A METRIC INTERPRETATION OF REFLEXIVITY FOR BANACH SPACES

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Abstract

We define two metrics $d_{1,\alpha}$ and $d_{\infty,\alpha}$ on each Schreier family $\mathcal{S}_\alpha$, $\alpha < \omega_1$, with which we prove the following metric characterization of the reflexivity of a Banach space X: X is reflexive if and only if there is an $\alpha < \omega_1$ such that there is no mapping $\Phi : \mathcal{S}_\alpha \to X$ for which

$$c d_{\infty,\alpha}(A, B) \le \|\Phi(A) - \Phi(B)\| \le C d_{1,\alpha}(A, B) \quad \text{for all } A, B \in \mathcal{S}_\alpha.$$

Additionally we prove, for separable and reflexive Banach spaces X and certain countable ordinals α, that $\max(\mathrm{Sz}(X), \mathrm{Sz}(X^*)) \le \alpha$ if and only if $(\mathcal{S}_\alpha, d_{1,\alpha})$ does not bi-Lipschitzly embed into X. Here $\mathrm{Sz}(Y)$ denotes the Szlenk index of a Banach space Y.

Contents

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1. Introduction and statement of the main results

In this article we seek a metric characterization of the reflexivity of Banach spaces. By a metric characterization of a property of a Banach space we mean a characterization which refers only to the metric structure of that space but not its linear structure. In 1976 Ribe [31] showed that two Banach spaces that are uniformly homeomorphic have uniformly linearly isomorphic finite-dimensional subspaces. In particular, this means that the finite-dimensional or *local properties* of a Banach space are determined by its metric structure. Based on this result Bourgain [10] suggested the "Ribe program," which asks to find metric descriptions of finite-dimensional invariants of Banach spaces. Bourgain [10] proved the following characterization of superreflexivity: a Banach space X is superreflexive if and only if the binary trees B_n of height at most n, $n \in \mathbb{N}$, endowed with their graph metric, are not uniformly bi-Lipschitzly embedded into X. A binary tree of height at most n is the set $B_n = \bigcup_{k=0}^{n}\{-1,1\}^k$, with the *graph* or *shortest path metric* $d(\sigma,\sigma') = i + j - 2\max\{t \geq 0 : \sigma_s = \sigma'_s, s = 1,2,\dots,t\}$, for $\sigma = (\sigma_s)_{s=1}^{i} \neq \sigma' = (\sigma'_s)_{s=1}^{j}$ in $\bigcup_{k=0}^{n}\{-1,1\}^k$. A new and shorter proof of this result was recently obtained by Kloeckner [19]. Baudier [6] extended this result and proved that a Banach space X is superreflexive if and only if the infinite binary tree $B_\infty = \bigcup_{n=0}^{\infty}\{-1,1\}^n$ (with the graph distance) does not bi-Lipschitzly embed into X. Nowadays this result can be deduced from Bourgain's result and Ostrovskii's [27, Theorem 1.2], which states that a locally finite metric space A embeds bi-Lipschitzly into a Banach space X if all of its finite subsets uniformly bi-Lipschitzly embed into X. Johnson and Schechtman [17] characterized superflexivity, using the *diamond graphs*, D_n, $n \in \mathbb{N}$, and proved that a Banach space X is superreflexive if and only if the D_n, $n \in \mathbb{N}$ do not uniformly bi-Lipschitzly embed into X. There are several other local properties, that is, properties of the finite-dimensional subspaces of Banach spaces, for which metric characterizations have been found. The following are some examples: Bourgain, Milman, and Wolfson [11] characterized having nontrivial type by using *Hamming cubes* (the sets $\{-1,1\}^n$, together with the ℓ_1-norm), and Mendel and Naor [22], [23] presented metric characterizations of Banach spaces with type p, $1 < p \leq 2$, and cotype q, $2 \leq q < \infty$. For a more extensive account on the Ribe program we refer the reader to the survey articles [5] and [24] and the book [28].

Instead of only asking for metric characterizations of local properties, one can also ask for metric characterizations of other properties of Banach spaces, properties which might not be determined by the finite-dimensional subspaces. A result in this direction was obtained by Baudier, Kalton, and Lancien [7]. They showed that a reflexive Banach space X has a renorming which is *asymptotically uniformly convex* (AUC) and *asymptotically uniformly smooth* (AUS) if and only if the branching trees T_n, of length $n \in \mathbb{N}$, do not uniformly bi-Lipschitzly embed into X. Here

$T_n = \bigcup_{k=0}^n \mathbb{N}^k$, together with the graph metric, that is, $d(a,b) = i + j - \max\{t \geq 0 : a_s = b_s, s = 1,2,\dots,t\}$, for $a = (a_1,a_2,\dots,a_i) \neq b = (b_1,b_2,\dots,b_j)$ in T_n. Among the many equivalent conditions for a reflexive Banach space X to be AUC- and AUS-renormable (see [25]), one of them states that $\mathrm{Sz}(X) = \mathrm{Sz}(X^*) = \omega$, where $\mathrm{Sz}(Z)$ denotes the *Szlenk index* of a Banach space Z (see Section 5 for the definition and properties of the Szlenk index). Dilworth, Kutzarova, Lancien, and Randrianarivony [13, Theorem 6.3] showed that a separable Banach space X is reflexive and AUC- and AUS-renormable if and only if X admits an equivalent norm for which X has *Rolewicz's β-property*. According to [20] a Banach space X has Rolewicz's β-property if and only if

$$\bar{\beta}_X(t) = 1 - \sup\Big\{\inf\Big\{\frac{\|x - x_n\|}{2} : n \geq 1\Big\} : (x_n)_{n=1}^\infty \subset B_X, \mathrm{sep}\big[(x_n)\big] \geq t, x \in B_X\Big\} > 0,$$

for all $t > 0$, where $\mathrm{sep}[(z_n)] = \inf_{m \neq n} \|z_m - z_n\|$, for a sequence $(z_n) \subset X$. The function $\bar{\beta}_X$ is called the *β-modulus of X*. Using the equivalence between the positivity of the β-modulus and the property that a separable Banach space is reflexive and AUC- and AUS-renormable, Baudier and Zhang [8] were able to establish a new and shorter proof of the above-cited result from [7]. Metric descriptions of other nonlocal Banach space properties, for example, the Radon–Nikodým property, can be found in [29].

