Fall 2016, Math 409, Section 502

Final Exam (Practice)
Monday, May 9 2016

Last name:

First name:

UIN:

Signature:

"An Aggie does not lie, cheat or steal or tolerate those who do."

This exam consists of **four** problems, the total point value of which is 100 points.

The answer to each question must be justified in detail.

The time length of this exam is two hours.

The use of electronic devices, such as cellphones, tables, laptops, and calculators is prohibited.

Good luck!

Problem 1	Problem 2	Problem 3	Problem 4	Total
25	25	25	25	100

You may use the remaining space on this page.

Problem 1. In this problem you will need to use that the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = e^x$ satisfies $g'(x) = g(x)$ and $g(-x) = 1/g(x)$ for all $x \in \mathbb{R}$ and $\lim\limits_{x \to +\infty} g(x) = +\infty$.

(i) For each $n \in \mathbb{N} \cup \{0\}$ let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined by the rule

$$f_n(x) = \begin{cases} \dfrac{e^{-1/x}}{x^n} & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

Prove that f_n is continuous for all $n \in \mathbb{N} \cup \{0\}$. *6 pts*
Hint: prove first $\lim\limits_{x \to +\infty} x^n e^{-x} = 0$ for all $n \in \mathbb{N}$.

(ii) Let $(f_n)_{n=0}^{\infty}$ be the sequence of functions from question (ii). Prove that for all $n \in \mathbb{N} \cup \{0\}$ and all $x \in \mathbb{R}$ we have $f_n'(x) = f_{n+2}(x) - n f_{n+1}(x)$. (Don't forget $x = 0$.) *6 pts*

(iii) Prove that $f = f_0$ has derivatives of order n, for all $n \in \mathbb{N}$ and that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$ (here, $f^{(n)}$ denotes the derivative of f of order n). *6 pts*

(iv) Prove that for all $n \in \mathbb{N}$, $\sup\{f^{(n)}(x) : x \in [0,1]\} \geqslant (n!)/e$. *7 pts*
Hint: use Taylor's formula centered around 0.

Problem 2.

(i) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Give the definition of the upper and lower Riemann sums of f with respect to some partition P of $[a, b]$. *6 pts*

(ii) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Give the definition of the upper and lower Riemann integrals of f over $[a, b]$ and prove that they are equal if and only if for all $\varepsilon > 0$ there exists a partition P of $[a, b]$ so that $U(f, P) - L(f, P) \leqslant \varepsilon$. *6 pts*

(iii) Let $f : [a, b] \to \mathbb{R}$ be continuous function. Prove that f is Riemann integrable. *6 pts*

(iv) Let $f : [a, b] \to \mathbb{R}$ be a differentiable function so that $f' : [a, b] \to \mathbb{R}$ is Riemann integrable. Prove that $\int_a^b f'(x)\mathrm{d}x = f(b) - f(a)$. *7 pts*

Problem 3.

(i) Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function and let $\lambda \in \mathbb{R}$ so that for all $a \leqslant c < d \leqslant b$ we have $\inf\{f(x) : x \in [c, d]\} \leqslant \lambda$. Prove $\displaystyle\int_a^b f(x)\mathrm{d}x \leqslant \lambda(b - a)$. *6 pts*

Hint: use lower Riemann sums.

(ii) Let $f : [a,b] \to \mathbb{R}$ be a **non-negative** Riemann integrable function and let $\lambda \in \mathbb{R}$ so that for there exist $a \leqslant c < d \leqslant b$ with $f(x) \geqslant \lambda$ for all $x \in [c,d]$. Prove that $\displaystyle\int_a^b f(x)\mathrm{d}x \geqslant \lambda(d-c)$ *Hint:* use lower Riemann sums. *6 pts*

(iii) Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. We define *7 pts*

$$M = \sup\{\inf\{|f(x)| : x \in [c, d]\} : a \leqslant c < d \leqslant b\}.$$

If $M > 0$, prove that for every $M > \varepsilon > 0$ there exists $r > 0$ so that for all $n \in \mathbb{N}$ we have

$$(M - \varepsilon)^n r \leqslant \int_a^b |f(x)|^n \mathrm{d}x \leqslant M^n (b - a).$$

(iv) Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Prove that *6 pts*

$$\lim_n \left(\int_a^b |f(x)|^n \mathrm{d}x \right)^{1/n} = \sup\{\inf\{|f(x)| : x \in [c, d]\} : a \leqslant c < d \leqslant b\}.$$

Problem 4.

(i) Let $L : [1, \infty) \to \mathbb{R}$ be the function defined by the rule

$$L(x) = \begin{cases} 0 & \text{if } x = 1 \\ \displaystyle\int_1^x \frac{1}{t}\mathrm{d}t & \text{if } x > 1. \end{cases}$$

Prove that for $n \in \mathbb{N}$ we have $L(x^n) = nL(x)$. *6 pts*

Hint: use the change of variable $\phi : [1, x] \to [1, x^n]$, $\phi(t) = t^n$.

(ii) Let s be a real number with $s \geqslant 1$. Use L'Hôpital's rule to evaluate *7 pts*

$$\lim_{x\to\infty} xL\left(1+\frac{s}{x}\right).$$

(iii) Let s be a real number with $s \geqslant 1$. Evaluate $\lim_{n} L\left(\left(1+\frac{s}{n}\right)^{n}\right)$. *6 pts*

(iv) Let s be a real number with $s \geqslant 1$. Use an appropriate continuity theorem and (iii) to show that $\lim_n \left(1 + \frac{s}{n}\right)^n$ exists in $\mathbb{R}$. *6 pts*

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