Fall 2016, Math 409, Section 502 **Second Midterm Exam (Practice)**

Friday, April 8 2016

Last name:

First name:

UIN:

Signature:

"An Aggie does not lie, cheat or steal or tolerate those who do."

This exam consists of **three** problems, the total point value of which is 100 points.

The answer to each question must be justified in detail.

The time length of this exam is 50 minutes.

The use of electronic devices, such as cellphones, tables, laptops, and calculators is prohibited.

Good luck!

Problem 1. (**four** questions)

(i) If $(x_n)_n$ is a real sequence, give the definition of $\limsup_n x_n$ and $\liminf_n x_n$.

(ii) Let E be a non-empty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ be a function. If $a \in E$, give the definition of continuity of f at a.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $(x_n)_n$ be a bounded real sequence. Prove that

$$(*) \qquad f(\limsup_n x_n) \leqslant \limsup_n f(x_n) \text{ and } \liminf_n f(x_n) \leqslant f(\liminf_n x_n).$$

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and assume that for every bounded sequence $(x_n)_n$ in $\mathbb{R}$ $(*)$ holds. Show that f is continuous.

Problem 2. (**four** questions)

(i) Let $f : [0, +\infty) \to \mathbb{R}$ be uniformly continuous. Prove that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ so that for all $x, y \in [0, +\infty)$ with $|x - y| \leqslant 1/n$, one has $|f(x) - f(y)| < \varepsilon$.

(ii) Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$ with $x \geqslant 1/n$.
Show that there exists $m \in \mathbb{N}$ with $m \leqslant nx$ and real numbers $0 = x_0 < x_1 < \cdots < x_m = x$ so that $|x_k - x_{k-1}| \leqslant 1/n$ for $k = 1, \ldots, m$.

(iii) Let $f : [0, +\infty) \to \mathbb{R}$ be a uniformly continuous function. Prove that for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $x > 1/n$, one has $|f(x) - f(0)| \leqslant (n\varepsilon)x$.

(iv) Let $f : [0, +\infty) \to \mathbb{R}$ be a uniformly continuous function. Prove that there exist positive constants C and M so that for all $x \geqslant 0$ one has $|f(x)| \leqslant Cx + M$.

Problem 3. (**four** questions)

(i) Let $f : [a, b] \to \mathbb{R}$ be a continuous function and assume that for some $x_0 \in [a, b]$, $f(x_0) > 0$. Prove that there exists $\delta > 0$, so that $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$.

(ii) Let $f : [a, b] \to \mathbb{R}$ a continuous function that is differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, show that f is constant.

(iii) Let $f : [a, b] \to \mathbb{R}$ be a continous function that is differentiable on (a, b). If $f'(x) > 0$ for all $x \in (a, b)$, show that f is stictly increasing.

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f' : \mathbb{R} \to \mathbb{R}$ is continuous, prove that there exist real numbers $a < b$, so that f restricted on $[a, b]$ is either constant or strictly monotone.