In our article we concentrate on metric descriptions of the property that a Banach space is reflexive and subclasses of reflexive Banach spaces. Ostrovskii [29] established a *submetric* characterization of reflexivity. Let T be the set of all pairs (x, y) in $\ell_1 \times \ell_1$ for which $\|x - y\|_1 \leq 2\|x - y\|_s$, where $\|\cdot\|_1$ denotes the usual norm on ℓ_1 and $\|\cdot\|_s$ denotes the *summing norm*, that is, $\|z\|_s = \sup_{k\in\mathbb{N}} |\sum_{j=1}^k z_j|$, for $z = (z_j) \in \ell_1$. Theorem 3.1 of [29] states that a Banach space X is not reflexive if and only if there are a map $f : \ell_1 \to X$ and a number $0 < c \leq 1$ such that $c\|x - y\|_1 \leq \|f(x) - f(y)\| \leq \|x - y\|_1$ for all $(x, y) \in T$. In Section 12 we will formulate a similar result, using a discrete subset of $\ell_1 \times \ell_1$, witnessing the same phenomena. Recently, Procházka [30, Theorem 3] formulated an interesting metric description of reflexivity. He constructed a uniformly discrete metric space M_R with the following properties. If M_R bi-Lipschitzly embeds into a Banach space X with distortion less than 2, then X is nonreflexive. The *distortion* of a bi-Lipschitz embedding f of one metric space into another is the product of the Lipschitz constant of f and the Lipschitz constant of f^{-1}. Conversely, if X is nonreflexive, then there exists a renorming $|\cdot|$ of X such that M_R embeds into $(X, |\cdot|)$ isometrically.

Our article has the goal of finding a metric characterization of reflexivity. An optimal result would be a statement, similar to Bourgain's result, of the form "all members of a certain family $(M_i)_{i\in I}$ of metric spaces embed uniformly bi-Lipschitzly into a space X if and only if X is not reflexive." In the language introduced by Ostrovskii [28], this would mean that $(M_i)_{i\in I}$ is a *family of test spaces for reflexivity*. Instead, our result will be of the form (see Theorem A below) "there is a family of sets $(M_i)_{i\in I}$, and for $i \in I$, there are metrics $d_{\infty,i}$ and $d_{1,i}$ on M_i with the property that a given space X is nonreflexive if and only if there are injections $\Phi_i : M_i \to X$ and $0 < c \leq 1$ such that $cd_{\infty,i}(x,y) \leq \|x-y\| \leq d_{1,i}(x,y)$, for all $x, y \in M_i$." In Section 12 we will discuss the difficulties in obtaining a characterization of reflexivity of the first form. Nevertheless, if we restrict ourselves to the class of reflexive spaces, we obtain a metric characterization for the *complexity* of a given space, which we measure by the Szlenk index, using test spaces. Roughly speaking, the higher the Szlenk index is of a given Banach space, the more averages of a given weakly null sequence that one has to take to obtain a norm null sequence. For a precise formulation of this statement we refer to Theorem 5.3. For the class of separable and reflexive spaces we will introduce an uncountable family of metric spaces $(M_\alpha)_{\alpha<\omega_1}$ for which we will show that the higher the complexity of a given reflexive and separable space X or its dual X^* is, the more members of $(M_\alpha)_{\alpha<\omega_1}$ can be uniformly bi-Lipschitzly embedded into X.

The definition of the *Schreier families* $\mathcal{S}_\alpha \subset [\mathbb{N}]^{<\omega}$, for $\alpha < \omega_1$, will be recalled in Section 2, the *Szlenk index* $\mathrm{Sz}(X)$ for a Banach space X will be defined in Section 5, and the two metrics $d_{1,\alpha}$ and $d_{\infty,\alpha}$ on $\mathcal{S}_\alpha$ will be defined in Section 7. The statements of our main results are as follows.

THEOREM A

A separable Banach space X is reflexive if and only if there is an $\alpha < \omega_1$ for which there does not exist a map $\Phi : \mathcal{S}_\alpha \to X$, with the property that for some numbers $C \geq c > 0$

$$cd_{\infty,\alpha}(A,B) \leq \big\|\Phi(A) - \Phi(B)\big\| \leq Cd_{1,\alpha}(A,B) \quad \textit{for all } A, B \in \mathcal{S}_\alpha. \tag{1}$$

Definition 1.1

Assume that X is a Banach space, $\alpha < \omega_1$, and $C \geq c > 0$. We call a map $\Phi : \mathcal{S}_\alpha \to X$ with the property that, for all $A, B \in \mathcal{S}_\alpha$,

$$cd_{\infty,\alpha}(A,B) \leq \big\|\Phi(A) - \Phi(B)\big\| \leq Cd_{1,\alpha}(A,B) \tag{2}$$

a *c-lower-$d_{\infty,\alpha}$ and C-upper-$d_{1,\alpha}$ embedding of $\mathcal{S}_\alpha$ into X*. If $\mathcal{A}$ is a subset of $\mathcal{S}_\alpha$ and $\Phi : \mathcal{A} \to X$ is a map which satisfies (2) for all $A, B \in \mathcal{A}$, we call it a *c-lower-$d_{\infty,\alpha}$ and C-upper-$d_{1,\alpha}$ embedding of $\mathcal{A}$ into X*.

Our next result extends one direction (the "easy direction") of [7, Main Result] to spaces with higher-order Szlenk indices. As in [7] reflexivity is not needed here.

