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A reflexive hereditarily indecomposable space with the hereditary invariant subspace property

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Dedicated to the memory of Edward Odell

Abstract

A separable Banach space X satisfies the invariant subspace property (ISP) if every bounded linear operator $T \in \mathcal{L}(X)$ admits a non-trivial closed invariant subspace. In this paper, we present the first known example of a reflexive Banach space $\mathfrak{X}_{\mathrm{ISP}}$ satisfying the ISP. Moreover, this is the first known example of a Banach space satisfying the hereditary ISP, namely every infinite-dimensional subspace of it satisfies the ISP. The space $\mathfrak{X}_{\mathrm{ISP}}$ is hereditarily indecomposable (HI) and every operator $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ is of the form $\lambda I + S$ with S a strictly singular operator. The critical property of the strictly singular operators of $\mathfrak{X}_{\mathrm{ISP}}$ is that the composition of any three of them is a compact one. The construction of $\mathfrak{X}_{\mathrm{ISP}}$ is based on saturation methods and it uses as an unconditional frame Tsirelson space. The new ingredient in the definition of the space is the saturation under constraints, a method initialized in a fundamental work of Edward Odell and Thomas Schlumprecht.

Introduction

The invariant subspace problem asks whether every bounded linear operator on an infinite-dimensional separable Banach space admits a non-trivial closed invariant subspace. A classical result of M. Aronszajn and K.T. Smith [**11**] asserts that the problem has a positive answer for compact operators. This result was extended by V. Lomonosov [**23**] for operators on complex Banach spaces that commute with a non-trivial compact operator. N.D. Hooker [**21**] and G. Sirotkin [**33**] have presented a version of Lomonosov's theorem for real spaces. It is also known that the problem, in its full generality, has a negative answer. Indeed P. Enflo [**15**] and subsequently C.J. Read [**28**, **29**] have provided several examples of operators on non-reflexive Banach spaces that do not admit a non-trivial invariant subspace. In particular, in a profound study concerning spaces admitting operators without non-trivial invariant subspaces, Read has proved that every separable Banach space that contains either c_0 or a complemented subspace isomorphic to ℓ_1 or J_∞, admits an operator without non-trivial closed invariant subspaces [**30**]. A comprehensive study of Read's methods of constructing operators with no non-trivial invariant subspaces can be found in [**19**, **20**]. Also recently a non-reflexive hereditarily indecomposable (HI) Banach space $\mathfrak{X}_K$ with the 'scalar plus compact' property has been constructed [**7**]. This is a $\mathcal{L}_\infty$ space with separable dual, resulting from a combination of HI techniques with the fundamental J. Bourgain and F. Delbaen [**12**] construction. As a consequence, the space $\mathfrak{X}_K$ satisfies the invariant subspace property (ISP). Moreover, recently a $\mathcal{L}_\infty$ space, containing ℓ_1 isomorphically, with the scalar plus compact property, has been constructed in a forthcoming work by R.G. Haydon, Th. Raikoftsalis and the first named author. Let us point out, that the latter shows that Read's result concerning separable spaces containing ℓ_1 as a complemented subspace is not extended to separable spaces containing ℓ_1.

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All the above results provide no information in either direction within the class of reflexive Banach spaces. The importance of a result in this class, concerning the invariant subspace problem, is reflected in the concluding phrase of Read [**30**] where the following is stated. 'It is clear that we cannot go much further until and unless we solve the invariant subspace problem on a reflexive Banach space.'

The aim of this work is to construct the first example of a reflexive Banach space $\mathfrak{X}_{\rm ISP}$ with the ISP, which is also the example of a Banach space with the hereditary ISP. This property is not proved for the aforementioned space $\mathfrak{X}_K$. It is notable that no subspace of $\mathfrak{X}_{\rm ISP}$ has the 'scalar plus compact' property. More precisely, the strictly singular (A bounded linear operator is called strictly singular, if its restriction on any infinite dimensional subspace is not an into isomorphism.) non-compact operators on every subspace Y of $\mathfrak{X}_{\rm ISP}$ form a non-separable subset of $\mathcal{L}(Y)$.

The space $\mathfrak{X}_{\rm ISP}$ is a hereditarily indecomposable (HI) space and every operator $T \in \mathcal{L}(\mathfrak{X}_{\rm ISP})$ is of the form $T = \lambda I + S$ with S strictly singular. We recall that there are strictly singular operators in Banach spaces without non-trivial invariant subspaces. More precisely, as is shown in [**31**], for a strictly increasing sequence $\{p_i\}_i$ of real numbers strictly greater than 2, the space $X = \ell_2 \oplus (\sum_i \oplus J_{p_i})_2$ admits strictly singular operators without non-trivial invariant subspaces. On the other hand, there are spaces where the ideal of strictly singular operators does not coincide with the corresponding one of compact operators and every strictly singular operator admits a non-trivial invariant subspace. The most classical spaces with this property are $L^p[0,1], 1 \leqslant p < \infty$ and $C[0,1]$. This is a combination of Lomonosov–Sirotkin theorem and the classical result, due to Milman [**24**], that the composition TS is a compact operator, for any T, S strictly singular operators, on any of the above spaces. In [**2**] Tsirelson-like spaces satisfying similar properties are presented. The possibility of constructing a reflexive space with the ISP without the 'scalar plus compact' property emerged from an earlier version of [**2**].

The following describes the main properties of the space $\mathfrak{X}_{\rm ISP}$.

THEOREM. *There exists a reflexive space $\mathfrak{X}_{\rm ISP}$ with a Schauder basis $\{e_n\}_{n\in\mathbb{N}}$ satisfying the following properties.*

(i) *The space $\mathfrak{X}_{\rm ISP}$ is HI.*

(ii) *Every seminormalized weakly null sequence $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence generating either ℓ_1 or c_0 as a spreading model. Moreover every infinite-dimensional subspace Y of $\mathfrak{X}_{\rm ISP}$ admits both ℓ_1 and c_0 as spreading models.*

(iii) *For every Y infinite-dimensional closed subspace of $\mathfrak{X}_{\rm ISP}$ and every $T \in \mathcal{L}(Y, \mathfrak{X}_{\rm ISP})$, $T = \lambda I_{Y,\mathfrak{X}_{\rm ISP}} + S$ with S strictly singular.*

(iv) *For every Y infinite-dimensional subspace of $\mathfrak{X}_{\rm ISP}$ the ideal $\mathcal{S}(Y)$ of the strictly singular operators is non-separable.*

(v) *For every Y subspace of $\mathfrak{X}_{\rm ISP}$ and every Q, S, T in $\mathcal{S}(Y)$ the operator QST is compact. Hence for every $T \in \mathcal{S}(Y)$ either $T^3 = 0$ or T commutes with a non-zero compact operator.*

(vi) *For every Y infinite-dimensional closed subspace of X and every $T \in \mathcal{L}(Y)$, T admits a non-trivial closed invariant subspace. In particular, every $T \neq \lambda I_Y$, for $\lambda \in \mathbb{R}$ admits a non-trivial hyperinvariant subspace.*

It is not clear to us whether the number of operators in property (v) can be reduced. For defining the space $\mathfrak{X}_{\rm ISP}$, we use classical ingredients such as the coding function σ, the interaction between conditional and unconditional structure, but also some new ones which we are about to describe.

In all previous HI constructions, one had to use a mixed Tsirelson space as the unconditional frame on which the HI norm is built. Mixed Tsirelson spaces appeared with Schlumprecht [**32**] space, twenty years after Tsirelson construction [**34**]. They became an inevitable ingredient for any HI construction, starting with the Gowers and Maurey [18] celebrated example, and followed by myriads of others [4, 10], etc. The most significant difference in the construction of $\mathfrak{X}_{\mathrm{ISP}}$ from the classical ones is that it uses as an unconditional frame the Tsirelson space itself.

As is clear to the experts, HI constructions based on Tsirelson space are not possible if we deal with a complete saturation of the norm. Thus, the second ingredient involves saturation under constraints. This method was introduced by Odell and Schlumprecht [25, 26] for defining heterogeneous local structure in HI spaces, a method also used in [2]. By saturation under constraints we mean that the operations $(1/2^n, \mathcal{S}_n)$ (see Remark 1.5) are applied on very fast growing families of averages, which are either α-averages or β-averages. The α-averages have also been used in [25, 26], while β-averages are introduced to control the behaviour of special functionals. It is notable that although the α, β-averages do not contribute to the norm of the vectors in $\mathfrak{X}_{\mathrm{ISP}}$, they are able to neutralize the action of the operations $(1/2^n, \mathcal{S}_n)$ on certain sequences and thus c_0 spreading models become abundant. This significant property yields the structure of $\mathfrak{X}_{\mathrm{ISP}}$ described in the above theorem.

Let us briefly describe some further structural properties of the space $\mathfrak{X}_{\mathrm{ISP}}$.

The first and the most crucial one is that for a (n, ε) special convex combination (see Definition 1.9) $\sum_{i\in F} c_i x_i$, with $\{x_i\}_{i\in F}$ a finite normalized block sequence, we have that

$$\left\|\sum_{i\in F} c_i x_i\right\| \leqslant \frac{6}{2^n} + 12\varepsilon.$$

This evaluation is due to the fact that the space is built on Tsirelson space and differs from the classical asymptotic ℓ_1 HI spaces (that is, [4, 10]) where seminormalized (n, ε) special convex combinations exist in every block subspace. A consequence of the above is that the frequency of the appearance of RIS sequences is significantly increased, which among others yields the following. Every strictly singular operator maps sequences generating c_0 spreading models to norm null ones. Furthermore, we classify weakly null sequences into sequences of rank 0, namely norm null ones, sequences of rank 1, namely sequences generating c_0 as a spreading model and sequences of rank 2 or 3, namely sequences generating ℓ_1 as a spreading model. The main result concerning these ranks is the following. If Y is an infinite-dimensional closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$ and T is a strictly singular operator on Y, then it maps sequences of non-zero rank, to sequences of strictly smaller rank. Combining the above properties, we conclude property (v) of the above theorem.

1. *The norming set of the space* $\mathfrak{X}_{\mathrm{ISP}}$

In this section, we define the norming set W of the space $\mathfrak{X}_{\mathrm{ISP}}$. This set is defined with the use of the sequence $\{\mathcal{S}_n\}_n$ which we remind below and also families of $\mathcal{S}_n$-admissible functionals.

As we have mentioned in the introduction, the set W will be a subset of the norming set W_T of the Tsirelson space.

1.1. *The Schreier families*

The Schreier families is an increasing sequence of families of finite subsets of the naturals, which first appeared in [1], and is inductively defined in the following manner.

Set $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\}$ and $\mathcal{S}_1 = \{F \subset \mathbb{N} : \#F \leqslant \min F\}$.

Suppose that $\mathcal{S}_n$ has been defined and set $\mathcal{S}_{n+1} = \{F \subset \mathbb{N} : F = \bigcup_{j=1}^k F_j$, where $F_1 < \cdots < F_k \in \mathcal{S}_n$ and $k \leqslant \min F_1\}$.

If for $n, m \in \mathbb{N}$, we set $\mathcal{S}_n * \mathcal{S}_m = \{F \subset \mathbb{N} : F = \bigcup_{j=1}^k F_j$, where $F_1 < \cdots < F_k \in \mathcal{S}_m$ and $\{\min F_j : j = 1, \ldots, k\} \in \mathcal{S}_n\}$, then it is well known [4] and follows easily by induction that $\mathcal{S}_n * \mathcal{S}_m = \mathcal{S}_{n+m}$.

NOTATION. A sequence of vectors $x_1 < \cdots < x_k$ in c_{00} is said to be $\mathcal{S}_n$-admissible if $\{\min \operatorname{supp} x_i : i = 1, \ldots, k\} \in \mathcal{S}_n$.

Let $G \subset c_{00}$. A vector $f \in G$ is said to be an average of size $s(f) = n$, if there exist $f_1, \ldots, f_d \in G, d \leqslant n$, such that $f = (1/n)(f_1 + \cdots + f_d)$.

A sequence $\{f_j\}_j$ of averages in G is said to be very fast growing, if $f_1 < f_2 < \cdots$, $s(f_j) > 2^{\max \operatorname{supp} f_{j-1}}$ and $s(f_j) > s(f_{j-1})$ for $j > 1$.

1.2. *The coding function*

Choose $L = \{\ell_k : k \in \mathbb{N}\}, \ell_1 > 2$ an infinite subset of the naturals such that:

(i) For any $k \in \mathbb{N}$, we have that $\ell_{k+1} > 2^{2\ell_k}$ and
(ii) $\sum_{k=1}^{\infty}(1/2^{\ell_k}) < \frac{1}{1000}$.

Define a partition of L into two infinite subsets L_1 and L_2. Set

$$\mathcal{Q} = \{((f_1, n_1), \ldots, (f_m, n_m)) : m \in \mathbb{N}, \{n_k\}_{k=1}^m \subset \mathbb{N}, f_1 < \cdots < f_m \in c_{00} \text{ with } f_k(i) \in \mathbb{Q}, \text{ for } i \in \mathbb{N},\ k = 1, \ldots, m\}.$$

Choose a one-to-one function $\sigma : \mathcal{Q} \to L_2$, called the coding function, such that for any $((f_1, n_1), \ldots, (f_m, n_m)) \in \mathcal{Q}$, we have that

$$\sigma((f_1, n_1), \ldots, (f_m, n_m)) > 2^{n_m} \cdot \max \operatorname{supp} f_m.$$

REMARK 1.1. For any $n \in \mathbb{N}$ we have that $\#L \cap \{n, \ldots, 2^{2n}\} \leqslant 1$.

1.3. *The norming set*

The norming set W is defined to be the smallest subset of c_{00} satisfying the following properties.

1. The set $\{\pm e_n\}_{n \in \mathbb{N}}$ is a subset of W, for any $f \in W$, we have that $-f \in W$, for any $f \in W$ and any I interval of the naturals we have that $If \in W$ and W is closed under rational convex combinations. Any $f = \pm e_n$ will be called a functional of type 0.

We call an α-average any average $\alpha \in W$ of the form $\alpha = (1/n)\sum_{j=1}^d f_j,\ d \leqslant n$, where $f_1 < \cdots < f_d \in W$.

2. The set W contains any functional f which is of the form $f = (1/2^n)\sum_{j=1}^d \alpha_j$, where $\{\alpha_j\}_{j=1}^d$ is an $\mathcal{S}_n$-admissible and very fast growing sequence of α-averages in W. If I is an interval of the naturals, then $g = \pm If$ is called a functional of type I_α, of weight $w(g) = n$.

3. The set W contains any functional f which is of the form $f = \frac{1}{2}\sum_{j=1}^d f_j$, where $\{f_j\}_{j=1}^d$ is an $\mathcal{S}_1$-admissible special sequence of type I_α functionals. This means that $w(f_1) \in L_1$ and $w(f_j) = \sigma((f_1, w(f_1)), \ldots, (f_{j-1}, w(f_{j-1})))$, for $j > 1$. If I is an interval of the naturals, then $g = \pm If$ is called a functional of type II with weights $\hat{w}(g) = \{w(f_j) : \operatorname{ran} f_j \cap I \neq \emptyset\}$.

We call a β-average any average $\beta \in W$ of the form $\beta = (1/n)\sum_{j=1}^d f_j, d \leqslant n$, where $f_1, \ldots, f_d \in W$ are functionals of type II, with disjoint weights $\hat{w}(f_j)$.

4. The set W contains any functional f which is of the form $f = (1/2^n)\sum_{j=1}^d \beta_j$, where $\{\beta_j\}_{j=1}^d$ is an $\mathcal{S}_n$-admissible and very fast growing sequence of β-averages in W. If I is an interval of the naturals, then $g = \pm If$ is called a functional of type I_β, of weight $w(g) = n$.

In general, we call a convex combination any $f \in W$ that is not of type 0, I_α, I_β or II.

For $x \in c_{00}$ define $\|x\| = \sup\{f(x) : f \in W\}$ and $\mathfrak{X}_{\mathrm{ISP}} = \overline{(c_{00}(\mathbb{N}), \|\cdot\|)}$. Evidently, $\mathfrak{X}_{\mathrm{ISP}}$ has a bimonotone basis.

One may also describe the norm on $\mathfrak{X}_{\mathrm{ISP}}$ with an implicit formula. Indeed, for some $x \in \mathfrak{X}_{\mathrm{ISP}}$, we have that

$$\|x\| = \max\left\{\|x\|_0,\ \|x\|_{II},\ \sup\left\{\frac{1}{2^n}\sum_{j=1}^{d}\|E_jx\|_{k_j}^{\alpha}\right\},\ \sup\left\{\frac{1}{2^n}\sum_{j=1}^{d}\|E_jx\|_{k_j}^{\beta}\right\}\right\},$$

where the inner suprema are taken over all $n \in \mathbb{N}$, all $\mathcal{S}_n$-admissible intervals $\{E_j\}_{j=1}^d$ of the naturals and $k_1 < \cdots < k_d$ such that $k_j > 2^{\max E_{j-1}}$ for $j > 1$.

By $\|x\|_{II}$, we denote $\|x\|_{II} = \sup\{f(x) : f \in W$ is a functional of type II$\}$ whereas for $j \in \mathbb{N}$, by $\|x\|_j^{\alpha}$ we denote $\|x\|_j^{\alpha} = \sup\{\alpha(x) : \alpha \in W$ is an α-average of size $s(\alpha) = j\}$

Similarly, by $\|x\|_j^{\beta}$ we denote $\|x\|_j^{\beta} = \sup\{\beta(x) : \beta \in W$ is a β-average of size $s(\beta) = j\}$.

Remark 1.2. Very fast growing sequences of α-averages have been considered by Odell and Schlumprecht [**25**, **26**] and were also used in [**2**]. However, β-averages are a new ingredient, introduced to control the behaviour of type II functionals on block sequences. The β-averages can also be used to provide an alternative and simpler approach of the main result in [**26**].

As we have mentioned in the introduction, the $\|z\|_j^{\alpha}$ and $\|z\|_j^{\beta}$, which are averages, do not contribute to the norm of the vector z. On the other hand, the $\{\|\cdot\|_j^{\alpha}\}_j$ and $\{\|\cdot\|_j^{\beta}\}_j$ have a significant role for the structure of the space $\mathfrak{X}_{\mathrm{ISP}}$.

Remark 1.3. The norming set W can be inductively constructed to be the union of subsets $\{W_m\}_{m=0}^{\infty}$ of c_{00}, where W_0 is the rational convex hull of $\{\pm e_n\}_n$. If W_m has been constructed, then set W_{m+1}^I to be the enlargement of W_m defined by including all type I_α and type I_β functionals constructed using elements of W_m, set W_{m+1}^{II} to be the enlargement of W_{m+1}^I defined by including all type II functionals constructed using elements of W_{m+1}^I and finally set W_{m+1} to be the rational convex hull of W_{m+1}^{II}.

1.4. *Tsirelson space*

Tsirelson's initial definition [**34**] of the first Banach space not containing any $\ell_p, 1 \leqslant p < \infty$ or c_0, concerned the dual of the so-called Tsirelson norm which was introduced by T. Figiel and W. B. Johnson [**17**] and satisfies the following implicit formula:

$$\|x\|_T = \max\left\{\|x\|_0,\ \sup\left\{\frac{1}{2}\sum_{j=1}^{d}\|E_jx\|_T\right\}\right\},$$

where $x \in c_{00}$ and the inner supremum is taken over all successive subsets of the naturals $d \leqslant E_1 < \cdots < E_d$. Tsirelson space T is defined to be the completion of $(c_{00}, \|\cdot\|_T)$. In the sequel by Tsirelson norm and Tsirelson space, we will mean the norm and the corresponding space from [**17**].

