$ is compact.*

The situation becomes more involved when considering quotients of the space \mathfrak{X}_{nr} . The spaces \mathcal{Y}_ξ defined in the previous section are reflexively saturated versions of the space \mathfrak{X}_{nr} , however they are not quotients of \mathfrak{X}_{nr} ; in fact, every bounded linear operator $T : \mathfrak{X}_{\text{nr}} \rightarrow \mathcal{Y}_\xi$ is compact. It is possible to obtain spaces similar to the spaces \mathcal{Y}_ξ , $\xi < \omega_1$ which are indeed quotients of \mathfrak{X}_{nr} , however the construction is a little more delicate. For $2 \leq \xi < \omega_1$, consider the tree \mathcal{U}_ξ as in Section 7. Now consider the set of all those weights $(m_j, n_j)_{j \in L_\xi}$, which are weights of elements $\gamma \in \bar{\Gamma}$, that appear in elements (finite sequences of pairs) of \mathcal{U}_ξ . Now, construct a self-determined subset of Γ which allows one to build averages and coordinates using precisely those weights. Note that within Γ , the notion of an α_c -average is predetermined. The resulting quotient \mathcal{Y}_ξ is a reflexively saturated \mathcal{L}_∞ space with the scalar-plus-compact property. We observe that the conclusion of Theorem 7.1 is false in the class of spaces $\{\mathcal{Y}_\xi : 2 \leq \xi \leq \omega_1\}$. The reason for this is that if $\xi < \zeta$ is such that $\mathcal{S}_\xi \subset \mathcal{S}_\zeta$, then \mathcal{Y}_ξ is a quotient of \mathcal{Y}_ζ . What is interesting, however, is that although \mathfrak{X}_{nr} contains no reflexive subspaces, it admits reflexively saturated quotients. Recall also that \mathfrak{X}_{nr} is a quotient of a reflexively saturated space, e.g., \mathfrak{B}_{mT} or of a version of \mathfrak{X}_{AH} . Summarizing the preceding discussion we reach the conclusion stated in the result below, which ought to be compared to a classical theorem proved by Johnson and Zippin stating that every quotient of c_0 is isomorphic to a subspace of c_0 [JZ]. Although quotients of classical \mathcal{L}_∞ -spaces have structure similar to those spaces, this does not happen in nonclassical \mathcal{L}_∞ -spaces.

Theorem 8.3. *There exists a triple of infinite-dimensional Banach spaces X_1 , X_2 and X_3 so that X_1 and X_3 are reflexively saturated, X_2 contains no reflexive subspace, X_2 is a quotient of X_1 , and X_3 is a quotient of X_2 . All three spaces are ℓ_1 -preduals with the scalar-plus-compact property.*

A classical result asserts that every quotient of a $C(K)$ space is either reflexive or contains c_0 isomorphically. This invites the following question that, as far as we know, is open.

Problem. Let X be a \mathcal{L}_∞ -space, and let Y be a quotient of X . Does Y have to be reflexive or contain a \mathcal{L}_∞ -subspace?

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