THEOREM B

Assume that X is a separable Banach space and that $\max(\mathrm{Sz}(X), \mathrm{Sz}(X^*)) > \omega^\alpha$, *for some countable ordinal α. Then $(\mathcal{S}_\alpha, d_{1,\alpha})$ embeds bi-Lipschitzly into X and X^*.*

We will deduce one direction of Theorem A from James's [16] characterization of reflexive Banach spaces and show that for any nonreflexive Banach space X and any $\alpha < \omega_1$ there is a map $\Phi_\alpha : \mathcal{S}_\alpha \to X$ which satisfies (1). The converse will follow from the following result.

THEOREM C

Assume that X is a reflexive and separable Banach space. Let $\xi < \omega_1$, and put $\beta = \omega^{\omega^\xi}$. If for some numbers $C > c > 0$ there exist a c-lower-d_{∞,β^2} and C-upper-d_{1,β^2} embedding of $\mathcal{S}_{\beta^2}$ into X, then $\mathrm{Sz}(X) > \omega^\beta$ *or* $\mathrm{Sz}(X^*) \geq \beta$.

Theorem C and, thus, the missing part of Theorem A will be shown in Section 11 in Theorem 11.6. Combining Theorems B and C, we obtain the following characterization of certain bounds of the Szlenk index of X and its dual X^*. This result extends [7, Main Result] to separable and reflexive Banach spaces with higher-order Szlenk indices.

COROLLARY 1.2

Assume that $\omega < \alpha < \omega_1$ is an ordinal for which $\omega^\alpha = \alpha$. Then the following statements are equivalent for a separable and reflexive space X.

(a) $\max(\mathrm{Sz}(X), \mathrm{Sz}(X^*)) \leq \alpha$.

(b) $(\mathcal{S}_\alpha, d_{1,\alpha})$ *is not bi-Lipschitzly embeddable into X.*

Corollary 1.2 and a result in [26] yield the following corollary. We thank Christian Rosendal, who pointed it out to us.

COROLLARY 1.3

If $\alpha < \omega_1$ with $\alpha = \omega^\alpha$, then the class of separable and reflexive Banach spaces X for which $\max(\mathrm{Sz}(X), \mathrm{Sz}(X^*)) \leq \alpha$ *is Borel in the Effros–Borel structure of closed subspaces of* $C[0, 1]$.

A proof of Corollaries 1.2 and 1.3 will be presented at the end of Section 11. For the proof of our main results we will need to introduce some notation and to make several preliminary observations. The reader who is at first only interested in understanding our main results will only need the definition of the *Schreier families* $\mathcal{S}_\alpha$, $\alpha < \omega_1$, given in Section 2.2, the definition of *repeated averages* stated at the beginning of Section 3, and the definition of the two metrics $d_{1,\alpha}$ and $d_{\infty,\alpha}$ on $\mathcal{S}_\alpha$ introduced in Section 7.

2. Regular families, Schreier families, and fine Schreier families

In this section we first recall the definition of general *regular subfamilies of* $[\mathbb{N}]^{<\omega}$. Then we recall the definition of the *Schreier families* $\mathcal{S}_\alpha$ and the *fine Schreier families* $\mathcal{F}_{\beta,\alpha}$, $\alpha \le \beta < \omega_1$ (see [1]). The recursive definition of both families depends on choosing for every limit ordinal a sequence (α_n) which increases to α. To ensure that our proof will work out, we need (α_n) to satisfy certain conditions.

2.1. Regular families in $[\mathbb{N}]^{<\omega}$

For a set S we denote the subsets, the finite subsets, and the countably infinite subsets by $[S]$, $[S]^{<\omega}$, and $[S]^\omega$, respectively. We always write subsets of $\mathbb{N}$ in increasing order. Thus, if we write $A = \{a_1, a_2, \dots, a_n\} \in [\mathbb{N}]^{<\omega}$ or $A = \{a_1, a_2, \dots\} \in [\mathbb{N}]^\omega$, we always assume that $a_1 < a_2 < \cdots$. Identifying the elements of $[\mathbb{N}]$ in the usual way with elements of $\{0,1\}^\omega$, we consider on $[\mathbb{N}]$ the product topology of the discrete topology on $\{0,1\}$. Note that it follows for a sequence $(A_n) \subset [\mathbb{N}]^{<\omega}$ and $A \in [\mathbb{N}]^{<\omega}$ that (A_n) converges to A if and only if for all $k \ge \max A$ there is an m so that $A_n \cap [1,k] = A$, for all $n \ge m$.

For $A \in [\mathbb{N}]^{<\omega}$ and $B \in [\mathbb{N}]$ we write $A < B$ if $\max(A) < \min(B)$. As a matter of convention we put $\max(\emptyset) = 0$ and $\min(\emptyset) = \infty$, and thus, $A < \emptyset$ and $A > \emptyset$ is true for all $A \in [\mathbb{N}]^{<\omega}$. For $m \in \mathbb{N}$ we write $m \le A$ or $m < A$ if $m \le \min(A)$ or $m < \min(A)$, respectively.

We denote by $\preceq$ the partial order of *extension* on $[\mathbb{N}]^{<\omega}$; that is, $A = \{a_1, a_2, \dots, a_l\} \preceq B = \{b_1, b_2, \dots, b_m\}$ if $l \le m$ and $a_i = b_i$, for $i = 1, 2, \dots, l$, and we write $A \prec B$ if $A \preceq B$ and $A \ne B$. We say that $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is *closed under taking restrictions* if $A \in \mathcal{F}$ whenever $A \prec B$ and $B \in \mathcal{F}$ and is *hereditary* if $A \in \mathcal{F}$ whenever $A \subset B$ and $B \in \mathcal{F}$, and $\mathcal{F}$ is called *compact* if it is compact in the product topology. Note that a family which is closed under restrictions is compact if and only if it is *well founded*, that is, if it does not contain strictly ascending chains with respect to extensions. Given $n, a_1 < \cdots < a_n, b_1 < \cdots < b_n$ in $\mathbb{N}$ we say that $\{b_1, \dots, b_n\}$ is *a spread* of $\{a_1, \dots, a_n\}$ if $a_i \le b_i$ for $i = 1, \dots, n$. A family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is called *spreading* if every spread of every element of $\mathcal{F}$ is also in $\mathcal{F}$. We sometimes have to pass from a family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ to the subfamily $\mathcal{F} \cap [N]^{<\omega} = \{A \in \mathcal{F} : A \subset N\}$, where $N \subset \mathbb{N}$ is infinite. A second way to pass to a subfamilies is the following. Assume that $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ and $N = \{n_1, n_2, \dots\} \in [\mathbb{N}]^\omega$; then we call

$$\mathcal{F}^N = \big\{\{n_j : j \in A\} : A \in \mathcal{F}\big\}$$

the *spread of* $\mathcal{F}$ *onto* N.