As is well known (see [**14**]), a norming set W_T of Tsirelson space is the smallest subset of c_{00} satisfying the following properties.

1. The set $\{\pm e_n\}_{n\in\mathbb{N}}$ is a subset of W_T, for any $f \in W_T$, we have that $-f \in W_T$, for any $f \in W_T$ and any E subset of the naturals we have that $Ef \in W_T$ and W_T is closed under rational convex combinations.

2. The set W_T contains any functional f which is of the form $f = \frac{1}{2}\sum_{j=1}^d f_j$, where $\{f_j\}_{j=1}^d$ is a $\mathcal{S}_1$ admissible sequence in W_T.

Remark 1.4. The following are well-known facts about Tsirelson space.

(i) The norming set W_T can be inductively constructed to be the union of an increasing sequence of subsets $\{W_T^m\}_{m=0}^\infty$ of c_{00}, in a similar manner as above.

(ii) The set W_T', which is the smallest subset of c_{00} satisfying the following properties, also is a norming set for Tsirelson space.

1. The set $\{\pm e_n\}_{n\in\mathbb{N}}$ is a subset of W_T', for any $f \in W_T'$ we have that $-f \in W_T'$ and for any $f \in W_T'$ and any E subset of the naturals we have that $Ef \in W_T'$.
2. The set W_T' contains any functional f which is of the form $f = \frac{1}{2}\sum_{j=1}^d f_j$, where $\{f_j\}_{j=1}^d$ is a $\mathcal{S}_1$ admissible sequence in W_T'.

Remark 1.5. It is easy to check that the norming set W_T of Tsirelson space is closed under $(1/2^n, \mathcal{S}_n)$ operations, namely for any $f_1 < \cdots < f_d$ in W_T $\mathcal{S}_n$-admissible, the functional $(1/2^n)\sum_{j=1}^d f_j \in W_T$. This explains that the norming set W of the space $\mathfrak{X}_{\mathrm{ISP}}$ is a subset of W_T. Therefore, Tsirelson space is the unconditional frame on which the norm of $\mathfrak{X}_{\mathrm{ISP}}$ is built. As we mentioned in in the introduction, $\mathfrak{X}_{\mathrm{ISP}}$ is the first HI construction which uses Tsirelson space instead of a mixed Tsirelson one.

As is shown in [**13**] (see also [**14**]), an equivalent norm on Tsirelson space is described by the following implicit formula. For $x \in c_{00}$ set

$$|||x||| = \max\left\{\|x\|_0,\ \sup\left\{\frac{1}{2}\sum_{j=1}^{2d}|||E_jx|||\right\}\right\},$$

where the inner supremum is taken over all successive subsets of the naturals $d \leqslant E_1 < \cdots < E_{2d}$. Then, for any $\{c_k\}_{k=1}^n \subset \mathbb{R}$, the following holds:

$$\left\|\sum_{k=1}^n c_ke_k\right\|_T \leqslant \left|\left|\left|\sum_{k=1}^n c_ke_k\right|\right|\right| \leqslant 3\left\|\sum_{k=1}^n c_ke_k\right\|_T. \tag{1}$$

Remark 1.6. A norming set $W_{(T,|||\cdot|||)}$ for $(T, |||\cdot|||)$ is also defined in a similar manner as W_T.

1.5. *Special convex combinations*

Next, we remind the notion of the (n,ε) special convex combinations (see [**4**, **6**, **10**]) which is one of the main tools used in the sequel.

Definition 1.7. Let $x = \sum_{k\in F} c_ke_k$ be a vector in c_{00}. Then x is said to be a (n,ε) basic special convex combination (or a (n,ε) basic s.c.c.) if the following conditions are satisfied.

(i) The set F is in $\mathcal{S}_n$, $c_k \geqslant 0$, for $k \in F$ and $\sum_{k\in F} c_k = 1$.

(ii) For any $G \subset F, G \in \mathcal{S}_{n-1}$, we have that $\sum_{k\in G} c_k < \varepsilon$.

The next result is from [**8**]. For a proof see [**10**, Chapter 2, Proposition 2.3].

PROPOSITION 1.8. *For any M infinite subset of the naturals, any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $F \subset M, \{c_k\}_{k\in F}$, such that $x = \sum_{k\in F} c_k e_k$ is a (n, ε) basic s.c.c.*

DEFINITION 1.9. Let $x_1 < \cdots < x_m$ be vectors in c_{00} and $\psi(k) = \min \operatorname{supp} x_k$, for $k = 1, \ldots, m$. Then $x = \sum_{k=1}^{m} c_k x_k$ is said to be a (n, ε) special convex combination (or (n, ε) s.c.c.), if $\sum_{k=1}^{m} c_k e_{\psi(k)}$ is a (n, ε) basic s.c.c.

2. *Basic evaluations for special convex combinations*

In this section we prove the basic inequality for block sequences in $\mathfrak{X}_{\mathrm{ISP}}$, with the auxiliary space actually being Tsirelson space. This will allow us to evaluate the norm of (n, ε) special convex combinations and is critical throughout the rest of the paper.

DEFINITION 2.1. Let $f \in W$ be a functional of type I_α or I_β, of weight $w(f) = n$, $f = (1/2^n) \sum_{j=1}^{d} f_j$. Then, by definition, there exist $F_1 < \cdots < F_p$ successive intervals of the naturals such that:

(i) $\bigcup_{i=1}^{p} F_i = \{1, \ldots, d\}$;
(ii) $\{\min \operatorname{supp} f_j : j \in F_i\} \in \mathcal{S}_{n-1}$, for $i = 1, \ldots, p$;
(iii) $\{\min \operatorname{supp} f_{\min F_i} : i = 1, \ldots, p\} \in \mathcal{S}_1$.

Set $g_i = (1/2^{n-1}) \sum_{j\in F_i} f_j$, for $i = 1, \ldots, p$. We call $\{g_i\}_{i=1}^{p}$ a Tsirelson analysis of f.

REMARK 2.2. If $f \in W$ is a functional of type I_α or I_β and $\{f_i\}_{i=1}^{p}$ is a Tsirelson analysis of f, then $f_i \in W$, $\{f_i\}_{i=1}^{p}$ is $\mathcal{S}_1$-admissible and $f = \frac{1}{2} \sum_{i=1}^{p} f_i$, although $\{f_i\}_{i=1}^{p}$ may not be a very fast growing sequence of α-averages or β-averages. Moreover, if $w(f) > 1$, then f_i is of the same type as f and $w(f_i) = w(f) - 1$ for $i = 1, \ldots, p$.

2.1. *The tree analysis of a functional* $\mathbf{f} \in \mathbf{W}$

A key ingredient for evaluating the norm of vectors in $\mathfrak{X}_{\mathrm{ISP}}$ is the analysis of the elements f of the norming set W. This is similar to the corresponding concept that has occurred in almost all previous HI and related constructions (that is, [**3**, **4**, **7**, **10**]). Next, we briefly describe the tree analysis in our context.

For any functional $f \in W$, we associate a family $\{f_\lambda\}_{\lambda\in\Lambda}$, where Λ is a finite tree which is inductively defined as follows.

Set $f_\emptyset = f$, where $\emptyset$ denotes the root of the tree to be constructed. If f is of type 0, then the tree analysis of f is $\{f_\emptyset\}$. Otherwise, suppose that the nodes of the tree and the corresponding functionals have been chosen up to a height p and let λ be a node of height $|\lambda| = p$. If f_λ is of type 0, then do not extend any further and λ is a maximal node of the tree.

If f_λ is of type I_α or I_β, set the immediate successors of λ to be the elements of the Tsirelson analysis of f_λ.

If f_λ is of type II, $f = \frac{1}{2} \sum_{j=1}^{d} f_j$, set the immediate successors of λ to be the $\{f_j\}_{j=1}^{d}$.

If f_λ is a convex combination, which includes α-averages and β-averages, $f_\lambda = \sum_{j=1}^{d} c_j f_j$, set the immediate successors of λ to be the $\{f_j\}_{j=1}^{d}$.

By Remark 1.3, it follows that the inductive construction ends in finitely many steps and that the tree Λ is finite.

Remark 2.3. Let $f \in W$ and $\{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of f. Then for any $\lambda \in \Lambda$ not a maximal node, such that f_λ is not a convex combination, we have that $f_\lambda = \frac{1}{2}\sum_{\mu\in\mathrm{succ}(\lambda)} f_\mu$, where $\{f_\mu\}_{\mu\in\mathrm{succ}(\lambda)}$ are $\mathcal{S}_1$-admissible and by $\mathrm{succ}(\lambda)$ we denote the immediate successors of λ in Λ.

Remark 2.4. In a similar manner, for any $f \in W_T'$ (see Remark 1.4(ii)), the tree analysis of f is defined.

Proposition 2.5. *Let $x = \sum_{k\in F} c_k e_k$ be a (n,ε) basic s.c.c. and $G \subset F$. Then the following holds*:

$$\Big\|\sum_{k\in G} c_k e_k\Big\|_T \leqslant \frac{1}{2^n}\sum_{k\in G} c_k + \varepsilon.$$

Proof. Let $f \in W_T'$. We may assume that $\mathrm{supp}\, f \subset G$. Set $G_1 = \{k \in \mathrm{supp}\, f : |f(e_k)| \leqslant 1/2^n\}, G_2 = \mathrm{supp}\, f \setminus G_1$. Then clearly $|G_1 f(\sum_{k\in G} c_k e_k)| \leqslant 1/2^n \sum_{k\in G} c_k$.

We will show by induction that $G_2 \in \mathcal{S}_{n-1}$. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of $G_2 f$. Then it is easy to see that $h(\Lambda) \leqslant n-1$. For λ a maximal node in Λ, we have that $\mathrm{supp}\, f_\lambda \in \mathcal{S}_0$. Assume that for any $\lambda \in \Lambda, |\lambda| = k > 0$, we have that $\mathrm{supp}\, f_\lambda \in \mathcal{S}_{n-1-k}$ and let $\lambda \in \Lambda$, such that $|\lambda| = k-1$. Then $f_\lambda = \frac{1}{2}\sum_{j=1}^d f_{\lambda_j}$, where $|\lambda_j| = k$, $\mathrm{supp}\, f_{\lambda_j} \in \mathcal{S}_{n-k-1}$ for $j = 1,\ldots,d$ and $\{\min\mathrm{supp}\, f_{\lambda_j} :\ j = 1,\ldots,d\} \in \mathcal{S}_1$. Then $\mathrm{supp}\, f_\lambda = \bigcup_{j=1}^d \mathrm{supp}\, f_{\lambda_j} \in \mathcal{S}_{n-1-(k-1)}$.

The induction is complete and it follows that $G_2 = \mathrm{supp}\, G_2 f \in \mathcal{S}_{n-1}$ and therefore $G_2 f(\sum_{k\in G} c_k e_k) \leqslant \sum_{k\in G_2} c_k < \varepsilon$. Hence, $|f(\sum_{k\in G} c_k e_k)| < (1/2^n)\sum_{k\in G} c_k + \varepsilon$. □

Proposition 2.6 Basic inequality. *Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\mathrm{ISP}}$ such that $\|x_k\| \leqslant 1$, for all k and let $f \in W$. Set $\phi(k) = \max\mathrm{supp}\, x_k$, for all k. Then there exists $g \in W_{(T,|||\cdot|||)}$ (see Remark 1.6) such that $2g(e_{\phi(k)}) \geqslant f(x_k)$, for all k.*

Proof. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of f. We will inductively construct $\{g_\lambda\}_{\lambda\in\Lambda}$ such that for any $\lambda \in \Lambda$ the following are satisfied.

(i) The functional g_λ is in $W_{(T,|||\cdot|||)}$ and $2g_\lambda(e_{\phi(k)}) \geqslant f_\lambda(x_k)$, for any k.
(ii) The set $\mathrm{supp}\, g_\lambda$ is a subset of $\{\phi(k) : \mathrm{ran}\, f_\lambda \cap \mathrm{ran}\, x_k \neq \emptyset\}$.

For $\lambda \in \Lambda$ a maximal node, if there exists k such that $\mathrm{ran}\, f_\lambda \cap \mathrm{ran}\, x_k \neq \emptyset$, set $g_\lambda = e^*_{\phi(k)}$. Otherwise set $g_\lambda = 0$.

Let $\lambda \in \Lambda$ be a non-maximal node, and suppose that $\{g_\mu\}_{\mu>\lambda}$ have been chosen. We distinguish two cases.

Case 1: f_λ is a convex combination (that is, f_λ is not of type 0, I_α, I_β or II).

If $f_\lambda = \sum_{\mu\in\mathrm{succ}(\lambda)} c_\mu f_\mu$, set $g_\lambda = \sum_{\mu\in\mathrm{succ}(\lambda)} c_\mu g_\mu$.

Case 2 : f_λ is not a convex combination.

If $f_\lambda = \frac{1}{2}\sum_{j=1}^d f_{\mu_j}$, where $\mathrm{succ}(\lambda) = \{\mu_j\}_{j=1}^d$ such that $f_{\mu_1} < \cdots < f_{\mu_d}$, set

$$\begin{aligned}
G_\lambda &= \{k : \mathrm{ran}\, f_\lambda \cap \mathrm{ran}\, x_k \neq \emptyset\},\\
G_1 &= \{k \in G_\lambda :\ \text{there exists at most one } j \text{ with } \mathrm{ran}\, f_{\mu_j} \cap \mathrm{ran}\, x_k \neq \emptyset\},\\
G_2 &= \{k \in G_\lambda :\ \text{there exist at least two } j \text{ with } \mathrm{ran}\, f_{\mu_j} \cap \mathrm{ran}\, x_k \neq \emptyset\},\\
I_j &= \{k \in G_1 : \mathrm{ran}\, x_k \cap \mathrm{ran}\, f_{\mu_j} \neq \emptyset\} \quad \text{for } j = 1,\ldots,d.
\end{aligned}$$

Observe that $\#G_2 \leqslant d-1$.

For $j = 1, \ldots, d$ set $g'_j = g_{\mu_j}|_{\phi(I_j)}$ and for $k \in G_2$ set $g_k = e^*_{\phi(k)}$. It is easy to check that if we set $g_\lambda = \frac{1}{2}(\sum_{j=1}^{d} g'_j + \sum_{k \in G_2} g_k)$, then g_λ is the desired functional. □

The induction is complete. Set $g = g_\emptyset$. □

REMARK 2.7. In the previous constructions (see [**3**, **4**, **7**, **10**]), the basic inequality is used for estimating the norm of linear combinations of block vectors which are RIS. In this paper, the basic inequality is stronger, as it is able to provide upper estimations for any block vectors. Moreover, RIS sequences are defined in a different manner as in previous constructions and they also play a different role, which will be discussed in the sequel.

COROLLARY 2.8. *Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\mathrm{ISP}}$ such that $\|x_k\| \leqslant 1, \{c_k\}_k \subset \mathbb{R}$ and $\phi(k) = \max \operatorname{supp} x_k$ for all k. Then*:

$$\left\| \sum_k c_k x_k \right\| \leqslant 6 \left\| \sum_k c_k e_{\phi(k)} \right\|_T .$$

Proof. Let $f \in W$. Apply the basic inequality and take $g \in W_{(T,\|\!|\cdot|\!\|)}$, such that if $\phi(k) = \max \operatorname{supp} x_k$ and $y_k = \operatorname{sgn}(c_k) x_k$ for all k, we have that $2g(e_{\phi(k)}) \geqslant f(y_k)$, for any k. It follows that

$$2g\left(\sum_k |c_k| e_{\phi(k)}\right) \geqslant f\left(\sum_k c_k x_k\right).$$

Therefore, applying (1), we obtain

$$\left\| \sum_k c_k x_k \right\| \leqslant 2 \left|\!\left|\!\left| \sum_k |c_k| e_{\phi(k)} \right|\!\right|\!\right| = 2 \left|\!\left|\!\left| \sum_k c_k e_{\phi(k)} \right|\!\right|\!\right| \leqslant 2 \cdot 3 \| \sum_k c_k e_{\phi(k)} \|_T . \qquad \square$$

COROLLARY 2.9. *Let $x = \sum_{k=1}^{m} c_k x_k$ be a (n, ε) s.c.c. in $\mathfrak{X}_{\mathrm{ISP}}$, such that $\|x_k\| \leqslant 1$, for $k = 1, \ldots, m$. If $F \subset \{1, \ldots, m\}$, then*

$$\left\| \sum_{k \in F} c_k x_k \right\| \leqslant \frac{6}{2^n} \sum_{k \in F} c_k + 12\varepsilon.$$

In particular, we have that $\|x\| \leqslant 6/2^n + 12\varepsilon$.

Proof. Set $\phi(k) = \max \operatorname{supp} x_k, \psi(k) = \min \operatorname{supp} x_k$. Corollary 2.8 yields that $\| \sum_{k \in F} c_k x_k \| \leqslant 6 \| \sum_{k \in F} c_k e_{\phi(k)} \|_T$.

Since, according to the assumption, $\sum_{k \in F} c_k e_{\psi(k)}$ is a (n, ε) basic s.c.c., it easily follows that $\sum_{k \in F} c_k e_{\phi(k)}$ is a $(n, 2\varepsilon)$ basic s.c.c.

By Proposition 2.5, the result follows. □

COROLLARY 2.10. *The basis of $\mathfrak{X}_{\mathrm{ISP}}$ is shrinking.*

Proof. Suppose that it is not. Then there exist $x^* \in \mathfrak{X}^*_{\mathrm{ISP}}, \|x^*\| = 1$, a normalized block sequence $\{x_k\}_{k \in \mathbb{N}}$ in $\mathfrak{X}_{\mathrm{ISP}}$ and $\delta > 0$, such that $x^*(x_k) > \delta$, for all $k \in \mathbb{N}$.

Choose $n \in \mathbb{N}$, such that $1/2^n < \delta/12$ and $\varepsilon > 0$, such that $\varepsilon < \delta/24$. By Proposition 1.8, there exists F a subset of $\mathbb{N}$, such that $x = \sum_{k \in F} c_k x_k$ is a (n, ε) s.c.c.

By Corollary 2.9, we have that $\delta > \|x\| \geqslant x^*(x) > \delta$. A contradiction, which completes the proof. □

PROPOSITION 2.11. *The basis of $\mathfrak{X}_{\mathrm{ISP}}$ is boundedly complete.*

Proof. Assume that it is not. Then there exist $\varepsilon > 0$ and $\{x_k\}_{k\in\mathbb{N}}$ a block sequence in $\mathfrak{X}_{\mathrm{ISP}}$, such that $\|x_k\| > \varepsilon$ and $\|\sum_{k=\ell}^{\ell+m} x_k\| \leqslant 1$, for all $\ell, m \in \mathbb{N}$.