A family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is called *regular* if it is hereditary, compact, and spreading. Note that if $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is compact, spreading, and closed under restriction, then it is also hereditary and thus regular. Indeed, if $B = \{b_1, b_2, \dots, b_l\} \in \mathcal{F}$ and $1 \le i_1 < i_2 < \cdots < i_k \le l$, then $A = \{b_{i_1}, b_{i_2}, \dots, b_{i_k}\}$ is a spread of $B' = \{b_1, b_2, \dots, b_k\}$, and since $B' \in \mathcal{F}$, it also follows that $A \in \mathcal{F}$.

If $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$, we denote the maximal elements of $\mathcal{F}$, that is, the elements $A \in \mathcal{F}$ for which there is no $B \in \mathcal{F}$ with $A \prec B$, by $\mathrm{MAX}(\mathcal{F})$. Note that if $\mathcal{F}$ is compact, then every element in $\mathcal{F}$ can be extended to a maximal element in $\mathcal{F}$.

For $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ and $A \in [\mathbb{N}]^{<\omega}$ we define

$$\mathcal{F}(A) = \big\{B \in [\mathbb{N}]^{<\omega} : A < B, A \cup B \in \mathcal{F}\big\}.$$

Note that if $\mathcal{F}$ is compact, spreading, closed under restrictions, or hereditary, then so is $\mathcal{F}(A)$.

If $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is compact, we denote by $\mathrm{CB}(\mathcal{F})$ its *Cantor–Bendixson* index, which is defined as follows. We first define the *derivative* of $\mathcal{F}$ by

$$\begin{aligned}\mathcal{F}' &= \big\{A \in \mathcal{F} : \exists (A_n) \subset \mathcal{F} \setminus \{A\}, A_n \to_{n\to\infty} A\big\} \\ &= \mathcal{F} \setminus \{A \in \mathcal{F} : A \text{ is isolated in } \mathcal{F}\}.\end{aligned}$$

Every maximal element A of $\mathcal{F}$ is not in $\mathcal{F}'$, and if $\mathcal{F}$ is spreading, then $\mathcal{F}' = \mathcal{F} \setminus \mathrm{MAX}(\mathcal{F})$. For $A \in [\mathbb{N}]^{<\omega}$ it easily follows that

$$\mathcal{F}'(A) = \big(\mathcal{F}(A)\big)'. \tag{3}$$

By transfinite induction we define for each ordinal α the *αth derivative of $\mathcal{F}$* by

$$\begin{aligned}&\mathcal{F}^{(0)} = \mathcal{F}, \\ &\mathcal{F}^{(\alpha)} = (\mathcal{F}^{(\gamma)})' \quad \text{if } \alpha = \gamma + 1, \qquad \text{and} \\ &\mathcal{F}^{(\alpha)} = \bigcap_{\gamma<\alpha} \mathcal{F}^{(\gamma)} \quad \text{if } \alpha \text{ is a limit ordinal.}\end{aligned}$$

It follows that $\mathcal{F}^{(\beta)} \subset \mathcal{F}^{(\alpha)}$ if $\alpha \le \beta$. By transfinite induction, (3) generalizes to

$$\mathcal{F}^{(\alpha)}(A) = \big(\mathcal{F}(A)\big)^{(\alpha)}, \quad \text{for all } A \in [\mathbb{N}]^{<\omega} \text{ and ordinal } \alpha. \tag{4}$$

Assume that $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is compact. Since $\mathcal{F}$ is countable and since every countable and compact metric space has isolated points, it follows that for some $\alpha < \omega_1$ the αth derivative of $\mathcal{F}$ is empty, and we define

$$\mathrm{CB}(\mathcal{F}) = \min\{\alpha : \mathcal{F}^{(\alpha)} = \emptyset\}.$$

Note that $\mathrm{CB}(\mathcal{F})$ is always a successor ordinal. Indeed, if $\alpha < \omega_1$ is a limit ordinal and $\mathcal{F}^{(\gamma)} \ne \emptyset$ for all $\gamma < \alpha$, then it follows that $\mathcal{F}^{(\alpha)} = \bigcap \mathcal{F}^{(\gamma)} \ne \emptyset$.

Definition 2.1
For $\mathcal{F}, \mathcal{G} \subset [\mathbb{N}]$ we define

$$\mathcal{F} \sqcup_< \mathcal{G} := \{A \cup B : A \in \mathcal{F}, B \in \mathcal{G}, \text{ and } A < B\}, \tag{5}$$

$$\mathcal{F}[\mathcal{G}] := \left\{ \bigcup_{i=1}^n B_i : \begin{array}{c} n \in \mathbb{N}, B_1 < B_2 < \cdots < B_n, B_i \in \mathcal{G}, i = 1, 2, \ldots, n, \\ \text{and } \{\min(B_i) : i = 1, 2, \ldots, n\} \in \mathcal{F} \end{array} \right\}. \tag{6}$$

It is not hard to see that if $\mathcal{F}$ and $\mathcal{G}$ are regular families, then so are $\mathcal{F} \sqcup_< \mathcal{G}$ and $\mathcal{F}[\mathcal{G}]$.