Choose k_0 such that $d = \min\operatorname{supp} x_{k_0} > 2/\varepsilon$. Set $F_1 = \{k_0\}$ and inductively choose $F_1, \ldots, F_d$, intervals of the naturals such that

(i) $\max F_j + 1 = \min F_{j+1}$, for $j < d$ and
(ii) $\#F_j > \max\{\#F_{j-1}, 2^{\max\operatorname{supp} x_{\max F_{j-1}}}\}$, for $1 < j \leqslant d$.

Then, if we set $y_j = \sum_{k\in F_j} x_k$, we have that $\|\sum_{j=1}^{d} y_j\| \leqslant 1$.

On the other hand, note that for $j = 1, \ldots, d$, there exists α_j an α-average in W, such that

(i) $\operatorname{ran}\alpha_j \subset \operatorname{ran} y_j$, therefore $\{\alpha_j\}_{j=1}^d$ is $\mathcal{S}_1$-admissible.
(ii) $s(\alpha_j) = \#F_j$, therefore $\{\alpha_j\}_{j=1}^d$ is very fast growing.
(iii) $\alpha_j(y_j) > \varepsilon$.

From the above, it follows $f = \frac{1}{2}\sum_{j=1}^{d}\alpha_j$ is a functional of type I_α in W and $f(\sum_{j=1}^{d} y_j) > \varepsilon \cdot d/2 > 1$. Since this cannot be the case, the proof is complete. □

These last two results and a well-known result due to James [**22**] allow us to conclude the following.

COROLLARY 2.12. *The space $\mathfrak{X}_{\mathrm{ISP}}$ is reflexive.*

DEFINITION 2.13. Let F be either $\mathbb{N}$ or an initial segment of the natural numbers. A block sequence $\{x_k\}_{k\in F}$ is said to be a $(C, \{n_k\}_{k\in F})$ α-rapidly increasing sequence (or $(C, \{n_k\}_{k\in F})$ α-RIS), for a positive constant $C \geqslant 1$ and a strictly increasing sequence of naturals $\{n_k\}_{k\in F}$, if $\|x_k\| \leqslant C$ for all $k \in F$ and the following conditions are satisfied.

(i) For any $k \in F$, for any functional f of type I_α of weight $w(f) = j < n_k$ we have that $|f(x_k)| < C/2^j$.
(ii) For any $k \in F$, we have that $1/2^{n_{k+1}} \max\operatorname{supp} x_k < 1/2^{n_k}$.

REMARK 2.14. Let $\{x_k\}_{k\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\mathrm{ISP}}$, such that there exist a positive constant C and $\{n_k\}_{k\in\mathbb{N}}$ strictly increasing naturals, such that $\|x_k\| \leqslant C$ for all k and condition (i) from Definition 2.13 is satisfied. Then $\{x_k\}_{k\in\mathbb{N}}$ admits a subsequence which is a $(C, \{n_k\}_{k\in\mathbb{N}})$ α-RIS.

DEFINITION 2.15. Let $n \in \mathbb{N}, C \geqslant 1, \theta > 0$. A vector $x \in \mathfrak{X}_{\mathrm{ISP}}$ is called a (C, θ, n) vector if the following holds: there exist $0 < \varepsilon < 1/36C2^{3n}$ and $\{x_k\}_{k=1}^m$ a block sequence in $\mathfrak{X}_{\mathrm{ISP}}$ with $\|x_k\| \leqslant C$ for $k = 1, \ldots, m$ such that the following holds.

(i) We have that $\min\operatorname{supp} x_1 \geqslant 8C2^{2n}$.
(ii) There exist $\{c_k\}_{k=1}^m \subset [0,1]$ such that $\sum_{k=1}^m c_k x_k$ is a (n, ε) s.c.c.
(iii) The vector x is of the form $x = 2^n \sum_{k=1}^m c_k x_k$ and $\|x\| \geqslant \theta$.

If, moreover, there exist $\{n_k\}_{k=1}^m$ strictly increasing natural numbers with $n_1 > 2^{2n}$ such that $\{x_k\}_{k=1}^m$ is a $(C, \{n_k\}_{k=1}^m)$ α-RIS, then x is called a (C, θ, n) exact vector.

REMARK 2.16. Let x be a (C, θ, n) vector in $\mathfrak{X}_{\mathrm{ISP}}$. Then, using Corollary 2.9 we conclude that $\|x\| < 7C$.

3. *The α and β indices*

To each block sequence, we will associate two indices related to α and β averages. In this section, we will show that every normalized block sequence $\{x_n\}_n$ has a further normalized block sequence $\{y_n\}_n$ such that on it both indices α and β are equal to zero. As we will show in the next section, this is sufficient for a sequence to have a subsequence generating a c_0 spreading model.

DEFINITION 3.1. Let $\{x_k\}_{k\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\mathrm{ISP}}$ that satisfies the following: for any $n \in \mathbb{N}$, for any very fast growing sequence $\{\alpha_q\}_{q\in\mathbb{N}}$ of α-averages in W and for any $\{F_k\}_{k\in\mathbb{N}}$ increasing sequence of subsets of the natural numbers, such that $\{\alpha_q\}_{q\in F_k}$ is $\mathcal{S}_n$-admissible, for any $\{x_{n_k}\}_{k\in\mathbb{N}}$ subsequence of $\{x_k\}_{k\in\mathbb{N}}$, we have that $\lim_k \sum_{q\in F_k} |\alpha_q(x_{n_k})| = 0$.
Then we say that the α-index of $\{x_k\}_{k\in\mathbb{N}}$ is zero and write $\alpha(\{x_k\}_k) = 0$. Otherwise, we write $\alpha(\{x_k\}_k) > 0$.

DEFINITION 3.2. Let $\{x_k\}_{k\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\mathrm{ISP}}$ that satisfies the following: for any $n \in \mathbb{N}$, for any very fast growing sequence $\{\beta_q\}_{q\in\mathbb{N}}$ of β-averages in W and for any $\{F_k\}_{k\in\mathbb{N}}$ increasing sequence of subsets of the natural numbers, such that $\{\beta_q\}_{q\in F_k}$ is $\mathcal{S}_n$-admissible, for any $\{x_{n_k}\}_{k\in\mathbb{N}}$ subsequence of $\{x_k\}_{k\in\mathbb{N}}$, we have that $\lim_k \sum_{q\in F_k} |\beta_q(x_{n_k})| = 0$.
Then we say that the β-index of $\{x_k\}_{k\in\mathbb{N}}$ is zero and write $\beta(\{x_k\}_k) = 0$. Otherwise, we write $\beta(\{x_k\}_k) > 0$.

PROPOSITION 3.3. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\mathrm{ISP}}$. Then the following assertions are equivalent.*

(i) $\alpha(\{x_k\}_k) = 0$.
(ii) *For any $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for any $j \geqslant j_0$ there exists $k_j \in \mathbb{N}$ such that for any $k \geqslant k_j$, and for any $\{\alpha_q\}_{q=1}^d$ $\mathcal{S}_j$-admissible and very fast growing sequence of α-averages such that $s(\alpha_q) > j_0$, for $q = 1, \dots, d$, we have that $\sum_{q=1}^d |\alpha_q(x_k)| < \varepsilon$.*

Proof. It is easy to prove that (i) follows from (ii), therefore we shall only prove the inverse. Suppose that (i) is true and (ii) is not.

Then there exists $\varepsilon > 0$ such that for any $j_0 \in \mathbb{N}$ there exists $j \geqslant j_0$, such that for any $k_0 \in \mathbb{N}$, there exists $k \geqslant k_0$ and $\{\alpha_q\}_{q=1}^d$ a $\mathcal{S}_j$-admissible and very fast growing sequence of α-averages with $s(\alpha_q) > j_0$, for $q = 1, \dots, d$, such that $\sum_{q=1}^d |\alpha_q(x_k)| \geqslant \varepsilon$.

We will inductively choose a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ and $\{\alpha^i\}_{i\in\mathbb{N}}$ a very fast growing sequence of α-averages, such that $|\alpha^i(x_{n_i})| > \varepsilon/2$, for any i. This evidently yields a contradiction.

For $j_0 = 1$, there exists $j_1 \geqslant 1$, such that there exists a subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$, a sequence $\{\alpha_q\}_{q\in\mathbb{N}}$ of α-averages with $s(\alpha_q) > 1$ for all $q \in \mathbb{N}$ and $\{F_j\}_{j\in\mathbb{N}}$ a sequence of increasing intervals of the naturals, such that the following holds.

(i) $\{\alpha_q\}_{q\in F_j}$ is very fast growing and $\mathcal{S}_{j_1}$-admissible.
(ii) $\sum_{q\in F_j} |\alpha_q(x_{k_j})| \geqslant \varepsilon$.
(iii) If $F_j' = F_j \setminus \{\min F_j\}$, then $\{\alpha_q\}_{q\in\bigcup_j F_j'}$ is very fast growing.

Since $\alpha(\{x_k\}_k) = 0$, we have that $\lim_j \sum_{q\in F'_j} |\alpha_q(x_{k_j})| = 0$. Choose j such that $|\alpha_{\min F_j}(x_{k_j})| > \varepsilon/2$ and set $n_1 = k_j, \alpha^1 = \alpha_{\min F_j}$.

Suppose that we have chosen $n_1 < \cdots < n_p$ and $\{a^i\}_{i=1}^p$ a very fast growing sequence of α-averages, such that $|\alpha^i(x_{n_i})| > \varepsilon/2$, for $i = 1, \ldots, p$.

Set $j_0 = \max\{s(\alpha^p),\ 2^{\max\operatorname{supp}\alpha^p}\}$ and repeat the first inductive step to find an α-average α with $s(\alpha) > j_0$ and $x_k > x_{n_p}$, $x_k > \alpha^p$, such that $|\alpha(x_k)| \geqslant \varepsilon/2$. Set $x_{n_{p+1}} = x_k$ and $\alpha^{p+1} = \alpha|_{\operatorname{ran} x_k}$. The inductive construction is complete and so is the proof. □

The proof of the next proposition is identical to the proof of the previous one.

Proposition 3.4. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\mathrm{ISP}}$. Then the following assertions are equivalent.*

(i) $\beta(\{x_k\}_k) = 0$.
(ii) *For any $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that for any $j \geqslant j_0$ there exists $k_j \in \mathbb{N}$ such that for any $k \geqslant k_j$, and for any $\{\beta_q\}_{q=1}^d$ $\mathcal{S}_j$-admissible and very fast growing sequence of β-averages such that $s(\beta_q) > j_0$, for $q = 1, \ldots, d$, we have that $\sum_{q=1}^d |\beta_q(x_k)| < \varepsilon$.*

Proposition 3.5. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a seminormalized block sequence in $\mathfrak{X}_{\mathrm{ISP}}$, such that either $\alpha(\{x_k\}_k) > 0$, or $\beta(\{x_k\}_k) > 0$.*

Then there exists $\theta > 0$ and a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$, that generates an ℓ_1^n spreading model, with a lower constant $\theta/2^n$, for all $n \in \mathbb{N}$.

More precisely, for every $n \in \mathbb{N}$ and $F \subset \mathbb{N}$ with $\{\min\operatorname{supp} x_{n_k} : k \in F\} \in \mathcal{S}_n$ and $\{c_k\}_k \subset \mathbb{R}$, we have that $\|\sum_{k\in F} c_k x_{n_k}\| \geqslant (\theta/2^n)\sum_{k\in F}|c_k|$.

In particular, for any $k_0, n \in \mathbb{N}$, there exists F a finite subset of $\mathbb{N}$ with $\min F \geqslant k_0$ and $\{c_k\}_{k\in F}$, such that $x = 2^n \sum_{k\in F} c_k x_{n_k}$ is a (C, θ, n) vector, where $C = \sup\{\|x_k\| : k \in \mathbb{N}\}$.

If moreover $\{x_k\}_k$ is a $(C', \{n_k\}_k)$ α-RIS, then x can be chosen to be a (C'', θ, n) exact vector, where $C'' = \max\{C, C'\}$.

Proof. Assume that $\alpha(\{x_k\}_k) > 0$. Then there exist $\ell \in \mathbb{N}, \varepsilon > 0, \{\alpha_q\}_{q\in\mathbb{N}}$ a very fast growing sequence of α-averages, $\{F_k\}_{k\in\mathbb{N}}$ increasing subsets of the naturals such that $\{\alpha_q\}_{q\in F_k}$ is $\mathcal{S}_\ell$-admissible for all $k \in \mathbb{N}$ and $\{x_{n_k}\}_{k\in\mathbb{N}}$ a subsequence of $\{x_k\}_{k\in\mathbb{N}}$, such that $\sum_{q\in F_k} |\alpha_q(x_{n_k})| > \varepsilon$, for all $k \in \mathbb{N}$. Pass, if necessary, to a subsequence, again denoted by $\{x_{n_k}\}_{k\in\mathbb{N}}$, generating some spreading model.

By changing the signs and restricting the ranges of the α_q, we may assume that $\sum_{q\in F_k} \alpha_q(x_{n_k}) > \varepsilon$, for all $k \in \mathbb{N}$ and $\operatorname{ran}\alpha_q \subset \operatorname{ran} x_{n_k}$ for any $q \in F_k$ and $k \in \mathbb{N}$. Set $\theta = \varepsilon/2^\ell$.

Let $k_0, n \in \mathbb{N}$ and choose $0 < \eta < 1/36C2^{3n}$. By Proposition 1.8, there exists F a finite subset of $\{n_k : k \geqslant \max\{k_0, 8C2^{2n}\}\}$ and $\{c_k\}_{k\in F}$, such that $x' = \sum_{k\in F} c_k x_{n_k}$ is a (n, η) s.c.c.

Set $f = (1/2^{\ell+n})\sum_{k\in F}\sum_{q\in F_{n_k}} \alpha_q$. Then f is a functional of type I_α in W and $f(x') > \varepsilon/2^{\ell+n} = \theta/2^n$. Therefore, $x = 2^n x'$ is the desired (C, θ, n) vector.

If moreover $\{x_k\}_k$ is a $(C', \{n_k\}_k)$ α-RIS, obvious modifications yield that x can be chosen to be a (C'', θ, n) exact vector.

Arguing in the same way, for any $n \in \mathbb{N}$, for any $F \subset \mathbb{N}$ with $\{\min\operatorname{supp} x_{n_k} : k \in F\} \in \mathcal{S}_n$ and $\{c_k\}_k \subset \mathbb{R}$, we have that $\|\sum_{k\in F} c_k x_{n_k}\| > (\theta/2^n)\sum_{k\in F}|c_k|$.

The proof is exactly the same if $\beta(\{x_k\}_k) > 0$. □

3.1. *Block sequences with α-index zero*

In this subsection we show that sequences $\{x_k\}_{k\in\mathbb{N}}$ with x_k a (C,θ,n_k) vector, with $\{n_k\}_k$ strictly increasing have α-index zero. We also prove that sequences with α-index zero have subsequences which are α-RIS.

PROPOSITION 3.6. *Let $\{x_k\}_k$ be a bounded block sequence in $\mathfrak{X}_{\mathrm{ISP}}$ with $\alpha(\{x_k\}_k)=0$. Then it has a subsequence that is a $(2C,\{n_k\}_k)$ α-RIS, where $C=\sup\{\|x_k\|:k\in\mathbb{N}\}$.*

Proof. Applying Proposition 3.3 we have the following. There exists $j_0\in\mathbb{N}$ such that for every $j\geqslant j_0$ there exists $k_j\in\mathbb{N}$ such that for every $k\geqslant k_j$ and $\{\alpha_q\}_{q=1}^d$ very fast growing and $\mathcal{S}_j$ admissible sequence of α-averages with $s(\alpha_q)>j_0$ for $q=1,\dots,d$, we have that $\sum_{q=1}^d|\alpha_q(x_k)|<C$.

We shall show that for every $j\geqslant j_0$ and $k_0\in\mathbb{N}$, there exists $k\geqslant k_0$ such that for every $f\in W$ of type I_α and $w(f)=n<j$, we have that $|f(x_k)|<2C/2^n$. If this is shown to be true, then by Remark 2.14 we are done.

Fix $j\geqslant j_0$ and $k_0\in\mathbb{N}$. Set $k=\max\{j_0,k_0,k_{j_0}\}$ and let $f\in W$ with $w(f)=n<j$. Then f is of the form $f=(1/2^n)\sum_{q=1}^d\alpha_q$, where $\{\alpha_q\}_{q=1}^d$ is a very fast growing and $\mathcal{S}_n$ admissible sequence of α-averages. We may clearly assume that $\operatorname{ran}\alpha_1\cap\operatorname{ran}x_k\neq\emptyset$. Then $\{\alpha_q\}_{q=2}^d$ is very fast growing with $s(\alpha_q)>\min\operatorname{supp}x_k\geqslant j_0$ and it is S_j admissible, as it is $\mathcal{S}_n$ admissible and $n<j$. We conclude the following.

$$|f(x_k)|\leqslant\frac{1}{2^n}\Big(|\alpha_1(x_k)|+\sum_{q=2}^d|\alpha_q(x_k)|\Big)<\frac{1}{2^n}(C+C).\qquad\square$$

LEMMA 3.7. *Let $x=2^n\sum_{k=1}^m c_kx_k$ be a (C,θ,n) vector in $\mathfrak{X}_{\mathrm{ISP}}$. Let also α be an α-average in W and set $G_\alpha=\{k:\operatorname{ran}\alpha\cap\operatorname{ran}x_k\neq\emptyset\}$. Then the following holds:*

$$|\alpha(x)|<\min\left\{\frac{C2^n}{s(\alpha)}\sum_{k\in G_\alpha}c_k,\ \frac{6C}{s(\alpha)}\sum_{k\in G_\alpha}c_k+\frac{1}{3\cdot2^{2n}}\right\}+2C2^n\max\{c_k:k\in G_\alpha\}.$$

Proof. If $\alpha=(1/p)\sum_{j=1}^d f_j$, set

$$\begin{aligned}E_1&=\{k\in G_\alpha:\ \text{there exists at most one } j \text{ with } \operatorname{ran}f_j\cap\operatorname{ran}x_k\neq\emptyset\},\\E_2&=\{1,\dots,m\}\setminus E_1,\\J_k&=\{j:\operatorname{ran}f_j\cap\operatorname{ran}x_k\neq\emptyset\}\quad\text{for } k\in E_2.\end{aligned}$$

Then it is easy to see that

$$\left|\alpha\left(\sum_{k\in E_1}c_kx_k\right)\right|\leqslant\frac{C}{p}\sum_{k\in G_\alpha}c_k. \tag{2}$$

Moreover,

$$\left|\alpha\left(\sum_{k\in E_2}c_kx_k\right)\right|<2C\max\{c_k:k\in G_\alpha\}. \tag{3}$$

To see this, note that

$$\left|\alpha\left(\sum_{k\in E_2}c_kx_k\right)\right|\leqslant\frac{1}{p}\sum_{k\in E_2}c_k\left(\sum_{j\in J_k}|f_j(x_k)|\right)<\max\{c_k:k\in G_\alpha\}\frac{2Cp}{p}.$$

Set $J = \{j : \text{there exists } k \in E_1 \text{ such that } \operatorname{ran} f_j \cap \operatorname{ran} x_k \neq \emptyset\}$ and for $j \in J$ set $G_j = \{k \in E_1 : \operatorname{ran} f_j \cap \operatorname{ran} x_k \neq \emptyset\}$. Then the G_j are pairwise disjoint and $\bigcup_{j\in J} G_j = E_1$.