2.2. *The Schreier families*

We define the *Schreier families* $\mathcal{S}_\alpha \subset [\mathbb{N}]^{<\omega}$ by transfinite induction for all $\alpha < \omega_1$ as follows:

$$\mathcal{S}_0 = \big\{\{n\} : n \in \mathbb{N}\big\} \cup \{\emptyset\}; \tag{7}$$

if $\alpha = \gamma + 1$, we let

$$\begin{aligned} \mathcal{S}_\alpha &= \mathcal{S}_1[\mathcal{S}_\gamma] \\ &= \Big\{ \bigcup_{j=1}^n E_j : n \le \min(E_1), E_1 < E_2 < \cdots < E_n, E_j \in \mathcal{S}_\gamma, j = 1, 2, \ldots, n \Big\}; \end{aligned} \tag{8}$$

and if α is a limit ordinal, we choose a fixed sequence $(\lambda(\alpha, n) : n \in \mathbb{N}) \subset [1, \alpha)$ which increases to α and put

$$\mathcal{S}_\alpha = \big\{E : \exists k \le \min(E), \text{ with } E \in \mathcal{S}_{\lambda(\alpha,k)}\big\}. \tag{9}$$

An easy induction shows that $\mathcal{S}_\alpha$ is a hereditary, compact, and spreading family for all $\alpha < \omega_1$. It is not very hard to see by transfinite induction that $\mathcal{S}_\alpha$ is in the following very limited sense *backward spreading*:

$$\text{if } A = \{a_1, a_2, \ldots, a_n\} \in \mathcal{S}_\alpha, \quad \text{then } \{a_1, a_2, \ldots, a_{n-1}, a_n - 1\} \in S_\alpha. \tag{10}$$

So, in particular, if $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$ is not maximal, then $(A \cup \{k\})_{k > \max(A)} \subset \mathcal{S}_\alpha$.

Additionally, by transfinite induction we can easily prove that $\mathcal{S}_\alpha$ is "almost" increasing in α in the following sense.

PROPOSITION 2.2
For all ordinals $\alpha < \beta < \omega_1$, there is an $n \in \mathbb{N}$ such that

$$\mathcal{S}_\alpha \cap [n, \infty)^{<\omega} \subset \mathcal{S}_\beta.$$

The following formula for $\mathrm{CB}(\mathcal{S}_\alpha)$ is well known and can easily be shown by transfinite induction for all $\alpha < \omega_1$.

PROPOSITION 2.3
For $\alpha < \omega_1$ *we have* $\mathrm{CB}(\mathcal{S}_\alpha) = \omega^\alpha + 1$.

We now make further assumptions on the approximating sequence $(\lambda(\alpha, n)) \subset [1, \alpha)$ that we had chosen to define the Schreier family S_α, for limit ordinals $\alpha < \omega_1$. We will choose $(\lambda(\alpha, n))$ recursively. Assume that α is a countable limit ordinal and that we have defined $(\lambda(\gamma, n))$, for all limit ordinals $\gamma < \alpha$, and thus $\mathcal{S}_\gamma$ for all $\gamma < \alpha$.

Recall that α can be represented uniquely in its *Cantor normal form*

$$\alpha = \omega^{\xi_k} m_k + \omega^{\xi_{k-1}} m_{k-1} + \cdots + \omega^{\xi_1} m_1, \tag{11}$$

where $\xi_k > \xi_{k-1} > \cdots > \xi_1$, $m_k, m_{k-1}, \ldots, m_1 \in \mathbb{N}$, and since α is a limit ordinal, $\xi_1 \geq 1$.

We distinguish between three cases.

Case 1: $k \geq 2$ *or* $m_1 \geq 2$. In this case we put for $n \in \mathbb{N}$

$$\lambda(\alpha, n) = \omega^{\xi_k} m_k + \omega^{\xi_{k-1}} m_{k-1} + \cdots + \omega^{\xi_1}(m_1 - 1) + \lambda(\omega^{\xi_1}, n). \tag{12}$$

Before considering the next cases we need to make the following observation.

PROPOSITION 2.4
Assume that for all limit ordinals $\gamma \leq \alpha$ *satisfying Case 1 the approximating sequences* $(\lambda(\gamma, n) : n \in \mathbb{N})$ *satisfy the above condition (12). It follows for all* $\gamma \leq \alpha$, *with*

$$\gamma = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \cdots + \omega^{\xi_1} m_1$$

being the Cantor normal form, that

$$\begin{aligned} &\mathcal{S}_\gamma = \mathcal{S}_{\gamma_2}[\mathcal{S}_{\gamma_1}], \quad \textit{where for some } j = 1, \ldots, l, \\ &\gamma_1 = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \cdots + \omega^{\xi_j} m_j^{(1)} \qquad \textit{and} \\ &\gamma_2 = \omega^{\xi_j} m_j^{(2)} + \omega^{\xi_{j-1}} m_{j-1} + \cdots + \omega^{\xi_1} m_1, \\ &\quad \textit{with } m_j^{(1)}, m_j^{(2)} \in \mathbb{N} \cup \{0\}, \ m_j = m_j^{(1)} + m_j^{(2)}. \end{aligned} \tag{13}$$

Proof
We will show (13) by transfinite induction for all $\gamma \leq \alpha$. Assume that (13) holds for all $\tilde{\gamma} < \gamma$. If $\gamma = \omega^\xi$, then (13) is trivially satisfied. Indeed, in that case $\gamma = \gamma + 0 = 0 + \gamma$

are the only two choices for writing γ as the sum of two ordinals, and we observe that $\mathcal{S}_0[\mathcal{S}_\gamma] = \mathcal{S}_\gamma[\mathcal{S}_0] = \mathcal{S}_\gamma$.

It is left to verify (13) in the case in which $l \geq 2$ or $m_l \geq 2$. Let $\gamma = \gamma_1 + \gamma_2$ be a decomposition of γ as in the statement of (13). We can without loss of generality assume that $\gamma_2 > 0$.

If $\gamma_2 = \beta + 1$ for some β (which implies that γ itself is a successor ordinal), it follows from the induction hypothesis and (8) that $\mathcal{S}_{\gamma_1+\beta+1} = \mathcal{S}_1[\mathcal{S}_\beta[\mathcal{S}_{\gamma_1}]]$, so we need to show that

$$\mathcal{S}_1\big[\mathcal{S}_\beta[\mathcal{S}_{\gamma_1}]\big] = \mathcal{S}_{\beta+1}[\mathcal{S}_{\gamma_1}].$$

If $A \in \mathcal{S}_1[\mathcal{S}_\beta[\mathcal{S}_{\gamma_1}]]$, we can write A as $A = \bigcup_{i=1}^m A_i$ with $m \leq A_1 < A_2 < \cdots < A_n$ and $A_i \in \mathcal{S}_\beta[\mathcal{S}_{\gamma_1}]$, for $i = 1, \ldots, n$, which in turn means that $A_i = \bigcup_{j=1}^{l_i} A_{(i,j)}$, where $A_{(i,1)} \leq A_{(i,2)} \leq \cdots \leq A_{(i,l_i)}$, $A_{(i,j)} \in \mathcal{S}_{\gamma_1}$, for $j = 1, 2, \ldots, l_i$, and $\{\min A_{(i,j)} : j = 1, 2, \ldots, l_i\} \in \mathcal{S}_\beta$, for $i = 1, 2, \ldots, m$. This means that $\{\min A_{(i,j)} : j = 1, 2, \ldots, l_i, i = 1, 2, \ldots, m\}$ is in $\mathcal{S}_{\beta+1}$, and thus, we conclude that $A \in \mathcal{S}_{\beta+1}[\mathcal{S}_{\gamma_1}]$. Conversely, we can show in a similar way that $\mathcal{S}_{\beta+1}[\mathcal{S}_{\gamma_1}] \subset \mathcal{S}_1[\mathcal{S}_\beta[\mathcal{S}_{\gamma_1}]]$.

If γ_2 is a limit ordinal, we first observe that

$$\lambda(\gamma, n) = \lambda(\gamma_1 + \gamma_2, n) = \gamma_1 + \lambda(\gamma_2, n).$$

If $A \in \mathcal{S}_{\gamma_1+\gamma_2}$, then it follows that there is an $n \leq \min A$ such that, using the induction hypothesis, we have

$$A \in \mathcal{S}_{\gamma_1+\lambda(\gamma_2,n)} = \mathcal{S}_{\lambda(\gamma_2,n)}[\mathcal{S}_{\gamma_1}].$$

This means that $A = \bigcup_{j=1}^m A_j$ with $A_1 < A_2 < \cdots < A_m$, $\{\min(A_j) : j = 1, 2, \ldots, m\} \in \mathcal{S}_{\lambda(\gamma_2,n)}$, and $A_j \in \mathcal{S}_{\gamma_1}$, for $j = 1, 2, \ldots, m$. Since $n \leq \min(A) = \min(A_1)$, it follows that $\{\min(A_j) : j = 1, 2, \ldots, m\} \in \mathcal{S}_{\gamma_2}$ and, thus, that $A \in \mathcal{S}_{\gamma_2}[\mathcal{S}_{\gamma_1}]$. Conversely, we can similarly show that if $A \in \mathcal{S}_{\gamma_2}[\mathcal{S}_{\gamma_1}]$, then it follows that $A \in \mathcal{S}_{\gamma_1+\gamma_2}$. □

If Case 1 does not hold, α must be of the form $\alpha = \omega^\gamma$.

Case 2: $\alpha = \omega^{\omega^\kappa}$, for some $\kappa < \omega_1$. In this case we make the following requirement on the sequence $(\lambda(\alpha, n) : n \in \mathbb{N})$:

$$\mathcal{S}_{\lambda(\alpha,n)} \subset \mathcal{S}_{\lambda(\alpha,n+1)}, \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

We can ensure that (14) holds as follows: first choose any sequence $\lambda'(\alpha, n)$ which increases to α. Then we notice that Proposition 2.2 yields that for a fast enough increasing sequence $(l_n) \subset \mathbb{N}$, it follows that $\mathcal{S}_{\lambda'(\alpha,n)+l_n} \subset \mathcal{S}_{\lambda'(\alpha,n+1)+l_{n+1}}$. Indeed, we first note that the only set $A \in \mathcal{S}_\gamma$, $\gamma < \alpha$, which contains 1 must be the singleton

$A=\{1\}$. This follows easily by induction. Additionally, we note that by (8) it follows that $[\{2,3,\dots,n\}]\subset\mathcal{S}_{\gamma+n}$, for each $\gamma<\alpha$ and $n\in\mathbb{N}$, which yields our claim. Set $\lambda(\alpha,n)=\lambda'(\alpha,n)+l_n$.

The remaining case is the following.

Case 3: $\alpha=\omega^{\omega^\kappa+\xi}$, *where* $1\le\xi\le\omega^\kappa$. We first observe that in this case κ and ξ are uniquely defined.

LEMMA 2.5
Let α be an ordinal number such that there are ordinal numbers κ, ξ with $\xi\le\omega^\kappa$ and $\alpha=\omega^{\omega^\kappa+\xi}$. Then for every κ', ξ' with $\xi'\le\omega^{\kappa'}$ so that $\alpha=\omega^{\omega^{\kappa'}+\xi'}$, we have $\kappa=\kappa'$ and $\xi=\xi'$.