For $j \in J$, Corollary 2.9 yields that

$$\left| f_j\left(\sum_{k\in G_j} c_k x_k\right)\right| \leqslant \frac{6C}{2^n}\sum_{k\in G_j} c_k + \frac{1}{3\cdot 2^{3n}}.$$

Therefore,

$$\left|\alpha\left(\sum_{k\in E_1} c_k x_k\right)\right| \leqslant \frac{1}{p}\sum_{j\in J}\left| f_j\left(\sum_{k\in G_j} c_k x_k\right)\right| \leqslant \frac{6C}{2^n p}\sum_{k\in G_\alpha} c_k + \frac{1}{3\cdot 2^{3n}}. \tag{4}$$

Then (2) and (4) yield the following.

$$\left|\alpha\left(\sum_{k\in E_1} c_k x_k\right)\right| \leqslant \min\left\{\frac{C}{s(\alpha)}\sum_{k\in G_\alpha} c_k,\ \frac{6C}{2^n s(\alpha)}\sum_{k\in G_\alpha} c_k + \frac{1}{3\cdot 2^{3n}}\right\}. \tag{5}$$

By summing up (3) and (5) the result follows. □

LEMMA 3.8. *Let x be a (C,θ,n) vector in $\mathfrak{X}_{\mathrm{ISP}}$. Let also $\{a_q\}_{q=1}^d$ be a very fast growing and $\mathcal{S}_j$-admissible sequence of α-averages, with $j < n$. Then the following holds*:

$$\sum_{q=1}^{d} |\alpha_q(x)| < \frac{6C}{s(\alpha_1)} + \frac{1}{2^n}.$$

Proof. Assume that $x = 2^n \sum_{k=1}^m c_k x_k$ is such that the assumptions of Definition 2.15 are satisfied. Set $q_1 = \min\{q : \operatorname{ran}\alpha_q \cap \operatorname{ran} x \neq \emptyset\}$. For convenience assume that $q_1 = 1$. Then by Lemma 3.7 we have that

$$|\alpha_1(x)| < \frac{6C}{s(\alpha_1)} + \frac{7}{18\cdot 2^{2n}}. \tag{6}$$

Set

$$\begin{aligned} J_1 &= \{q > 1 :\ \text{there exists at most one } k \text{ such that } \operatorname{ran}\alpha_q \cap \operatorname{ran} x_k \neq \emptyset\},\\ J_2 &= \{q > 1 : q \notin J_1\},\\ G^q &= \{k : \operatorname{ran}\alpha_q \cap \operatorname{ran} x_k \neq \emptyset\} \quad \text{for } q > 1,\\ G_1 &= \{k :\ \text{there exists } q \in J_1 \text{ with } \operatorname{ran}\alpha_q \cap \operatorname{ran} x_k \neq \emptyset\}. \end{aligned}$$

Then $\{\min\operatorname{supp} x_k : k \in G_1 \setminus \{\min G_1\}\} \in \mathcal{S}_j$, hence $\sum_{k\in G_1} c_k < \frac{1}{18C2^{3n}}$.

It is easy to check that

$$\sum_{q\in J_1} |\alpha_q(x)| \leqslant 2^j C 2^n \left\|\sum_{k\in G_1} c_k x_k\right\| < 2^{n-1} C 2^n \frac{1}{18C2^{3n}} = \frac{1}{36\cdot 2^n}. \tag{7}$$

For $q \in J_2$, Lemma 3.7 yields that

$$\begin{aligned} |\alpha_q(x)| &< \frac{C2^n}{s(\alpha_q)}\sum_{k\in G^q} c_k + 2C2^n \max\{c_k : k \in G^q\}\\ &< \frac{C2^n}{\min\operatorname{supp} x}\sum_{k\in G^q} c_k + 2C2^n c_{k_q}, \end{aligned}$$

where $k_q \in G^q$, such that $c_{k_q} = \max\{c_k : k \in G^q\}$.

Then $\{\min\operatorname{supp} x_{k_q} : q \in J_2 \setminus \{\min J_2\}\} \in \mathcal{S}_j$. By the above, we conclude that

$$\sum_{q\in J_2} |\alpha_q(x)| < \frac{2C2^n}{\min\operatorname{supp} x} + \frac{8}{36\cdot 2^{2n}} < \frac{1}{4\cdot 2^n} + \frac{8}{36\cdot 2^{2n}}. \tag{8}$$

Summing up (6)–(8), the desired result follows. □

PROPOSITION 3.9. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a block sequence of (C,θ,n_k) vectors in $\mathfrak{X}_{\mathrm{ISP}}$ with $\{n_k\}_k$ strictly increasing. Then $\alpha(\{x_k\}_k) = 0$.*

Proof. We shall make use of Proposition 3.3. Let $\varepsilon > 0$ and choose $j_0 \in \mathbb{N}$ such that $6C/j_0 < \varepsilon/2$. For $j \geqslant j_0$, choose k_j, such that $1/2^{n_{k_j}} < \varepsilon/2$. For $k \geqslant k_j$, Lemma 3.8 yields that if $\{\alpha_q\}_{q=1}^d$ is a very fast growing and $\mathcal{S}_j$-admissible sequence of α-averages and $s(\alpha_q) > j_0$, for $q = 1,\dots,d$, we have that

$$\sum_{q=1}^{d} |\alpha_q(x_k)| < \frac{6C}{j_0} + \frac{1}{2^{n_k}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

PROPOSITION 3.10. *Let x be a (C,θ,n) vector in $\mathfrak{X}_{\mathrm{ISP}}$. Then for any $f \in W$ functional of type I_α, such that $w(f) = j < n$, we have that $|f(x)| < 7C/2^j$.*

Proof. Let $f = (1/2^j)\sum_{q=1}^d \alpha_j$ be a functional of type I_α with weight $w(f) = j < n$. Then Lemma 3.8 yields that

$$|f(x)| \leqslant \frac{1}{2^j}\left(\sum_{q=1}^{d} |\alpha_q(x)|\right) < \frac{1}{2^j}\left(\frac{6C}{s(\alpha_1)} + \frac{1}{2^n}\right) \leqslant \frac{7C}{2^j}. \qquad \square$$

The proposition below follows immediately from Proposition 3.10 and Remark 2.14.

PROPOSITION 3.11. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a block sequence of (C,θ,n_k) vectors in $\mathfrak{X}_{\mathrm{ISP}}$ with $\{n_k\}_k$ strictly increasing. Then some subsequence of $\{x_k\}_{k\in\mathbb{N}}$ is a $(7C,\{n_k\}_k)$ α-RIS.*

3.2. *Block sequences with β-index zero*

In this subsection, we first prove that every block sequence of (C,θ,n_k) exact vectors with $\{n_k\}_k$ strictly increasing, has β-index zero. This yields that every block sequence has a further block sequence with both α and β indices equal to zero. We start with the following technical lemma. Its meaning becomes more transparent in the following Corollary 3.13 and Lemmas 3.14 and 3.15.

NOTATION. Let $x = 2^n\sum_{k=1}^m c_k x_k$ be a (C,θ,n) exact vector, with $\{x_k\}_{k=1}^m$ a $(C,\{n_k\}_{k=1}^m)$ α-RIS. Let also $f = \frac{1}{2}\sum_{j=1}^d f_j$ be a type II functional. Set

$$I_0 = \{j : n \leqslant w(f_j) < 2^{2n}\},$$
$$I_1 = \{j : w(f_j) < n\},$$
$$I_2 = \{j : 2^{2n} \leqslant w(f_j) < n_1\},$$
$$J_k = \{j : n_k \leqslant w(f_j) < n_{k+1}\} \quad \text{for } k < m \quad \text{and} \quad J_m = \{j : n_m \leqslant w(f_j)\}.$$

Under the above notation the following lemma holds.

LEMMA 3.12. *Let $x = 2^n \sum_{k=1}^m c_k x_k$ be a (C,θ,n) exact vector in $\mathfrak{X}_{\mathrm{ISP}}, n \geqslant 2$, with $\{x_k\}_{k=1}^m$ a $(C,\{n_k\}_{k=1}^m)$ α-RIS. Let also $f = \frac{1}{2}\sum_{j=1}^d f_j$ be a functional of type II.*

Then there exists $F_f \subset \{k : \operatorname{ran} f \cap \operatorname{ran} x_k \neq \emptyset\}$ with $\{\min\operatorname{supp} x_k : k \in F_f\} \in \mathcal{S}_2$ such that

$$|f(x)| < 7C\#I_0 + \frac{C}{2}\left(\sum_{k=2}^m \sum_{j\in J_k} \frac{2^{n_k}}{2^{w(f_j)+n_{k-1}}} + \sum_{k=1}^{m-1}\sum_{j\in J_k}\frac{2^n}{2^{w(f_j)}} + \sum_{j\in I_1}\frac{7}{2^{w(f_j)}} + \sum_{j\in I_2}\frac{2^n}{2^{w(f_j)}}\right) + C2^n\sum_{k\in F_f} c_k.$$

Proof. Note that $\{J_k\}_{k=1}^m$ are disjoint intervals of $\{1,\dots,d\}$ and that $g_k = \frac{1}{2}\sum_{j\in J_k} f_j \in W$, for $k = 1,\dots,m$.

Set $F_f = \{k : \operatorname{ran} g_k \cap \operatorname{ran} x_k \neq \emptyset\}$. It easily follows that $\{\min\operatorname{supp} x_k : k \in F_f\} \in \mathcal{S}_2$ and that

$$\frac{2^n}{2}\sum_{k=1}^m \left|\sum_{j\in J_k} f_j(c_k x_k)\right| \leqslant C2^n \sum_{k\in F_f} c_k. \tag{9}$$

Let $k_0 \leqslant m, j \in J_{k_0}$. Then

$$2^n\left|f_j\left(\sum_{k<k_0} c_k x_k\right)\right| < C\frac{2^{n_{k_0}}}{2^{w(f_j)+n_{k_0-1}}} \quad\text{and}\quad 2^n\left|f_j\left(\sum_{k>k_0} c_k x_k\right)\right| < C\frac{2^n}{2^{w(f_j)}}. \tag{10}$$

Proposition 3.10 yields that for $j \in I_1$ we have that $|f_j(x)| < 7C/2^{w(f_j)}$ and hence

$$\frac{1}{2}\sum_{j\in I_1}|f_j(x)| < \frac{C}{2}\sum_{j\in I_1}\frac{7}{2^{w(f_j)}}. \tag{11}$$

For $j \in I_2$, we have that $|f_j(x)| < C2^n/2^{w(f_j)}$ and therefore

$$\frac{1}{2}\sum_{j\in I_2}|f_j(x)| < \frac{C}{2}\sum_{j\in I_2}\frac{2^n}{2^{w(f_j)}}. \tag{12}$$

By Remark 2.16 yields that $\|x\| < 7C$, and since I_0 is an interval, it follows that $\frac{1}{2}\sum_{j\in I_0} f_j \in W$. Therefore,

$$\frac{1}{2}\left|\sum_{j\in I_0} f_j(x)\right| < 7C. \tag{13}$$

Summing up (9)–(13) the desired result follows. □

The next corollary will be useful in the next sections, when we define the notion of dependent sequences.

COROLLARY 3.13. *Let x be a (C,θ,n) exact vector in $\mathfrak{X}_{\mathrm{ISP}}, n \geqslant 3$. Let also $f = \frac{1}{2}\sum_{j=1}^d f_j$ be a functional of type II. with $\hat{w}(f) \cap \{n,\dots,2^{2n}\} = \emptyset$. Then*

$$|f(x)| < \frac{C}{2^n} + \frac{C}{2^{2n}} + \sum_{\{j:\ w(f_j)<n\}} \frac{4C}{2^{w(f_j)}}.$$

Proof. Let $x = 2^n \sum_{k=1}^m c_k x_k$ with $\{x_k\}_{k=1}^m$ a $(C, \{n_k\}_{k=1}^m)$ α-RIS. Apply Lemma 3.12. Then the following holds:

$$|f(x)| < \frac{C}{2}\left(\sum_{k=2}^{m}\sum_{j\in J_k}\frac{2^{n_k}}{2^{w(f_j)+n_{k-1}}} + \sum_{k=1}^{m-1}\sum_{j\in J_k}\frac{2^n}{2^{w(f_j)}} + \sum_{j\in I_1}\frac{7}{2^{w(f_j)}} + \sum_{j\in I_2}\frac{2^n}{2^{w(f_j)}}\right) + \frac{1}{36\cdot 2^{2n}} \tag{14}$$

Note the following:

$$\sum_{k=2}^{m}\sum_{j\in J_k}\frac{2^{n_k}}{2^{w(f_j)+n_{k-1}}} \leqslant \frac{1}{2^{n_1}} < \frac{1}{2^{2n}}, \tag{15}$$

$$\sum_{j\in I_2}\frac{2^n}{2^{w(f_j)}} + \sum_{k=1}^{m-1}\sum_{j\in J_k}\frac{2^n}{2^{w(f_j)}} = 2^n\left(\sum_{\{j:w(f_j)\geqslant 2^{2n}\}}\frac{1}{2^{w(f_j)}}\right) \leqslant \frac{2}{2^n}. \tag{16}$$

Applying (15) and (16) to (14) the result follows. □

Lemma 3.14. *Let $x = 2^n \sum_{k=1}^m c_k x_k$ be a (C, θ, n) exact vector in $\mathfrak{X}_{\rm ISP}$, $n \geqslant 2$, with $\{x_k\}_{k=1}^m$ a $(C, \{n_k\}_{k=1}^m)$ α-RIS. Let also β be a β-average. Then there exists $F_\beta \subset \{k : \operatorname{ran}\beta \cap \operatorname{ran} x_k \neq \emptyset\}$ with $\{\min\operatorname{supp} x_k : k \in F_\beta\} \in \mathcal{S}_2$ such that*

$$|\beta(x)| < \frac{8C}{s(\beta)} + C2^n\sum_{k\in F_\beta} c_k.$$

Proof. If $\beta = (1/p)\sum_{q=1}^d g_q$, then by definition the g_q are functionals of type II with disjoint weights $\hat{w}(g_q)$.

For convenience, we may write $g_q = \frac{1}{2}\sum_{j\in G_q} f_j$, where the index sets $G_q, q = 1, \ldots, d$ are pairwise disjoint. Note that for $j_1, j_2 \in G, j_1 \neq j_2$ we have that $w(f_{j_1}) \neq w(f_{j_2})$.

By slightly modifying the previously used notation, set $G = \bigcup_{q=1}^d G_q$ and

$$\begin{aligned}
I_0 &= \{j \in G : n \leqslant w(f_j) < 2^{2n}\},\\
I_1 &= \{j \in G : w(f_j) < n\},\\
I_2 &= \{j \in G : 2^{2n} \leqslant w(f_j) < n_1\},\\
J_k &= \{j \in G : n_k \leqslant w(f_j) < n_{k+1}\} \quad \text{for } k < m \quad \text{and}\\
J_m &= \{j \in G : n_m \leqslant w(f_j)\}.
\end{aligned}$$

By Remark 1.1, there exists at most one $q_0 \leqslant d$, with $\hat{w}(f_{q_0}) \cap \{n, \ldots, 2^{2n}\} \neq \emptyset$ and if such a q_0 exists, then $\#\hat{w}(f_{q_0}) \cap \{n, \ldots, 2^{2n}\} \leqslant 1$.

Apply Lemma 3.12. Then for $q = 1, \ldots, d$ there exists $F_q \subset \{x_k : \operatorname{ran}\beta \cap \operatorname{ran} x_k \neq \emptyset\}$ with $\{\min\operatorname{supp} x_k : k \in F_q\} \in \mathcal{S}_2$ such that

$$\begin{aligned}
|\beta(x)| < \frac{7C}{p} + \frac{C}{2p}&\left(\sum_{k=2}^{m}\sum_{j\in J_k}\frac{2^{n_k}}{2^{w(f_j)+n_{k-1}}} + \sum_{k=1}^{m-1}\sum_{j\in J_k}\frac{2^n}{2^{w(f_j)}} + \sum_{j\in I_1}\frac{7}{2^{w(f_j)}} + \sum_{j\in I_2}\frac{2^n}{2^{w(f_j)}}\right)\\
&+ \frac{1}{p}C2^n\sum_{q=1}^{d}\sum_{k\in F_q} c_k.
\end{aligned} \tag{17}$$

Just as in the proof of Corollary 3.13, note the following:

$$\sum_{k=2}^{m}\sum_{j\in J_k}\frac{2^{n_k}}{2^{w(f_j)+n_{k-1}}}<\frac{1}{2^{2n}},\tag{18}$$

$$\sum_{j\in I_2}\frac{2^n}{2^{w(f_j)}}+\sum_{k=1}^{m-1}\sum_{j\in J_k}\frac{2^n}{2^{w(f_j)}}\leqslant\frac{2}{2^n}.\tag{19}$$

By the definition of the coding function σ, we obtain

$$\sum_{j\in I_1}\frac{7}{2^{w(f_j)}}<\frac{7}{1000},\tag{20}$$

$$\frac{1}{p}C2^n\sum_{q=1}^{d}\sum_{k\in F_q}c_k\leqslant C2^n\max\left\{\sum_{k\in F_q}c_k:q=1,\dots,p\right\}=C2^n\sum_{k\in F_{q_0}}c_k\tag{21}$$

for some $1\leqslant q_0\leqslant d$.

Set $F_\beta=F_{q_0}$ and apply (18)–(21) to (17) to derive the desired result. □

LEMMA 3.15. *Let x be a (C,θ,n) exact vector in $\mathfrak{X}_{\mathrm{ISP}}, n\geqslant 4$. Let also $\{\beta_q\}_{q=1}^d$ be a very fast growing and $\mathcal{S}_j$-admissible sequence of β-averages with $j\leqslant n-3$. Then we have that*

$$\sum_{q=1}^{d}|\beta_q(x)|<\sum_{q=1}^{d}\frac{8C}{s(\beta_q)}+\frac{1}{2^n}.$$

Proof. Set

$$\begin{aligned}J_1&=\{q:\ \text{there exists at most one } k \text{ such that } \operatorname{ran}\beta_q\cap\operatorname{ran}x_k\neq\emptyset\},\\ J_2&=\{1,\dots,d\}\setminus J_2,\\ G_1&=\{k:\ \text{there exists } q\in J_1 \text{ with } \operatorname{ran}\beta_q\cap\operatorname{ran}x_k\neq\emptyset\}.\end{aligned}$$

Then $\{\min\operatorname{supp}x_k:k\in G_1\setminus\{\min G_1\}\}\in\mathcal{S}_{j+1}$ and it is easy to check that

$$\sum_{q\in J_1}^{d}|\beta_q(x)|\leqslant 2^j2^n\left\|\sum_{k\in G_1}c_kx_k\right\|<\frac{2}{9\cdot 2^n}.\tag{22}$$

For $q\in J_2$, choose $F_q\subset\{1,\dots,m\}$ as in Lemma 3.14 and set $F=\bigcup_{q\in J_2}F_q$. Then $\{\min\operatorname{supp}x_k:k\in F\setminus\{\min F\}\}\in\mathcal{S}_{n-1}$, therefore $\sum_{q\in J_2}\sum_{k\in F_q}c_k<1/9C2^{3n}$.