Proof
Let $\alpha=\omega^{\omega^\kappa+\xi}=\omega^{\omega^{\kappa'}+\xi'}$ be as above. By [33, Section 7.2, Theorem 41] $\omega^\kappa+\xi=\omega^{\kappa'}+\xi'$. If $\kappa'<\kappa$, then $\omega^{\kappa'}+\xi'\le\omega^{\kappa'}2<\omega^{\kappa'}\omega=\omega^{\kappa'+1}\le\omega^\kappa\le\omega^\kappa+\xi$, which is a contradiction. We conclude that $\kappa\le\kappa'$, and therefore, by interchanging the roles of κ and κ' we obtain that $\kappa=\kappa'$. In conclusion, $\omega^\kappa+\xi=\omega^\kappa+\xi'$, and therefore $\xi=\xi'$ as well. □

We now choose a sequence $(\theta(\xi,n))_n$ of ordinal numbers increasing to ω^ξ so that

$$\mathcal{S}_{\omega^{\omega^\kappa}\theta(\xi,n)}\subset\mathcal{S}_{\omega^{\omega^\kappa}\theta(\xi,n+1)},\tag{15}$$

and we define

$$\lambda(\alpha,n)=\omega^{\omega^\kappa}\theta(\xi,n).\tag{16}$$

We describe how (15) can be obtained. Start with an arbitrary sequence $(\theta'(\xi,n))_n$ increasing to ω^ξ. We shall recursively choose natural numbers $(k_n)_{n\in\mathbb{N}}$, so that by setting $\theta(\xi,n)=\theta'(\xi,n)+k_n$, (15) is satisfied. Assuming that $k_1,\dots,k_n$ have been chosen, choose k_{n+1} as in the argument yielding (14), so that

$$\mathcal{S}_{\omega^{\omega^\kappa}\theta(\xi,n)}\subset\mathcal{S}_{\omega^{\omega^\kappa}\theta'(\xi,n+1)+k_{n+1}}.$$

We will show that this k_{n+1} is the desired natural number, that is, that

$$\mathcal{S}_{\omega^{\omega^\kappa}\theta(\xi,n)}\subset\mathcal{S}_{\omega^{\omega^\kappa}(\theta'(\xi,n+1)+k_{n+1})}.$$

First note that, by using finite induction and Proposition (2.4), it is easy to verify that for $\gamma<\alpha$, with $\gamma=\omega^\xi$, for some $\xi<\omega_1$, and for $n\in\mathbb{N}$

$$\mathcal{S}_{\gamma\cdot n}=\underbrace{\mathcal{S}_\gamma\big[\mathcal{S}_\gamma\cdots\mathcal{S}_\gamma[\mathcal{S}_\gamma]\big]}_{n\text{ times}},\tag{17}$$

and thus,

$$
\begin{aligned}
&\mathcal{S}_{\omega^{\omega^\kappa}(\theta'(\xi,n+1)+k_{n+1})}\\
&\quad= \mathcal{S}_{\omega^{\omega^\kappa}\Theta'(\xi,n+1)+\omega^{\omega^\kappa}k_{n+1}} = \underbrace{\mathcal{S}_{\omega^{\omega^\kappa}}\big[\cdots\big[\mathcal{S}_{\omega^{\omega^\kappa}}}_{k_{n+1}\text{ times}}[\mathcal{S}_{\omega^{\omega^\kappa}\theta'(\xi,n+1)}]\big]\big]\\
&\quad\supset \underbrace{\mathcal{S}_1\big[\cdots\big[\mathcal{S}_1}_{k_{n+1}\text{ times}}[\mathcal{S}_{\omega^{\omega^\kappa}\theta'(\xi,n+1)}]\big]\big] = \mathcal{S}_{\omega^{\omega^\kappa}\theta'(\xi,n+1)+k_{n+1}} \supset \mathcal{S}_{\omega^{\omega^\kappa}\theta(\xi,n)}.
\end{aligned}
$$

We point out that the sequence $(\theta(\xi,n))_n$ also depends on α.

PROPOSITION 2.6

Assume that the approximating sequences $(\lambda(\alpha,n):n\in\mathbb{N})$ satisfy the above conditions for all limit ordinals α. It follows for all $\gamma<\omega_1$, with

$$\gamma=\omega^{\xi_l}m_l+\omega^{\xi_{l-1}}m_{l-1}+\cdots+\omega^{\xi_1}m_1$$

being the Cantor normal form, that

$$\mathcal{S}_{\lambda(\gamma,n)}\subset\mathcal{S}_{\lambda(\gamma,n+1)}\quad \textit{for all } n\in\mathbb{N} \textit{ if } \gamma \textit{ is a limit ordinal}; \tag{18}$$

$$
\begin{aligned}
&\mathcal{S}_\gamma=\mathcal{S}_{\gamma_2}[\mathcal{S}_{\gamma_1}],\quad \textit{where for some } j=1,2,\dots,l,\\
&\gamma_1=\omega^{\xi_l}m_l+\omega^{\xi_{l-1}}m_{l-1}+\cdots+\omega^{\xi_j}m_j^{(1)}\qquad \textit{and}\\
&\gamma_2=\omega^{\xi_j}m_j^{(2)}+\omega^{\xi_{j-1}}m_{j-1}+\cdots+\omega^{\xi_1}m_1,\\
&\quad \textit{with } m_j^{(1)},\ m_j^{(2)}\in\mathbb{N}\cup\{0\},\ m_j=m_j^{(1)}+m_j^{(2)};
\end{aligned}\tag{19}
$$

and if $\beta=\omega^{\omega^\kappa}$ and γ is a limit ordinal with $\gamma\le\beta$, then

$$\textit{there is a sequence } \big(\eta(\gamma,n)\big)_n \textit{ increasing to } \gamma \textit{ so that } \lambda(\beta\gamma,n)=\beta\eta(\gamma,n). \tag{20}$$

(This sequence $(\eta(\gamma,n))_n$ can depend on β.)

Proof

We first prove (18) and (19) simultaneously for all $\gamma<\omega_1$. Assume that our claim is true for all $\tilde\gamma<\gamma$. Then (19) follows from Proposition 2.4.