Lemma 3.14 yields that

$$\sum_{q\in J_2}|\beta_q(x)|<\sum_{q\in J_2}\frac{8C}{s(\beta_q)}+\frac{1}{9\cdot 2^{2n}}.\tag{23}$$

Combining (22) and (23), the result follows. □

PROPOSITION 3.16. *Let $\{x_k\}_k$ be a block sequence of (C,θ,n_k) exact vectors in $\mathfrak{X}_{\mathrm{ISP}}$ with $\{n_k\}_k$ strictly increasing. Then $\alpha(\{x_k\}_k)=0$ as well as $\beta(\{x_k\}_k)=0$.*

Proof. Proposition 3.9 yields that $\alpha(\{x_k\}_k)=0$. To prove that $\beta(\{x_k\}_k)=0$, we shall make use of Proposition 3.4. Let $\varepsilon>0$ and choose $j_0\in\mathbb{N}$ such that

$$\frac{8C}{j_0}<\frac{\varepsilon}{4}.$$

For $j \geqslant j_0$ choose k_j, such that $n_{k_j} \geqslant j+3$ and $(1/2^{n_{k_j}}) < \varepsilon/4$. For $k \geqslant k_j$, Lemma 3.15 yields that if $\{\beta_q\}_{q=1}^d$ is a very fast growing and $\mathcal{S}_j$-admissible sequence of β-averages and $s(\beta_q) > j_0$, for $q = 1, \dots, d$, we have that

$$\sum_{q=1}^{d} |\beta_q(x_k)| < \sum_{q=1} \frac{8C}{s(\beta_q)} + \frac{1}{2^{n_k}} < \frac{\varepsilon}{4} + \sum_{j > \min\operatorname{supp} x_k} \frac{8C}{2^j} + \frac{\varepsilon}{4} < \varepsilon. \qquad \square$$

Corollary 3.17. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}_{\mathrm{ISP}}$. Then there exists a further normalized block sequence $\{y_k\}_{k\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$, such that $\alpha(\{y_k\}_k) = 0$ as well as $\beta(\{y_k\}_k) = 0$.*

Proof. If $\alpha(\{x_k\}_k) = 0$ and $\beta(\{x_k\}_k) = 0$, then there is nothing to prove. Otherwise, if $\alpha(\{x_k\}_k) > 0$ or $\beta(\{x_k\}_k) > 0$, apply Proposition 3.5 to construct a block sequence $\{z_k\}_{k\in\mathbb{N}}$ of $(1, \theta, n_k)$ vectors, with $\{n_k\}_k$ strictly increasing. Then by Proposition 3.9 $\alpha(\{z_k\}_k) = 0$ and Proposition 3.11 yields, that passing, if necessary, to a subsequence, we have that $\{z_k\}_{k\in\mathbb{N}}$ is a $(7, \{n_k\}_k)$ α-RIS.

If $\beta(\{z_k\}_k) = 0$, set $y_k = 1/\|z_k\| z_k$ and $\{y_k\}_{k\in\mathbb{N}}$ is the desired sequence.

Otherwise, if $\beta(\{z_k\}_k) > 0$, apply once more Proposition 3.5 to construct a block sequence $\{w_k\}_{k\in\mathbb{N}}$, of $(7, \theta', m_k)$ exact vectors, with $\{m_k\}_k$ strictly increasing. Proposition 3.16 yields that $\alpha(\{w_k\}_k) = 0$, as well as $\beta(\{w_k\}_k) = 0$. Set $y_k = 1/\|w_k\| w_k$ and $\{y_k\}_{k\in\mathbb{N}}$ is the desired sequence. $\square$

4. c_0 *spreading models*

This section is devoted to necessary conditions for a sequence $\{x_k\}_k$ to generate a c_0 spreading model. At the beginning a Ramsey-type result is proved concerning type II functionals acting on a block sequence $\{x_k\}_k$ with $\beta(\{x_k\}_k) = 0$. Then conditions are provided for a finite sequence to be equivalent to the basis of ℓ_∞^n. This is critical for establishing the HI property and the properties of the operators in the space. Moreover, it is shown that any block sequence $\{x_k\}_k$ with $\alpha(\{x_k\}_k) = 0$ and $\beta(\{x_k\}_k) = 0$ contains a subsequence generating a c_0 spreading model. Another critical property related to sequences generating c_0 spreading models is that increasing Schreier sums of them define α-RIS sequences.

4.1. *Evaluation of type II functionals on $\{x_k\}_k$ with $\beta(\{x_k\}_k) = 0$*

Definition 4.1. Let $x_1 < x_2 < x_3$ be vectors in $\mathfrak{X}_{\mathrm{ISP}}$, $f = \frac{1}{2}\sum_{j=1}^d f_j$ be a functional of type II, such that $\operatorname{supp} f \cap \operatorname{ran} x_i \neq \emptyset$, for $i = 1, 2, 3$ and $j_0 = \min\{j : \operatorname{ran} f_j \cap \operatorname{ran} x_2 \neq \emptyset\}$. If $\operatorname{ran} f_{j_0} \cap \operatorname{ran} x_3 = \emptyset$, then we say that f separates x_1, x_2 and x_3.

Definition 4.2. Let $i, j \in \mathbb{N}$. If there exists $f \in W$ a functional of type II, such that $i, j \in \hat{w}(f)$, then we say that i is compatible with j.

Lemma 4.3. *Let $x_1 < x_2 < \cdots < x_m$ be vectors in $\mathfrak{X}_{\mathrm{ISP}}$, such that there exist $\varepsilon > 0$ and $\{f_k\}_{k=2}^{m-1}$ functionals of type II satisfying the following conditions.*

(i) *The functional f_k separates x_1, x_k, x_m, for $k = 2, \dots, m-1$.*

(ii) *If $f_k = \frac{1}{2}\sum_{j=1}^{d_k} f_j^k$ and $j_k = \min\{j : \operatorname{ran} f_j^k \cap \operatorname{ran} x_k \neq \emptyset\}$, then $w(f_{j_k}^k)$ is not compatible with $w(f_{j_\ell}^\ell)$ for $k \neq \ell$.*

(iii) *We have that $|f_k(x_m)| > \varepsilon$ for $k = 2, \dots, m-1$.*

Then there exists a β-average β in W of size $s(\beta)=m-2$ such that $\beta(x_m)>\varepsilon$.

Proof. Set $g_k=\operatorname{sgn}(f_k(x_m))f_k|_{\operatorname{ran}x_m}$, for $k=2,\dots,m-1$. Then g_k is a functional of type II in W. We will show that the g_k have disjoint weights $\hat{w}(g_k)$.

Indeed, if this is not the case then there exist $2\leqslant k<\ell\leqslant m-1$ and $i\in\hat{w}(g_k)\cap\hat{w}(g_\ell)$. By (i) and the fact that for any special sequence $\{h_i\}_{i=1}^p$, the weight of h_p uniquely determines the sequence $\{h_i\}_{i=1}^{p-1}$, it follows that $f_k|_{[\min\operatorname{supp}x_2,\dots,\max\operatorname{supp}f^\ell_{j_\ell}]}=\epsilon f_\ell|_{[\min\operatorname{supp}x_2,\dots,\max\operatorname{supp}f^\ell_{j_\ell}]}$, with $\epsilon\in\{-1,1\}$. This contradicts (ii).

By the above and (iii), it follows that if we set $\beta=(1/(m-2))\sum_{k=2}^{m-1}g_k$, then β is the desired β-average. $\square$

LEMMA 4.4. *Let $x_1<x_2<\cdots<x_m$ be vectors in $\mathfrak{X}_{\mathrm{ISP}}$, such that there exist $\varepsilon>0$ and $\{f_k\}_{k=2}^{m-1}$ functionals of type II satisfying the following conditions.*

(i) *f_k separates x_1,x_k,x_m, for $k=2,\dots,m-1$.*
(ii) *If $f_k=\frac{1}{2}\sum_{j=1}^{d_k}f^k_j$ and $j_k=\min\{j:\operatorname{ran}f^k_j\cap\operatorname{ran}x_k\neq\emptyset\}$, then $w(f^k_{j_k})=w(f^\ell_{j_\ell})$ for $k\neq\ell$.*
(iii) *If $j'_k=\min\{j:\operatorname{ran}f^k_j\cap\operatorname{ran}x_m\neq\emptyset\}$, then $w(f^k_{j'_k})\neq w(f^\ell_{j'_\ell})$ for $k\neq\ell$.*
(iv) *$|f_k(x_m)|>\varepsilon$ for $k=2,\dots,m-1$.*

Then there exists a β-average β in W of size $s(\beta)=m-2$ such that $\beta(x_m)>\varepsilon$.

Proof. As before, set $g_k=\operatorname{sgn}(f_k(x_m))f_k|_{\operatorname{ran}x_m}$, for $k=2,\dots,m-1$. Then g_k is a functional of type II in W. We will show that the g_k have disjoint weights $\hat{w}(g_k)$.

Suppose that there exist $2\leqslant k<\ell\leqslant m-1$ and $i\in\hat{w}(g_k)\cap\hat{w}(g_\ell)$. By (i), (ii) and the fact that for any special sequence $\{h_i\}_{i=1}^p$, the weight of h_p uniquely determines the sequence $\{h_i\}_{i=1}^{p-1}$, it follows that

$$f_k|_{[\min\operatorname{supp}x_2,\dots,\min\operatorname{supp}x_m]}=\epsilon f_\ell|_{[\min\operatorname{supp}x_2,\dots,\min\operatorname{supp}x_m]}$$

with $\epsilon\in\{-1,1\}$. This leaves us no choice, but to conclude that $w(f^k_{j'_k})=w(f^\ell_{j'_\ell})$, which contradicts (iii).

By (iv), it follows that if we set $\beta=(1/(m-2))\sum_{k=2}^{m-1}g_k$, then β is the desired β-average. $\square$

PROPOSITION 4.5. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a bounded block sequence in $\mathfrak{X}_{\mathrm{ISP}}$, such that $\beta(\{x_k\}_k)=0$. Then for any $\varepsilon>0$, there exists M an infinite subset of the naturals, such that for any $k_1<k_2<k_3\in M$, for any functional $f\in W$ of type II that separates x_{k_1},x_{k_2},x_{k_3}, we have that $|f(x_{k_i})|<\varepsilon$, for some $i\in\{1,2,3\}$.*

Proof. Suppose that this is not the case. Then by using Ramsey theorem [**27**], we may assume that there exists $\varepsilon>0$ such that for any $k<\ell<m\in\mathbb{N}$, we have that there exists $f_{k,\ell,m}$ a functional of type II, that separates x_k,x_ℓ,x_m and $|f_{k,\ell,m}(x_k)|>\varepsilon,|f_{k,\ell,m}(x_\ell)|>\varepsilon$ and $|f_{k,\ell,m}(x_m)|>\varepsilon$.

For $1<k<m$, if $f_{1,k,m}=\frac{1}{2}\sum_{j=1}^{d_{k,m}}f^{k,m}_j$, set

$$\begin{aligned}i_{k,m}&=\min\{j:\operatorname{ran}f^{k,m}_j\cap\operatorname{ran}x_1\neq\emptyset\},\\ j_{k,m}&=\min\{j:\operatorname{ran}f^{k,m}_j\cap\operatorname{ran}x_k\neq\emptyset\},\\ j'_{k,m}&=\min\{j:\operatorname{ran}f^{k,m}_j\cap\operatorname{ran}x_m\neq\emptyset\}.\end{aligned}$$

Note that for $1 < k < m$, since $|f_{1,k,m}(x_1)| > \varepsilon$, it follows that

$$\frac{1}{2^{w(f^{k,m}_{i_{k,m}})}} > \frac{\varepsilon}{\|x_1\| \max\operatorname{supp} x_1}.$$

By applying Ramsey theorem once more, we may assume that there exists $n_1 \in \mathbb{N}$, such that for any $1 < k < m$, we have that $w(f^{k,m}_{i_{k,m}}) = n_1$.

Arguing in the same way and diagonalizing, we may assume that for any $k > 1$, there exists $n_k \in \mathbb{N}$ such that for any $m > k$, we have that $w(f^{k,m}_{j_{k,m}}) = n_k$. Set

$$\begin{aligned} A_1 &= \{\{k,\ell\} \in [\mathbb{N} \setminus \{1\}]^2 : n_k \neq n_\ell \text{ and } n_k \text{ is compatible with } n_\ell\},\\ A_2 &= \{\{k,\ell\} \in [\mathbb{N} \setminus \{1\}]^2 : n_k \neq n_\ell \text{ and } n_k \text{ is not compatible with } n_\ell\},\\ A_3 &= \{\{k,\ell\} \in [\mathbb{N} \setminus \{1\}]^2 : n_k = n_\ell\}. \end{aligned}$$

Once more, Ramsey theorem yields that there exists M an infinite subset of the naturals, such that $[M]^2 \subset A_1, [M]^2 \subset A_2$ or $[M]^2 \subset A_3$.

Assume that $[M]^2 \subset A_1$ and for convenience assume that $M = \mathbb{N} \setminus \{1\}$. Choose $k_0 > 1$ such that $k_0 > \max\operatorname{supp} x_1$. Since n_1 is compatible with n_2 and in general n_{k-1} is compatible with n_k, for $k > 1$, it follows that there exists a functional $f = \frac{1}{2}\sum_{j=1}^{d} f_j$ of type II in W, such that $\operatorname{ran} f \cap \operatorname{ran} x_1 \neq \emptyset$ and for $k = 1, \dots, k_0$ there exists j_k, with $w(f_{j_k}) = n_k$, for $k = 1, \dots, k_0$.

Since $\min\operatorname{supp} f_1 \leqslant \max\operatorname{supp} x_1$ it follows that $\{f_j\}_{j=1}^d$ cannot be $\mathcal{S}_1$-admissible, a contradiction.

Assume next that $[M]^2 \subset A_2$. Lemma 4.3 yields that $\beta(\{x_k\}_k) > 0$ and since this cannot be, we conclude that $[M]^2 \subset A_3$, therefore there exists $n_0 \in \mathbb{N}$, such that $n_k = n_0$, for any $k \in M$.

Assume once more that $M = \mathbb{N} \setminus \{1\}$ and set

$$B = \{\{k,\ell,m\} \in [\mathbb{N} \setminus \{1\}]^3 : w(f^{1,k,m}_{j'_{k,m}}) = w(f^{1,\ell,m}_{j'_{\ell,m}})\}.$$

If there exists M an infinite subset of the naturals, such that $[M]^3 \subset B^c$, Lemma 4.4 yields that $\beta(\{x_k\}_k) > 0$, therefore by one last Ramsey argument, there exists M an infinite subset of the naturals, such that $[M]^3 \subset B$.

By the above, we conclude that for $m \geqslant 4$, $\operatorname{ran} x_k \subset \operatorname{ran} f^{2,m}_{j_{2,m}}$ and $|f^{2,m}_{j_{2,m}}(x_k)| > 2\varepsilon$, for $k = 2, \dots, m-2$.

Set $f_m = f^{2,m}_{j_{2,m}}$ and let f be a w^* limit of some subsequence of $\{f_m\}_{m\in\mathbb{N}}$. Then $|f(x_k)| \geqslant 2\varepsilon$, for any $k \geqslant 2$. Corollary 2.10 yields a contradiction and this completes the proof. □

Remark 4.6. The proof of Proposition 4.5 is the only place where the condition $\beta(\{x_k\}_k) = 0$ is needed. This makes necessary to introduce the β-averages and their use in the definition of the norm.

4.2. *Finite sequences equivalent to ℓ^n_∞ basis*

Proposition 4.7. *Let $x_1 < \dots < x_n$ be a seminormalized block sequence in $\mathfrak{X}_{\mathrm{ISP}}$, such that $\|x_k\| \leqslant 1$ for $k = 1, \dots, n$ and there exist $n + 3 \leqslant j_1 < \dots < j_n$ strictly increasing naturals such that the following are satisfied.*

(i) *For any $k_0 \in \{1, \dots, n\}$, for any $k \geqslant k_0$, $k \in \{1, \dots, n\}$, for any $\{\alpha_q\}_{q=1}^d$ very fast growing and $\mathcal{S}_j$-admissible sequence of α-averages, with $j < j_{k_0}$ and $s(\alpha_1) > \min\operatorname{supp} x_{k_0}$, we have that $\sum_{q=1}^d |\alpha_q(x_k)| < 1/n \cdot 2^n$.*

(ii) *For any $k_0 \in \{1, \dots, n\}$, for any $k \geqslant k_0$, $k \in \{1, \dots, n\}$, for any $\{\beta_q\}_{q=1}^d$ very fast growing and $\mathcal{S}_j$-admissible sequence of β-averages, with $j < j_{k_0}$ and $s(\beta_1) > \min\operatorname{supp} x_{k_0}$, we have that $\sum_{q=1}^d |\beta_q(x_k)| < 1/n \cdot 2^n$.*

(iii) *For $k = 1, \dots, n-1$, the following holds: $(1/2^{j_{k+1}}) \max\operatorname{supp} x_k < 1/2^n$.*

(iv) *For any* $1 \leqslant k_1 < k_2 < k_3 \leqslant n$, *for any functional* $f \in W$ *of type II that separates* $x_{k_1}, x_{k_2}, x_{k_3}$, *we have that* $|f(x_{k_i})| < 1/n \cdot 2^n$, *for some* $i \in \{1,2,3\}$.

Then $\{x_k\}_{k=1}^n$ *is equivalent to the unit vector basis of* ℓ_∞^n, *with an upper constant* $3 + 3/2^n$. *Moreover, for any functional* $f \in W$ *of type* I_α *with weight* $w(f) = j < j_1$, *we have that* $|f(\sum_{k=1}^n x_k)| < (3 + 4/2^n)/2^j$.

Proof. By using Remark 1.3, we will inductively prove, that for any $\{c_k\}_{k=1}^n \subset [-1,1]$ the following hold.

(i) For any $f \in W$, we have that $|f(\sum_{k=1}^n c_k x_k)| < (3 + 3/2^n)\max\{|c_k| : k = 1,\dots,n\}$.
(ii) If f is of type I_α and $w(f) \geqslant 2$, then $|f(\sum_{k=1}^n c_k x_k)| < (1 + 2/2^n)\max\{|c_k| : k = 1,\dots,n\}$.
(iii) If f is of type I_α and $w(f) = j < j_1$, then $|f(\sum_{k=1}^n c_k x_k)| < (3 + 4/2^n)/2^j \max\{|c_k| : k = 1,\dots,n\}$.

For any functional $f \in W_0$ the inductive assumption holds. Assume that it holds for any $f \in W_m$ and let $f \in W_{m+1}$. If f is a convex combination, then there is nothing to prove.

Assume that f is of type I_α, $f = (1/2^j)\sum_{q=1}^d \alpha_q$, where $\{\alpha_q\}_{q=1}^d$ is a very fast growing and $\mathcal{S}_j$-admissible sequence of α-averages in W_m.

Set $k_1 = \min\{k : \operatorname{ran} f \cap \operatorname{ran} x_k \neq \emptyset\}$ and $q_1 = \min\{q : \operatorname{ran}\alpha_q \cap \operatorname{ran} x_{k_1} \neq \emptyset\}$.

We distinguish three cases.

Case 1 : $j < j_1$.

For $q > q_1$, we have that $s(\alpha_q) > \min\operatorname{supp} x_{k_1}$, therefore we conclude that

$$\sum_{q>q_1}\left|\alpha_q\left(\sum_{k=1}^n c_k x_k\right)\right| < \frac{1}{2^n}\max\{|c_k| : k = 1,\dots,n\}, \tag{24}$$

while the inductive assumption yields that

$$\left|\alpha_{q_1}\left(\sum_{k=1}^n c_k x_k\right)\right| < \left(3 + \frac{3}{2^n}\right)\max\{|c_k| : k = 1,\dots,n\}. \tag{25}$$

Then (24) and (25) allow us to conclude that

$$\left|f\left(\sum_{k=1}^n c_k x_k\right)\right| < \frac{3 + 4/2^n}{2^j}\max\{|c_k| : k = 1,\dots,n\}. \tag{26}$$

Hence, (iii) from the inductive assumption is satisfied.

Case 2 : There exists $k_0 < n$, such that $j_{k_0} \leqslant j < j_{k_0+1}$.

Arguing as previously we obtain that

$$\left|f\left(\sum_{k>k_0} c_k x_k\right)\right| < \frac{3 + 4/2^n}{2^{j_{k_0}}}\max\{|c_k| : k = 1,\dots,n\} \tag{27}$$

and

$$\left|f\left(\sum_{k<k_0} c_k x_k\right)\right| < \frac{1}{2^n}\max\{|c_k| : k = 1,\dots,n\}. \tag{28}$$

Using (27) and (28), the fact that $|f(x_{k_0})| \leqslant 1$ and $j_{k_0} \geqslant n + 3$, we conclude that

$$\left|f\left(\sum_{k=1}^n c_k x_k\right)\right| < \left(1 + \frac{2}{2^n}\right)\max\{|c_k| : k = 1,\dots,n\}. \tag{29}$$

Case 3 : $j \geqslant j_n$

By using the same arguments, we conclude that

$$\left|f\left(\sum_{k=1}^{n} c_k x_k\right)\right| < \left(1+\frac{1}{2^n}\right)\max\{|c_k| : k=1,\dots,n\}. \tag{30}$$

Then (26), (29) and (30) yield that (ii) from the inductive assumption is satisfied.

If f is of type I_β, then the proof is exactly the same, therefore assume that f is of type II, $f=\frac{1}{2}\sum_{j=1}^{d} f_j$, where $\{f_j\}_{j=1}^{d}$ is an $\mathcal{S}_1$-admissible sequence of functionals of type I_α in W_m. Set

$$E=\left\{k : |f(x_k)| \geqslant \frac{1}{n\cdot 2^n}\right\},$$
$$E_1=\{k\in E:\ \text{there exist at least two } j \text{ such that } \operatorname{ran} f_j\cap \operatorname{ran} x_k\neq\emptyset\}.$$

Then $\#E_1\leqslant 2$. Indeed, if $k_1<k_2<k_3\in E_1$, then f separates x_{k_1},x_{k_2} and x_{k_3} which contradicts our initial assumptions.

If moreover we set $J=\{j : \text{there exists } k\in E\setminus E_1 \text{ such that } \operatorname{ran} f_j\cap\operatorname{ran} x_k\neq\emptyset\}$, then for the same reasons we obtain that $\#J\leqslant 2$.

Since for any j, we have that $w(f_j)\in L$, we obtain that $w(f_j)>2$, therefore:

$$\left|f\left(\sum_{k\in E\setminus E_1}^{n} c_k x_k\right)\right| < \left(1+\frac{2}{2^n}\right)\max\{|c_k| : k=1,\dots,n\}, \tag{31}$$

$$\left|f\left(\sum_{k\in E_1}^{n} c_k x_k\right)\right| \leqslant 2\max\{|c_k| : k=1,\dots,n\}, \tag{32}$$

$$\left|f\left(\sum_{k\notin E}^{n} c_k x_k\right)\right| \leqslant n\cdot\frac{1}{n\cdot 2^n}\max\{|c_k| : k=1,\dots,n\}. \tag{33}$$

Finally, (31)–(33) yield the following.

$$\left|f\left(\sum_{k=1}^{n} c_k x_k\right)\right| < \left(3+\frac{3}{2^n}\right)\max\{|c_k| : k=1,\dots,n\}.$$

This means that (i) from the inductive assumption is satisfied and this completes the proof. □

4.3. *The spreading models of $\mathfrak{X}_{\mathrm{ISP}}$.*

In this subsection, we show that every seminormalized block sequence has a subsequence which generates either ℓ_1 or c_0 as a spreading model.

PROPOSITION 4.8. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a seminormalized block sequence in $\mathfrak{X}_{\mathrm{ISP}}$, such that $\|x_k\|\leqslant 1$ for all $k\in\mathbb{N}$ and $\alpha(\{x_k\}_k)=0$ as well as $\beta(\{x_k\}_k)=0$. Then it has a subsequence, again denoted by $\{x_k\}_{k\in\mathbb{N}}$ satisfying the following.*

(i) *The sequence $\{x_k\}_{k\in\mathbb{N}}$ generates a c_0 spreading model. More precisely, for any $n\leqslant k_1<\cdots<k_n$, we have that $\|\sum_{i=1}^{n} x_{k_i}\|\leqslant 4$.*

(ii) *There exists a strictly increasing sequence of naturals $\{j_n\}_{n\in\mathbb{N}}$, such that for any $n\leqslant k_1<\cdots<k_n$, for any functional f of type I_α with $w(f)=j<j_n$, we have that*

$$\left|f\left(\sum_{i=1}^{n} x_{k_i}\right)\right| < \frac{4}{2^j}.$$

Proof. By repeatedly applying Proposition 4.5 and diagonalizing, we may assume that for any $n \leqslant k_1 < k_2 < k_3$, for any functional f of type II that separates x_{k_1}, x_{k_2} and x_{k_3}, we have that $|f(x_{k_i})| < 1/n \cdot 2^n$, for some $i \in \{1,2,3\}$.

Use Propositions 3.3 and 3.4 to inductively choose a subsequence of $\{x_k\}_{k\in\mathbb{N}}$, again denoted by $\{x_k\}_{k\in\mathbb{N}}$ and $\{j_k\}_{k\in\mathbb{N}}$ a strictly increasing sequence of naturals with $j_k \geqslant k+3$ for all $k \in \mathbb{N}$, such that the following are satisfied.

(i) For any $k_0 \in \mathbb{N}$, for any $k \geqslant k_0$, for any $\{\alpha_q\}_{q=1}^d$ very fast growing and $\mathcal{S}_j$-admissible sequence of α-averages, with $j < j_{k_0}$ and $s(\alpha_1) > \min\operatorname{supp} x_{k_0}$, we have that $\sum_{q=1}^d |\alpha_q(x_k)| < 1/k_0 \cdot 2^{k_0}$.

(ii) For any $k_0 \in \mathbb{N}$, for any $k \geqslant k_0$, for any $\{\beta_q\}_{q=1}^d$ very fast growing and $\mathcal{S}_j$-admissible sequence of β-averages, with $j < j_{k_0}$ and $s(\beta_1) > \min\operatorname{supp} x_{k_0}$, we have that $\sum_{q=1}^d |\beta_q(x_k)| < 1/k_0 \cdot 2^{k_0}$.

(iii) For $k \in \mathbb{N}$, the following holds: $(1/2^{j_{k+1}})\max\operatorname{supp} x_k < 1/2^k$.

It is easy to check that for $n \leqslant k_1 < \cdots < k_n$, the assumptions of Proposition 4.7 are satisfied. ☐

Propositions 3.5 and 4.8 yield the following.

COROLLARY 4.9. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a normalized weakly null sequence in $\mathfrak{X}_{\mathrm{ISP}}$. Then it has a subsequence that generates a spreading model which is either equivalent to c_0, or to ℓ_1.*

DEFINITION 4.10. A pair $\{x, f\}$, where $x \in \mathfrak{X}_{\mathrm{ISP}}$ and $f \in W$ is called an $(n,1)$-exact pair, if the following hold.

(i) f is a functional of type I_α with $w(f) = n, \min\operatorname{supp} x \leqslant \min\operatorname{supp} f$ and $\max\operatorname{supp} x \leqslant \max\operatorname{supp} f$.

(ii) There exists $x' \in \mathfrak{X}_{\mathrm{ISP}}$ a $(4,1,n)$ exact vector, such that $\frac{28}{29} < f(x') \leqslant 1$ and $x = x'/f(x')$.

A pair $\{x, f\}$, where $x \in \mathfrak{X}_{\mathrm{ISP}}$ and $f \in W$ is called an $(n,0)$-exact pair, if the following hold.

(i) f is a functional of type I_α with $w(f) = n, \min\operatorname{supp} x \leqslant \min\operatorname{supp} f$ and $\max\operatorname{supp} x \leqslant \max\operatorname{supp} f$.

(ii) x is a $(4,1,n)$ exact vector and $f(x) = 0$.

REMARK 4.11. In the literature the condition $\max\operatorname{supp} x \leqslant \max\operatorname{supp} f$ is usually of the form $\max\operatorname{supp} f \leqslant \max\operatorname{supp} x$. In this paper, although the usual approach works as well, it is more convenient to use the above condition.

REMARK 4.12. If $\{x, f\}$ is an $(n,1)$-exact pair, then $f(x) = 1$ and by Remark 2.16 we have that $1 \leqslant \|x\| \leqslant 29$.

PROPOSITION 4.13. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}_{\mathrm{ISP}}$, that generates a c_0 spreading model. Then there exists $\{F_k\}_{k\in\mathbb{N}}$ an increasing sequence of subsets of the naturals such that $\#F_k \leqslant \min F_k$ for all $k \in \mathbb{N}$ and $\lim_k \#F_k = \infty$ such that by setting $y_k = \sum_{i\in F_k} x_k$, there exists a subsequence of $\{y_k\}_{k\in\mathbb{N}}$, which generates an ℓ_1^n spreading model, for all $n \in \mathbb{N}$.*

Furthermore, for any $k_0, n \in \mathbb{N}$, there exists F a finite subset of $\mathbb{N}$ with $\min F \geqslant k_0$ and $\{c_k\}_{k\in F}$, such that the following hold.

(i) *$x' = 2^n \sum_{k\in F} c_k y_k$ is a $(4,1,n)$ exact vector.*

(ii) *For any* $\eta > 0$ *there exists a functional* f_η *of type* I_α *of weight* $w(f_\eta) = n$ *such that* $f_\eta(x') > 1 - \eta, \min\operatorname{supp} x' \leqslant \min\operatorname{supp} f_\eta$ *and* $\max\operatorname{supp} f_\eta > \max\operatorname{supp} x'$.

In particular, if $f = f_{1/29},\ x = x'/f(x')$, *then* $\{x, f\}$ *is an* $(n, 1)$*-exact pair.*

Proof. Since $\{x_k\}_{k\in\mathbb{N}}$ generates a c_0 spreading model, Proposition 3.5 yields that $\alpha(\{x_k\}_k) = 0$ as well as $\beta(\{x_k\}_k) = 0$, therefore passing, if necessary, to a subsequence $\{x_k\}_{k\in\mathbb{N}}$, satisfies the conclusion of Proposition 4.8.

Choose $\{F_k\}_{k\in\mathbb{N}}$ an increasing sequence of subsets of the naturals, such that the following are satisfied.

(i) $\#F_k \leqslant \min F_k$ for all $k \in \mathbb{N}$.
(ii) $\#F_{k+1} > \max\{\#F_k,\ 2^{\max\operatorname{supp} x_{\max F_k}}\}$, for all $k \in \mathbb{N}$.

By Proposition 4.5 and Remark 2.14, we have that $1 \leqslant \|y_k\| \leqslant 4$, for all $k \in \mathbb{N}$ and passing, if necessary, to a subsequence, $\{y_k\}_{k\in\mathbb{N}}$ is a $(4, \{n_k\}_{k\in\mathbb{N}})$ α-RIS.

Moreover, it is easy to see that for any $k \in \mathbb{N}, \eta > 0$, there exists an α-average α of size $s(\alpha) = \#F_k$, such that $\alpha(y_k) > 1 - \eta$ and $\operatorname{ran}\alpha \subset y_k$.

This yields that $\alpha(\{y_k\}_k) > 0$, therefore we may apply Proposition 3.5 to conclude that $\{y_k\}_{k\in\mathbb{N}}$ has a subsequence generating an ℓ_1^n spreading model, for all $n \in \mathbb{N}$.

We now prove the second assertion. Let $k_0, n \in \mathbb{N}$ and fix $0 < \varepsilon < (36 \cdot 4 \cdot 2^{3n})^{-1}$. By taking a larger k_0, we may assume that $n_{k_0} > 2^{2n}$.

Set $\varepsilon' = \varepsilon(1 - \varepsilon)$ Proposition 1.8 yields that there exists $\{d_1, \ldots, d_m\}$ a finite subset of $\{k : k \geqslant k_0\}$ and $\{c_k\}_{k=1}^m$ such that $x'' = \sum_{k=1}^m c_k y_{d_k}$ is a (n, ε') s.c.c. It is straightforward to check that $x' = 2^n \sum_{k=1}^{m-1} (c_k/1 - c_m) y_{d_k}$ is a $(4, \theta, n)$ exact vector. If (ii) also holds, it will follow that $\theta \geqslant 1$.

For some $\eta > 0,\ k = 1, \ldots, m$, choose an α-average α_k of size $s(\alpha_k) = \#F_{d_k}$, such that $\alpha_k(y_{d_k}) > 1 - \eta$ and $\operatorname{ran}\alpha \subset y_k$. Set $f = (1/2^n)(\sum_{k=1}^m \alpha_k)$, which is a functional of type I_α of weight $w(f) = n$ such that $f(x') > 1 - \eta$ and $\max\operatorname{supp} f > \max\operatorname{supp} x$. □

Corollary 4.14. *Let* Y *be an infinite-dimensional closed subspace of* $\mathfrak{X}_{\mathrm{ISP}}$. *Then* Y *admits a spreading model equivalent to* c_0 *as well as a spreading model equivalent to* ℓ_1.

Proof. Assume first that Y is generated by some normalized block sequence $\{x_k\}_{k\in\mathbb{N}}$. Corollary 3.17 and Proposition 4.8 yield that it has a further normalized block sequence $\{y_k\}_{k\in\mathbb{N}}$, generating a spreading model equivalent to c_0.

Proposition 4.13 yields that $\{y_k\}_{k\in\mathbb{N}}$ has a further block sequence generating an ℓ_1 spreading model.

Since any subspace contains a sequence arbitrarily close to a block sequence, the result follows. □

Corollary 4.14 and Proposition 4.13 yield the following.

Corollary 4.15. *Let* X *be a block subspace of* $\mathfrak{X}_{\mathrm{ISP}}$. *Then for every* $n \in \mathbb{N}$ *there exists* $x \in X$ *and* $f \in W$, *such that* $\{x, f\}$ *is an* $(n, 1)$*-exact pair.*

We remind that, as Propositions 3.5 and 4.8 state, if a sequence generates an ℓ_1 spreading model, then passing, if necessary, to a subsequence, it generates an ℓ_1^k spreading model for any

$k \in \mathbb{N}$. However, as the next proposition states, the space $\mathfrak{X}_{\mathrm{ISP}}$ does not admit higher order c_0 spreading models.

PROPOSITION 4.16. *The space $\mathfrak{X}_{\mathrm{ISP}}$ does not admit c_0^2 spreading models.*

Proof. Towards a contradiction, assume that there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\mathfrak{X}_{\mathrm{ISP}}$, generating a c_0^2 spreading model. Then it must be weakly null and we may assume that it is a normalized block sequence. By Proposition 4.13, it follows that there exist $\{F_k\}_{k\in\mathbb{N}}$ increasing, Schreier admissible subsets of the naturals and $c > 0$ such that $\|\sum_{j=1}^n \sum_{i\in F_{k_j}} x_i\| \geqslant n \cdot c$ for any $n \leqslant k_1 < \cdots < k_n$. Since for any such $F_{k_1} < \cdots < F_{k_n}$ we have that $\bigcup_{j=1}^n F_{k_j} \in \mathcal{S}_2$, it follows that $\{x_k\}_{k\in\mathbb{N}}$ does not generate a c_0^2 spreading model. □

COROLLARY 4.17. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$. Then Y^* admits a spreading model equivalent to ℓ_1 as well as a spreading model equivalent to c_0^n, for any $n \in \mathbb{N}$.*

Proof. Since Y contains a sequence $\{x_k\}_{k\in\mathbb{N}}$ generating a spreading model equivalent to c_0, which we may assume is Schauder basic, then for any normalized $\{x_k^*\}_{k\in\mathbb{N}} \subset Y^*$, such that $x_k^*(x_m) = \delta_{n,m}$ for $n, m \in \mathbb{N}$, we have that $\{x_k^*\}_{k\in\mathbb{N}}$ generates a spreading model equivalent to ℓ_1.

To see that Y^* admits a spreading model equivalent to c_0^n for any $n \in \mathbb{N}$, take the previously used sequence $\{x_k\}_{k\in\mathbb{N}}$. Working just like in the proof of Proposition 4.13 find $\{F_k\}_{k\in\mathbb{N}}$ successive subsets of the natural such that $\min F_k \geqslant \#F_k$, for all $k \in \mathbb{N}$, if $y_k = \sum_{i\in F_k} x_i$ for all $k \in \mathbb{N}$, then $\{y_k\}_{k\in\mathbb{N}}$ is seminormalized and there exists a very fast growing sequence of α-averages $\{\alpha_k\}_{k\in\mathbb{N}} \subset W$ such that $\liminf \alpha_k(\sum_{i\in F_k} x_i) \geqslant 1$.

Then, if $c = \limsup_k \|y_k\|$, we evidently have that $\liminf_k \|\alpha_k\| \geqslant 1/c$ and since for any $n \in \mathbb{N}, F \in \mathcal{S}_n$, we have that $(1/2^n)\sum_{q\in F} \alpha_q$ is a functional of type I_α in W, it follows that $\|\sum_{q\in F}^d \alpha_q\| \leqslant 2^n$. This means that, $\{\alpha_k\}_{k\in\mathbb{N}}$ generates a spreading model equivalent to c_0^n, with an upper constant 2^n.

Let $I^* : \mathfrak{X}_{\mathrm{ISP}}^* \to Y^*$ be the dual operator of $I : Y \to \mathfrak{X}_{\mathrm{ISP}}$. Then $\{I^*(\alpha_k)\}_{k\in\mathbb{N}}$ generates a spreading model equivalent to c_0^n, for any $n \in \mathbb{N}$. To prove this, since $\|I^*\| = 1$, all that needs to be shown is that $\liminf_k \|I^*(\alpha_k)\| > 0$. Indeed,

$$\liminf_k \|I^*(\alpha_k)\| \geqslant \liminf_k (I^*\alpha_k)\left(\frac{\sum_{i\in F_k} x_i}{c}\right) = \liminf_k \alpha_k\left(\frac{\sum_{i\in F_k} x_i}{c}\right) \geqslant 1/c. \qquad \square$$

5. *Properties of $\mathfrak{X}_{\mathrm{ISP}}$ and $\mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$*

In this final section it is proved that $\mathfrak{X}_{\mathrm{ISP}}$ is HI and the properties of the operators acting on infinite-dimensional closed subspaces of $\mathfrak{X}_{\mathrm{ISP}}$ are presented.

5.1. *Dependent sequences and the HI property of $\mathfrak{X}_{\mathrm{ISP}}$.*

In the first part of this subsection, we introduce the dependent sequences, which are the main tool for proving the HI property of $\mathfrak{X}_{\mathrm{ISP}}$ and studying the structure of the operators.

DEFINITION 5.1. A sequence of pairs $\{x_k, f_k\}_{k=1}^n$ is said to be a 1-dependent sequence (respectively, a 0-dependent sequence) if the following are satisfied.

(i) $\{x_k, f_k\}$ is an $(m_k, 1)$-exact pair (respectively an $(m_k, 0)$-exact pair) for $k = 1, \dots, n$, with $m_1 > 4n2^{2n}$.

(ii) $\max \operatorname{supp} f_k < \min \operatorname{supp} x_{k+1}$ for $k = 1, \dots, n-1$.

(iii) $\{f_k\}_{k=1}^n$ is an $\mathcal{S}_1$-admissible special sequence of type I_α functionals in W, that is, $f = \frac{1}{2}\sum_{k=1}^n f_k$ is a functional of type II in W.

Proposition 5.2. *Let X be a block subspace of $\mathfrak{X}_{\mathrm{ISP}}$ and $n \in \mathbb{N}$. Then there exist $x_1, \dots, x_n$ in X and $f_1, \dots, f_n$ in W, such that $\{x_k, f_k\}_{k=1}^n$ is a 1-dependent sequence.*

Proof. Choose $m_1 \in L_1$ with $m_1 > 4n2^{2n}$. By Corollary 4.15, there exists $\{x_1, f_1\}$ an $(m_1, 1)$-exact pair in X. Then $\min \operatorname{supp} f_1 \geqslant \min \operatorname{supp} x_1 > n$.

Let $d < n$ and suppose that we have chosen $\{x_k, f_k\}$ $(m_k, 1)$-exact pairs for $k = 1, \dots, d$ such that $\{f_k\}_{k=1}^m$ is a special sequence and $\max \operatorname{supp} f_k < \min \operatorname{supp} x_{k+1}$ for $k = 1, \dots, d$.

Set $m_{d+1} = \sigma((f_1, m_1), \dots, (f_d, m_d))$. Then applying Corollary 4.15 once more, there exists $\{x_{d+1}, f_{d+1}\}$ an m_{d+1}-exact pair in X, such that $\max \operatorname{supp} f_d < \min \operatorname{supp} x_{d+1}$.

The inductive construction is complete and $\{x_k, f_k,\}_{k=1}^n$ is a 1-dependent sequence. $\square$

An immediate modification of the above proof yields the following.

Corollary 5.3. *If X and Y are block subspaces of $\mathfrak{X}_{\mathrm{ISP}}$ and $n \in \mathbb{N}$, then a 1-dependent sequence $\{x_k, f_k,\}_{k=1}^{2n}$ can be chosen, such that $x_{2k-1} \in X$ and $x_{2k} \in Y$ for $k = 1, \dots, n$.*

Proposition 5.4. *Let $\{(x_k, f_k)\}_{k=1}^{2n}$ be a 1-dependent sequence in $\mathfrak{X}_{\mathrm{ISP}}$ and set $y_k = x_{2k-1} - x_{2k}$, for $k = 1, \dots, n$. Then we have that:*

(i) $\|\sum_{k=1}^{2n} x_k\| \geqslant n$,

(ii) $\|\sum_{k=1}^{n} y_k\| \leqslant 232$.

Proof. Since $\frac{1}{2}\sum_{k=1}^{2n} f_k$ is a type II functional in W, it immediately follows that $\|\sum_{k=1}^{2n} x_k\| \geqslant \frac{1}{2}\sum_{k=1}^{2n} f_k(x_k) = n$.

By Remark 4.12 it follows that $1 \leqslant \|y_k\| \leqslant 58$, for $k = 1, \dots, n$.

Set $y'_k = \frac{1}{58} y_k$ and $j_k = m_{2k-1} - 2$ for $k = 1, \dots, n$. We will show that the assumptions of Proposition 4.7 are satisfied for the y'_k's. From this, it will follow that $\|\sum_{k=1}^n y_k\| \leqslant 58 \cdot 4$, which is the desired result.

The first and second assumptions of Proposition 4.7 follow from Lemmas 3.8 and 3.15, respectively, and the definition of the 1-dependent sequence.

The third assumption follows from the fact that, by the definition of the 1-dependent sequence, $\max \operatorname{supp} f_k > \max \operatorname{supp} x_k$, for $k = 1, \dots, 2n$ and the definition of the coding function σ.

It remains to be proved that the fourth assumption is also satisfied. Let $1 \leqslant k_1 < k_2 < k_3 \leqslant n$ and $g = \frac{1}{2}\sum_{j=1}^d g_j$ be a functional of type II that separates y'_{k_1}, y'_{k_2} and y'_{k_3}.

Set $j_0 = \min\{j : \operatorname{ran} g_j \cap \operatorname{ran} y'_{k_3} \neq \emptyset\}$ and assume first that $w(f_{j_0}) = m_{2k_3-1}$ Since $\operatorname{supp} g \cap \operatorname{supp} y_{k_1} \neq \emptyset$, it follows that $g_{j_0-1} = f_{2k_3-2}$ and there exists I an interval of the naturals, $\operatorname{ran} y'_{k_2} \subset I$, such that $g = I(\frac{1}{2}\sum_{k=1}^{j_0-1} f_k)$. This yields that $g(y'_{k_2}) = 0$.

Otherwise, if $w(f_{j_0}) \neq m_{2k_3-1}$, set $g' = g|_{\operatorname{ran} y'_{k_3}}$ and Corollary 3.13 yields the following.

$$|g'(y'_{k_3})| < \frac{2\cdot 4}{58}\frac{29}{28}\left(\frac{1}{2^{m_{2k_3-1}}} + \frac{1}{2^{2m_{2k_3-1}}} + \sum_{\substack{j\in\hat{w}(g'):\\ w(g_j)<n}} \frac{4}{2^{w(g_j)}}\right).$$

Since g separates y_{k_1}, y_{k_2} and y_{k_3}, we have that $\min \hat{w}(g') > p_0 = \min\operatorname{supp} x_1$, therefore

$$\sum_{\substack{j\in\hat{w}(g'):\\ w(g_j)<n}} \frac{1}{2^{w(g_j)}} < \sum_{p>p_0}\frac{1}{2^p} = \frac{1}{2^{p_0}} \leqslant \frac{1}{2^{32\cdot 2^{2m_1}}}.$$

By the choice of m_1, we conclude that $|g(y'_{k_3})| < 1/n2^n$, which means that the fourth assumption is satisfied. □

The next proposition is proved by using similar arguments.

Proposition 5.5. *Let $\{(x_k, f_k)\}_{k=1}^n$ be a 0-dependent sequence in $\mathfrak{X}_{\mathrm{ISP}}$. Then we have that:*

$$\left\|\sum_{k=1}^n x_k\right\| \leqslant 112.$$

We pass to the main structural property of $\mathfrak{X}_{\mathrm{ISP}}$.

Theorem 5.6. *The space $\mathfrak{X}_{\mathrm{ISP}}$ is hereditarily indecomposable.*

Proof. It is enough to show that for X and Y block subspaces of $\mathfrak{X}_{\mathrm{ISP}}$, for any $\varepsilon > 0$, there exist $x \in X$ and $y \in Y$, such that $\|x + y\| \geqslant 1$ and $\|x - y\| < \varepsilon$. Let $n \in \mathbb{N}$, such that $232/n < \varepsilon$.

By Corollary 5.3, there exists a 1-dependent sequence $\{x_k, f_k,\}_{k=1}^{2n}$, such that $x_{2k-1} \in X$ and $x_{2k} \in Y$ for $k = 1, \dots, n$.

Set $x = (1/n)\sum_{k=1}^n x_{2k-1}$ and $y = (1/n)\sum_{k=1}^n x_{2k}$. By applying Proposition 5.4, the result follows. □

5.2. *The structure of $\mathcal{L}(Y, \mathfrak{X}_{\mathrm{ISP}})$*

For Y a closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$ and $T : Y \to \mathfrak{X}_{\mathrm{ISP}}$ we show that $T = \lambda I_{Y,\mathfrak{X}_{\mathrm{ISP}}} + S$ with $S : Y \to \mathfrak{X}_{\mathrm{ISP}}$ strictly singular.

Proposition 5.7. *Let Y be a subspace of $\mathfrak{X}_{\mathrm{ISP}}$ and $T : Y \to \mathfrak{X}_{\mathrm{ISP}}$ be a linear operator, such that there exists $\{x_k\}_{k\in\mathbb{N}}$ a sequence in Y generating a c_0 spreading model and $\limsup \operatorname{dist}(Tx_k, \mathbb{R}x_k) > 0$. Then T is unbounded.*

Proof. Passing, if necessary, to a subsequence, there exists $1 > \delta > 0$, such that $\operatorname{dist}(Tx_k, \mathbb{R}x_k) > \delta$, for any $k \in \mathbb{N}$.

Since $\{x_k\}_{k\in\mathbb{N}}$ generates a c_0 spreading model, it is weakly null. Set $y_k = Tx_k$ and assume that T is bounded. It follows that passing, if necessary, to a subsequence of $\{x_k\}_{k\in\mathbb{N}}$, then $\{y_k\}_{k\in\mathbb{N}}$ also generates a c_0 spreading model.

We may assume that $\{x_k\}_{k\in\mathbb{N}}$, as well as $\{y_k\}_{k\in\mathbb{N}}$ are block sequences with rational coefficients, and that $\lim_k \|x_k\| = 1$, as well as $\lim_k \|y_k\| = 1$.

If this is not the case pass, if necessary, to a further subsequence of $\{x_k\}_{k\in\mathbb{N}}$, such that both $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ are equivalent to some block sequences with rational coefficients $\{x'_k\}_{k\in\mathbb{N}}, \{y'_k\}_{k\in\mathbb{N}}$, respectively, and moreover $\lim_k \|x'_k\| = 1$, as well as $\lim_k \|y'_k\| = 1$. Set $Y' = [\{x'_k\}_{k\in\mathbb{N}}]$ and $T' : Y' \to \mathfrak{X}_{\mathrm{ISP}}$, such that $T'x'_k = y'_k$. It is easy to check that T' is also bounded and $\operatorname{dist}(T'x'_k, \mathbb{R}x'_k) > \delta'$, for some $\delta' > 0$.

Set $I_k = \{\min\{\operatorname{ran} x_k \cup \operatorname{ran} y_k\}, \ldots, \max\{\operatorname{ran} x_k \cup \operatorname{ran} y_k\}\}$ and passing, if necessary, to a subsequence, we have that $\{I_k\}_{k\in\mathbb{N}}$ is an increasing sequence of intervals of the naturals. We will choose $\{f_k\}_{k\in\mathbb{N}} \subset W$, such that $f_k(y_k) > \delta/5$, $f_k(x_k) = 0$ and $\operatorname{ran} f_k \subset I_k$, for all $k \in \mathbb{N}$.

The Hahn–Banach Theorem yields that for all $k \in \mathbb{N}$, there exists $f'_k \in B_{\mathfrak{X}^*_{\mathrm{ISP}}}$, such that $f'_k(y_k) > \delta$, $f'_k(x_k) = 0$ and $\operatorname{ran} f'_k \subset I_k$, for all $k \in \mathbb{N}$. By the fact that $\mathfrak{X}_{\mathrm{ISP}}$ is reflexive, it follows that W is norm dense in $B_{\mathfrak{X}^*_{\mathrm{ISP}}}$, therefore there exists $f''_k \in W$ with $\|f'_k - f''_k\| < \delta/4$ and $\operatorname{ran} f''_k \subset I_k$, for all $k \in \mathbb{N}$. It follows that $f''_k(y_k) > 3\delta/4, |f''_k(x_k)| < \delta/4$ and $f''_k(x_k)$ is rational, for all $k \in \mathbb{N}$. Furthermore, there exists $g_k \in W$, such that $g_k(x_k) > 1 - \delta/4$, $g_k(x_k)$ is rational and $\operatorname{ran} g_k \subset I_k$, for all $k \in \mathbb{N}$. Set $f_k = \frac{1}{2}(f''_k - f''_k(x_k)/g_k(x_k)g_k)$. By doing some easy calculations, it follows that the f_k are the desired functionals.

For the rest of the proof, we may assume that the $\{x_k\}_k$ are normalized. By copying the proof of Proposition 4.13, for any $k_0 \in \mathbb{N}, n \in \mathbb{N}$, there exists F a finite subset of the naturals with $\min F \geqslant k_0$ and $\{c_k\}_{k\in F}$ such that

(i) $z = 2^n \sum_{k\in F} c_k x_k$ is a $(4, 1, n)$ exact vector.

(ii) There exists a functional f of type I_α with weight $w(f) = n$ such that $f(z) = 0, \max\operatorname{supp} f > \max\operatorname{supp} z$ and if $w = 2^n \sum_{k\in F} c_k y_k$, then $f(w) > \delta/5$.

Using the above fact and arguing in the same way as in the proof of Proposition 5.2, for some $n \in \mathbb{N}$, we construct a sequence $\{z_k\}_{k=1}^n$ and $\{g_k\}_{k=1}^n$ such that $\{(z_k, g_k)\}_{k=1}^n$ is 0-dependent and if $w_k = Tz_k$, then $g_k(w_k) > \delta/5$ and $\operatorname{ran} g_k \cap \operatorname{ran} w_m = \emptyset$, for $k \neq m$.

Then $f = \frac{1}{2}\sum_{k=1}^n g_k$ is a functional of type II in W and $\|\sum_{k=1}^n w_k\| \geqslant \frac{1}{2}\sum_{k=1}^n g_k(w_k) > n\delta/10$.

Moreover, Proposition 5.5 yields that $\|\sum_{k=1}^n z_k\| \leqslant 112$. It follows that $\|T\| > n \cdot \delta/1120$. Since n was arbitrary, T cannot be bounded, a contradiction which completes the proof. □

In [**16**], it is proved that if X is a HI complex Banach space, Y is a subspace of X and $T : Y \to X$ is a bounded linear operator, then there exists $\lambda \in \mathbb{C}$, such that $T - \lambda I_{Y,X} : Y \to X$ is strictly singular. Here, we prove a similar result for $\mathfrak{X}_{\mathrm{ISP}}$.

Theorem 5.8. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$ and $T : Y \to \mathfrak{X}_{\mathrm{ISP}}$ be a bounded linear operator. Then there exists $\lambda \in \mathbb{R}$, such that $T - \lambda I_{Y,\mathfrak{X}_{\mathrm{ISP}}} : Y \to \mathfrak{X}_{\mathrm{ISP}}$ is strictly singular.*

Proof. If T is strictly singular, then evidently $\lambda = 0$ is the desired scalar.

Otherwise, if T is not strictly singular, choose Z an infinite-dimensional closed subspace of Y, such that $T : Z \to \mathfrak{X}_{\mathrm{ISP}}$ is an into isomorphism. Choose $\{x_k\}_{k\in\mathbb{N}}$ a normalized sequence in Z generating a c_0 spreading model. Proposition 5.7 yields that $\lim_k \operatorname{dist}(Tx_k, \mathbb{R}x_k) = 0$. Choose $\{\lambda_k\}_{k\in\mathbb{N}}$ scalars, such that $\lim_k \|Tx_k - \lambda_k x_k\| = 0$ and λ a limit point of $\{\lambda_k\}_{k\in\mathbb{N}}$.

We will prove that $S = T - \lambda I_{Y,\mathfrak{X}_{\mathrm{ISP}}}$ is strictly singular. Towards a contradiction, suppose that this is not the case. Then there exists $\{y_k\}_{k\in\mathbb{N}}$ a normalized sequence in Y generating a c_0 spreading model and $\delta > 0$, such that $\|Sy_k\| = \|(T - \lambda I_{Y,\mathfrak{X}_{\mathrm{ISP}}})y_k\| > \delta$, for all $k \in \mathbb{N}$.

As previously, we may assume that $\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}}$ as well as $\{Sy_k\}_{k\in\mathbb{N}}$ are all normalized block sequences generating c_0 spreading models.

By Proposition 5.7 and passing, if necessary, to a subsequence, there exists $\mu \in \mathbb{R}$, such that $\lim_k \|Sy_k - \mu y_k\| = 0$. Evidently, $\mu \neq 0$, otherwise we would have that $\lim_k \|Sy_k\| = 0$. Pass, if necessary, to a further subsequence of $\{y_k\}_{k\in\mathbb{N}}$, such that $\sum_{k=1}^\infty \|Sy_k - \mu y_k\| < |\mu|/232$.

Observe that $\lim_k \|Sx_k\| = 0$ and therefore we may pass, if necessary, to a subsequence of $\{x_k\}_{k\in\mathbb{N}}$, such that $\sum_{k=1}^\infty \|Sx_k\| < |\mu|/232$.

Arguing in the same manner as in the proof of Proposition 5.2, for some $n \in \mathbb{N}$ construct $\{z_k\}_{k=1}^{2n}$ and $\{f_k\}_{k=1}^{2n}$ such that z_{2k-1} is a linear combination of $\{y_k\}_{k\in\mathbb{N}}$, z_{2k} is a linear combination of $\{x_k\}_{k\in\mathbb{N}}$ and $\{(z_k, f_k)\}_{k=1}^{2n}$ is a 1-dependent sequence. Set $f = \frac{1}{2}\sum_{k=1}^{2n} f_k$, which is a functional of type II in W and set $w_k = z_{2k-1} - z_{2k}$. Proposition 5.4 yields that $\|\sum_{k=1}^n w_k\| \leqslant 232$.

On the other hand, we have that

$$\begin{aligned}
\left\|\sum_{k=1}^n Sw_k\right\| &\geqslant \left(\left\|\sum_{k=1}^n Sz_{2k-1}\right\| - \left\|\sum_{k=1}^n Sz_{2k}\right\|\right)\\
&\geqslant \left(\left\|\sum_{k=1}^n \mu z_{2k-1}\right\| - \left\|\sum_{k=1}^n (Sz_{2k-1} - \mu z_{2k-1})\right\| - \frac{29|\mu|}{232}\right)\\
&\geqslant \left(\frac{|\mu|}{2}\sum_{k=1}^n f_{2k-1}(z_{2k-1}) - \frac{29|\mu|}{232} - \frac{29|\mu|}{232}\right)\\
&= \frac{n|\mu|}{2} - \frac{|\mu|}{4} \geqslant \frac{n|\mu|}{4}.
\end{aligned}$$

It follows that $\|S\| \geqslant n|\mu|/928$, where n was arbitrary. This means that S is unbounded, a contradiction completing the proof. □

5.3. *Strictly singular operators*

In this subsection we study the action of strictly singular operators on Schauder basic sequences in subspaces of $\mathfrak{X}_{\text{ISP}}$.

PROPOSITION 5.9. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and $T : Y \to \mathfrak{X}_{\text{ISP}}$ be a linear bounded operator. Then the following assertions are equivalent.*

(i) *The operator T is not strictly singular.*

(ii) *There exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in Y generating a c_0 spreading model, such that $\{Tx_k\}_{k\in\mathbb{N}}$ is not norm convergent to 0.*

Proof. Assume first that T is not strictly singular and let Z be an infinite-dimensional closed subspace of Y, such that $T|_Z$ is an isomorphism. Since any subspace of $\mathfrak{X}_{\text{ISP}}$ contains a sequence generating a c_0 spreading model, then so does Z. Since $T|_Z$ is an isomorphism, the second assertion is true.

Assume now that there exists $\{x_k\}_{k\in\mathbb{N}}$ a sequence in Y generating a c_0 spreading model, such that $\{Tx_k\}_{k\in\mathbb{N}}$ does not norm converge to 0. By Proposition 5.7 and passing, if necessary to a subsequence, there exists $\lambda \neq 0$, such that $\lim_k \|Tx_k - \lambda x_k\| = 0$. Passing, if necessary, to a further subsequence, we have that $\sum_{k=1}^\infty \|Tx_k - \lambda x_k\| < \infty$. But this means that $\{x_k\}_{k\in\mathbb{N}}$ is equivalent to $\{Tx_k\}_{k\in\mathbb{N}}$, therefore T is not strictly singular. □

Next we introduce four ranks which provide a complete classification of weakly null sequences. The sequences of rank 0 are the norm null sequences and the sequences of rank 1, 2 and 3 are seminormalized. The crucial property of this classification is the following: strictly singular operators map sequences of positive rank to sequences of strictly smaller rank.

Definition 5.10. Let $\{x_k\}_k$ be a normalized block sequence in $\mathfrak{X}_{\mathrm{ISP}}$. We say that $\{x_k\}_k$ is of rank 1 if $\alpha(\{x_k\}_k) = 0$ and $\beta(\{x_k\}_k) = 0$.

Definition 5.11. Let $\{x_k\}_k$ be a normalized block sequence in $\mathfrak{X}_{\mathrm{ISP}}$. We say that $\{x_k\}_k$ is of rank 2, if it satisfies one of the following.

(i) We have that $\alpha(\{x_k\}_k) = 0$ and for every L infinite subset of the natural numbers, $\beta(\{x_k\}_{k\in L}) > 0$.
(ii) For every L infinite subset of the natural numbers $\alpha(\{x_k\}_{k\in L}) > 0$ and for every $C \geqslant 1, \theta > 0, \{n_j\}_j$ strictly increasing sequence of natural numbers, $F_j \subset L$ and $\{c_k^j\}_{k\in F_j}, j \in \mathbb{N}$ such that $w_j = 2^{n_j}\sum_{k\in F_j} c_k^j x_k$ are (C, θ, n_j) vectors for every $j \in \mathbb{N}$, we have that $\beta(\{w_j\}_j) = 0$.

Definition 5.12. Let $\{x_k\}_k$ be a normalized block sequence. We say that $\{x_k\}_k$ is of rank 3, if for every L infinite subset of the natural numbers, $\alpha(\{x_k\}_{k\in L}) > 0$ and there exist $C \geqslant 1, \theta > 0, \{n_j\}_j$ strictly increasing sequence of natural numbers, $F_j \subset L$ and $\{c_k^j\}_{k\in F_j}, j \in \mathbb{N}$ such that $w_j = 2^{n_j}\sum_{k\in F_j} c_k^j x_k$ are (C, θ, n_j) are vectors for every $j \in \mathbb{N}$, and $\beta(\{w_j\}_j) > 0$.

Remark 5.13. It follows easily from the definitions above that every normalized block sequence has a subsequence that is of some rank. Moreover, if a normalized block sequence is of some rank, then any of its subsequences is of the same rank.

Definition 5.14. Let $\{x_k\}_k$ be a weakly null sequence in $\mathfrak{X}_{\mathrm{ISP}}$. If it is norm null, then we say that it is of rank 0. If it is seminormalized, we say that $\{x_k\}_k$ is of rank i, if there exists a normalized block sequence $\{x'_k\}_k$ which is of rank i, such that $\sum_k \|x_k/\|x_k\| - x'_k\| < \infty$.

Remark 5.15. Every weakly null sequence in $\{x_k\}_k$ has a subsequence which is of some rank. Moreover, Propositions 3.5 and 4.8 yield that $\{x_k\}_k$ has a subsequence which is of rank 1 if and only it admits a c_0 spreading model and it has a subsequence that is of rank 2 or 3 if and only if it admits ℓ_1 as a spreading model.

Proposition 5.9 yields the following.

Proposition 5.16. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$ and $T : Y \to \mathfrak{X}_{\mathrm{ISP}}$ be a strictly singular operator. Then for every $\{x_k\}_k$ weakly null sequence in Y which is of rank* 1*, we have that $\{Tx_k\}_k$ is of rank 0.*

Proposition 5.17. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$ and $T : Y \to \mathfrak{X}_{\mathrm{ISP}}$ be a strictly singular operator. Then for every $\{x_k\}_k$ weakly null sequence in Y which is of rank* 2*, $\{Tx_k\}_k$ has a subsequence of rank* 0 *or of rank* 1*.*

Proof. Towards a contradiction pass to a subsequence of $\{x_k\}_k$ and assume that there exist $\{x'_k\}_k$, $\{y_k\}_k$ normalized block sequences, with $\sum_k \|x_k/\|x_k\| - x'_k\| < \infty, \sum_k \|Tx_k/\|Tx_k\| -$

$y_k\| < \infty$, $\{x'_k\}_k$ satisfies either (i) or (ii) from Definition 5.11 and $\{y_k\}_k$ is of either rank 2 or 3. By Remark 5.15, we may assume that both $\{x'_k\}_k$, $\{y_k\}_k$ generate ℓ_1 as a spreading model.

Setting $T' : [\{x'_k\}_k] \to \mathfrak{X}_{\mathrm{ISP}}$ with $T'x'_k = y_k$ for all $k \in \mathbb{N}$, we have that T' is bounded and strictly singular. Arguing in a similar manner as in the proof of Proposition 3.5, we may choose $\{n_j\}_j$ a strictly increasing sequence of natural numbers, $\{F_j\}_j$ a strictly increasing sequence of natural numbers and $\{c_k^j\}_{k\in F_j}, j \in \mathbb{N}$ such that $z_j = 2^{n_j}\sum_{k\in F_j} c_k^j x'_k$ and $w_j = T'z_j = 2^{n_j}\sum_{k\in F_j} c_k^j y_k$ are $(1, \theta, n_j)$ vectors for every $j \in \mathbb{N}$. Proposition 3.9 yields that $\alpha(\{z_j\}_j) = 0$ as well as $\alpha(\{w_j\}_j) = 0$.

If $\{x'_k\}_k$ satisfies (i) from Definition 5.11, then by Proposition 3.6 we may assume that it is a $(2, \{m_k\}_k)$ α-RIS, therefore the z_j can in fact have been chosen to be $(2, \theta, n_j)$ exact vectors. Proposition 4.8 yields that $\beta(\{z_j\}_j) = 0$. We have concluded that $\{z_j\}_j$ is of rank 1 and by Proposition 5.16 we have that $\{w_j\}_j$ is norm null, which contradicts the fact that $\|w_j\| \geqslant \theta$.

If on the other hand, if $\{x'_k\}_k$ satisfies (ii) from Definition 5.11, then we have that $\beta(\{z_j\}_j) = 0$. Again, Proposition 5.16 yields that $\{w_j\}_j$ is norm null, which cannot be the case. □

PROPOSITION 5.18. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$ and $T : Y \to \mathfrak{X}_{\mathrm{ISP}}$ be a strictly singular operator. Then every $\{x_k\}_k$ weakly null sequence in Y, has a subsequence $\{x_{k_n}\}_n$ such that $\{Tx_{k_n}\}_n$ is of rank 0, of rank 1, or of rank 2.*

Proof. If $\{Tx_k\}_k$ has a norm null subsequence then there is nothing to prove. Otherwise, pass to a subsequence of $\{x_k\}_k$, again denoted by $\{x_k\}_k$ and choose $\{x'_k\}_k$ and $\{y_k\}_k$ normalized block sequences, with $\sum_k \|x_k/\|x_k\| - x'_k\| < \infty, \sum_k \|Tx_k/\|Tx_k\| - y_k\| < \infty$. By passing to a further subsequence and slightly perturbing the x'_k and y_k, we may assume that $\min\operatorname{supp} x'_k = \min\operatorname{supp} y_k$ for all $k \in \mathbb{N}$ and that $\{x'_k\}_k$, $\{y_k\}_k$ are of some rank.

Setting $T' : [\{x'_k\}_k] \to \mathfrak{X}_{\mathrm{ISP}}$ with $T'x'_k = y_k$ for all $k \in \mathbb{N}$, we have that T' is bounded and strictly singular. Towards a contradiction, assume that $\{y_k\}_k$ satisfies the assumption of Definition 5.12. By Remark 5.15, passing to a further subsequence, we have that $\{y_k\}_k$ generates an ℓ_1 spreading model and by the boundedness of T', we may assume that so does $\{x'_k\}_k$. Passing to an even further subsequence, we have that both $\{x'_k\}_k$ and $\{y_k\}_k$ satisfy the conclusion of Proposition 3.5.

Choose $C \geqslant 1, \theta > 0, \{n_j\}_j$ strictly increasing sequence of natural numbers, $F_j \subset L$ and $\{c_k^j\}_{k\in F_j}, j \in \mathbb{N}$ such that $w_j = 2^{n_j}\sum_{k\in F_j} c_k^j y_k$ are (C, θ, n_j) vectors for every $j \in \mathbb{N}$, and $\beta(\{w_j\}_j) > 0$.

Since $\min\operatorname{supp} x'_k = \min\operatorname{supp} y_k$ for all $k \in \mathbb{N}$, we have that $z_j = 2^{n_j}\sum_{k\in F_j} c_k^j x'_k$ are (C, θ, n_j) vectors for every $j \in \mathbb{N}$.

Proposition 3.9 yields that $\alpha(\{z_j\}_j) = 0$ as well as $\alpha(\{w_j\}_j) = 0$.

Since $\beta(\{w_j\}_j) > 0$, we may pass to a subsequence of $\{w_j\}_j$ that generates an ℓ_1 spreading model and if $w'_j = w_j/\|w_j\|$, then $\{w'_j\}_j$ satisfies (i) from Definition 5.11, it is therefore of rank 2.

Once more, the boundedness of T' yields that if $z'_j = z_j/\|w_j\|$, then $\{z'_j\}_j$ generates an ℓ_1 spreading model. Since $\alpha(\{z'_j\}_j) = 0$, we conclude that $\beta(\{z'_j\}_j) > 0$. We may therefore pass to a final subsequence of $\{z'_j\}_j$ which is of rank 2. Since $T'z'_j = w'_j$, Proposition 5.17 yields a contradiction. □

5.4. *The invariant subspace property*

THEOREM 5.19. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$ and $Q, S, T : Y \to Y$ be strictly singular operators. Then QST is compact.*

Proof. Since $\mathfrak{X}_{\text{ISP}}$ is reflexive, it is enough to show that for any weakly null sequence $\{x_k\}_{k\in\mathbb{N}}$, we have that $\{QSTx_k\}_{k\in\mathbb{N}}$ norm converges to zero.

Proposition 5.18 yields that passing, if necessary to a subsequence, $\{Tx_k\}_k$ is of rank 0, rank 1 or rank 2. If it is of rank 0, then it is norm null and we are done. If it is of rank 1, Proposition 5.16 yields that $\{STx_k\}_k$ is of rank 0 and as previously we are done. Otherwise, $\{Tx_k\}_k$ is of rank 2. By Proposition 5.17, we may pass to a further subsequence, such that $\{STx_k\}_k$ is either of rank 0, or rank 1. If it is not of rank 0, then applying Proposition 5.16 we have that $\{QSTx_k\}_{k\in\mathbb{N}}$ norm converges to zero and the proof is complete. □

REMARK 5.20. We would like to point out that we can neither prove nor disprove the existence of sequences of rank 3. The failure of the existence of such sequences, would yield that the composition of any two strictly singular operators defined on a subspace of $\mathfrak{X}_{\text{ISP}}$, is a compact one.

COROLLARY 5.21. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and $S : Y \to Y$ be a non-zero strictly singular operator. Then S admits a non-trivial closed hyperinvariant subspace (i.e. there a exists a non-trivial closed subspace Z of Y that is invariant for any bounded linear operator which commutes with S).*

Proof. Assume first that $S^3 = 0$. Then it is straightforward to check that $\ker S$ is a non-trivial closed hyperinvariant subspace of S.

Otherwise, if $S^3 \neq 0$, then Theorem 5.19 yields that S^3 is compact and non-zero. Since S commutes with its cube, by Theorem 2.1 from [**33**], it is enough to check that for any $\alpha, \beta \in \mathbb{R}$ such that $\beta \neq 0$, we have that $(\alpha I - S)^2 + \beta^2 I \neq 0$ (see also [**21**], Theorem 2). Since S is strictly singular, it is easy to see that this condition is satisfied. □

COROLLARY 5.22. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and $T : Y \to Y$ be a non-scalar operator. Then T admits a non-trivial closed hyperinvariant subspace.*

Proof. Theorem 5.8 yields that there exist $\lambda \in \mathbb{R}$, such that $S = T - \lambda I$ is strictly singular, and since T is not a scalar operator, we evidently have that S is not zero.

By Corollary 5.21, it follows that S admits a non-trivial closed hyperinvariant subspace Z. It is straightforward to check that Z also is a hyperinvariant subspace for T. □

In the final result, which is related to Proposition 3.1 from [**5**], we show that the 'scalar plus compact' property fails in every subspace of $\mathfrak{X}_{\text{ISP}}$.

PROPOSITION 5.23. *Let Y be an infinite-dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$. Then there exists a strictly singular, non-compact operator $S : Y \to Y$. In fact, the strictly singular non-compact operators on Y form a non-separable subset of $\mathcal{L}(Y)$.*

Proof. By Corollary 4.14, there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in Y that generates a spreading model equivalent to c_0, say with an upper constant c_1 and by Corollary 4.17, there exists a sequence $\{x_k^*\}_{k\in\mathbb{N}}$ in Y^* that also generates a spreading model equivalent to c_0, say with an upper constant c_2. Therefore, $\{x_k\}_{k\in\mathbb{N}}$ and $\{x_k^*\}_{k\in\mathbb{N}}$ are weakly null and we may assume that they are Schauder basic and that $\dim(Y/[x_k]_k) = \infty$. We may also assume that there exist $\{z_k\}_{k\in\mathbb{N}}$ in Y such that $\{x_k^*\}_{k\in\mathbb{N}}$ is almost biorthogonal to $\{z_k\}_{k\in\mathbb{N}}$.

For $\varepsilon > 0$, set $M_\varepsilon = 4c_1/\varepsilon$. Choose a strictly increasing sequence of naturals $\{q_j\}_{j\in\mathbb{N}}$, such that $q_j \geqslant M_{1/2^{j+1}}$. Set $S : Y \to Y$, such that $Sx = \sum_{k=1}^{\infty} x^*_{q_k}(x)x_k$. Then:

(i) S is bounded and non-compact.
(ii) S is strictly singular.

We first prove that it is bounded. Let $x \in Y, \|x\| = 1,\ x^* \in Y^*, \|x^*\| = 1$. For $j \geqslant 0$, set $B_j = \{k \in \mathbb{N} : 1/2^{j+1} < |x^*(x_k)| \leqslant 1/2^j\}$. Since $\{x_k\}_{k\in\mathbb{N}}$ generates c_0 as a spreading model, it follows that $B_j \leqslant M_{1/2^{j+1}} \leqslant q_j$. Set $C_j = \{k \in B_j : k \geqslant j\}, D_j = B_j \setminus C_j$. Evidently, $\#D_j \leqslant j$ and it is easy to see that $\#\{q_k : k \in C_j\} \leqslant \min\{q_k : k \in C_j\}$, therefore, since $\{x^*_k\}_{k\in\mathbb{N}}$ generates a spreading model equivalent to c_0, it follows that

$$\left|\sum_{k\in C_j} x^*(x_k)x^*_{q_k}(x)\right| < c_2 \max\{|x^*(x_k)| : k \in C_j\}.$$

Therefore, $|\sum_{k\in B_j} x^*_{q_k}(x)x^*(x_k)| \leqslant c_2 \max\{|x^*(x_k)| : k \in C_j\} + j/2^j \leqslant c_2/2^j + j/2^j$. From this it follows that

$$\|Sx\| \leqslant \sum_{j=0}^{\infty} \frac{j + c_2}{2^j}\|x\|.$$

We will show that S is non-compact. Consider the previously mentioned sequence $\{z_k\}_{k\in\mathbb{N}}$ which is almost biorthogonal to $\{x^*_{q_k}\}_{k\in\mathbb{N}}$. Then $\{z_k\}_{k\in\mathbb{N}}$ is a seminormalized sequence in Y and $\{Sz_k\}_{k\in\mathbb{N}}$ does not have a norm convergent subsequence.

We now prove that S is strictly singular. Suppose that it is not, then there exists $\lambda \neq 0$ such that $T = S - \lambda I$ is strictly singular. Since λI is a Fredholm operator and T is strictly singular, it follows that $S = T + \lambda I$ is also a Fredholm operator, therefore $\dim(Y/S[Y]) < \infty$. The fact that $S[Y] \subset [x_k]_k$ and $\dim(Y/[x_k]_k) = \infty$ yields a contradiction.

Moreover, for any further subsequence $\{x^*_k\}_{k\in L}$ of $\{x^*_{q_k}\}_{k\in\mathbb{N}}$, if we set $S_L x = \sum_{k\in L} x^*_k(x)x_k$, then S_L satisfies the same conditions. This yields that there exists an uncountable ε-separated set of strictly singular non-compact operators. □

5.5. *Some final remarks*

We would like to mention that the structure of the dual of $\mathfrak{X}_{\mathrm{ISP}}$ is unclear to us. In particular, we cannot determine whether $\mathfrak{X}^*_{\mathrm{ISP}}$ shares similar properties with $\mathfrak{X}_{\mathrm{ISP}}$. For example, we do not know whether $\mathfrak{X}^*_{\mathrm{ISP}}$ admits only c_0 and ℓ_1 as a spreading model. However, the following holds.

Proposition 5.24. *Every infinite-dimensional quotient of $\mathfrak{X}^*_{\mathrm{ISP}}$ satisfies ISP. More precisely, every non-scalar operator defined on an infinite-dimensional quotient of $\mathfrak{X}^*_{\mathrm{ISP}}$ admits a non-trivial closed hyperinvariant subspace.*

The proof of the above statement is an immediate consequence of the following classical and easy result. If X is a reflexive Banach space, then the following are equivalent.

(i) The space X satisfies the hereditary ISP.
(ii) Every infinite-dimensional quotient of X^* satisfies ISP.

If even more, every non-scalar operator defined on an infinite-dimensional closed subspace of X admits a non-trivial closed hyperinvariant subspace, then every non-scalar operator defined on an infinite-dimensional quotient of X^* admits a non-trivial closed hyperinvariant subspace.

We would also like to mention that the method of constructing HI Banach spaces with saturation under constraints, using Tsirelson space as an unconditional frame, can be used to yield further results. For example, in [**9**] a reflexive HI Banach space $\mathfrak{X}_{\text{usm}}$ is constructed having the following property. In every subspace Y of $\mathfrak{X}_{\text{usm}}$ there exists a weakly null normalized sequence $\{y_n\}_n$, such that every subsymmetric sequence $\{z_n\}_n$ in any Banach space is isomorphically generated as a spreading model of a subsequence of $\{y_n\}_n$.

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