If $l=m_1=1$, we deduce (18) from the choice of $\lambda(\gamma,n)$, $n\in\mathbb{N}$ (see (14), (15), (16)). If $l\ge 2$ or $m_2\ge 2$, we deduce from (13) and the induction hypothesis that

$$
\begin{aligned}
\mathcal{S}_{\lambda(\gamma,n)}&=\mathcal{S}_{\omega^{\xi_k}m_k+\cdots+\omega^{\xi_2}m_2+\omega^{\xi_1}m_1+\lambda(\omega^{\xi_1},n)}\\
&=\mathcal{S}_{\lambda(\omega^{\xi_1},n)}[\mathcal{S}_{\omega^{\xi_k}m_k+\cdots+\omega^{\xi_2}m_2+\omega^{\xi_1}m_1}]\\
&\subset\mathcal{S}_{\lambda(\omega^{\xi_1},n+1)}[\mathcal{S}_{\omega^{\xi_k}m_k+\cdots+\omega^{\xi_2}m_2+\omega^{\xi_1}m_1}]=\mathcal{S}_{\lambda(\gamma,n+1)},
\end{aligned}
$$

which verifies (18) also in that case.

To verify (20) let $\kappa < \omega_1$ with $\beta = \omega^{\omega^\kappa} \geq \gamma$. Recall that by (16) $\lambda(\omega^{\omega^\kappa + \xi_1}, n) = \omega^{\omega^\kappa}\theta(\xi_1, n)$. For each n, define $\eta(\gamma, n) = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \cdots + \omega^{\xi_1}(m_1 - 1) + \theta(\xi_1, n)$. We will show that $(\eta(\gamma, n))_{n\in\mathbb{N}}$ has the desired property. Note that the Cantor normal form of $\beta\gamma$ is $\beta\gamma = \omega^{\omega^\kappa+\xi_l} m_l + \omega^{\omega^\kappa+\xi_{l-1}} m_{l-1} + \cdots + \omega^{\omega^\kappa+\xi_1} m_1$. Hence, by (12)

$$\begin{aligned}
\lambda(\beta\gamma, n) &= \omega^{\omega^\kappa+\xi_l} m_l + \omega^{\omega^\kappa+\xi_{l-1}} m_{l-1} + \cdots + \omega^{\omega^\kappa+\xi_1}(m_1 - 1) + \lambda(\omega^{\omega^\kappa+\xi_1}, n) \\
&= \omega^{\omega^\kappa+\xi_l} m_l + \omega^{\omega^\kappa+\xi_{l-1}} m_{l-1} + \cdots + \omega^{\omega^\kappa+\xi_1}(m_1 - 1) + \omega^{\omega^\kappa}\theta(\xi_1, n) \\
&= \omega^{\omega^\kappa}\big(\omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \cdots + \omega^{\xi_1}(m_1 - 1) + \theta(\xi_1, n)\big) \\
&= \beta\eta(\gamma, n). \qquad \square
\end{aligned}$$

Remark

The proof of Proposition 2.6 (in particular, the definition of $(\eta(\gamma, n))_n$) implies the following. Let ξ be a countable ordinal number, and let $\gamma \leq \beta = \omega^{\omega^\xi}$ be a limit ordinal number. If $\gamma = \omega^{\xi_1}$, then

$$\eta(\gamma, n) = \theta(\xi_1, n), \quad \text{for all } n \in \mathbb{N}. \tag{21}$$

Otherwise, if the Cantor normal form of γ is

$$\gamma = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \cdots + \omega^{\xi_1} m_1,$$

$\gamma_1 = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \cdots + \omega^{\xi_j} m_j^{(1)}$, and $\gamma_2 = \omega^{\xi_j} m_j^{(2)} + \omega^{\xi_{j-1}} m_{j-1} + \cdots + \omega^{\xi_1} m_1$, with $m_j^{(1)}, m_j^{(2)} \in \mathbb{N} \cup \{0\}$, $m_j = m_j^{(1)} + m_j^{(2)}$, then we have

$$\eta(\gamma, n) = \gamma_1 + \eta(\gamma_2, n), \quad \text{for all } n \in \mathbb{N}. \tag{22}$$

COROLLARY 2.7

If $\alpha < \omega_1$ is a limit ordinal, then it follows that

$$\mathcal{S}_\alpha = \big\{A \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\} : A \in \mathcal{S}_{\lambda(\alpha, \min(A))}\big\} \cup \{\emptyset\}. \tag{23}$$

Remark 2.8

If we had defined $\mathcal{S}_\alpha$ by (23) for limit ordinals $\alpha < \omega_1$, where $(\lambda(\alpha, n) : n \in \mathbb{N})$ is any sequence increasing to α, then we would not have ensured that the family $\mathcal{S}_\alpha$ is a regular family.

2.3. *The fine Schreier families*

We will now define the *fine Schreier sets*. For that we will also need to choose appropriate approximating sequences for limit ordinals. We will define the fine Schreier sets as a doubly indexed family $\mathcal{F}_{\beta,\alpha} \subset [\mathbb{N}]^{<\omega}$, $\alpha \le \beta < \omega_1$. Later in the proof of Theorems A and C, we will fix β, depending on the Banach space X that we are considering.

Definition 2.9
For a countable ordinal number ξ and $\beta = \omega^{\omega^{\xi}}$, we recursively define a hierarchy of families of finite subsets of the natural numbers $(\mathcal{F}_{\beta,\gamma})_{\gamma \le \beta}$ as follows:

(a) $\mathcal{F}_{\beta,0} = \{\emptyset\}$;

(b) if $\gamma < \beta$, then $\mathcal{F}_{\beta,\gamma+1} = \{\{n\} \cup F : F \in \mathcal{F}_{\beta,\gamma}, n \in \mathbb{N}\}$ (i.e., $\mathcal{F}_{\beta,\gamma+1} = \mathcal{F}_{\beta,1} \sqcup_{<} \mathcal{F}_{\beta,\gamma}$); and

(c) if $\gamma \le \beta$ is a limit ordinal number, then $\mathcal{F}_{\beta,\gamma} = \bigcup_{n\in\mathbb{N}} (\mathcal{F}_{\beta,\eta(\gamma,n)} \cap [n,\infty)^{<\omega})$, where $(\eta(\gamma,n))_n$ is the sequence provided by Proposition 2.6 (and it depends on β).

Remark
It can be easily shown by transfinite induction that each family $\mathcal{F}_{\beta,\gamma}$ is regular. In the literature, fine Schreier families are usually defined recursively as a singly indexed family $(\mathcal{F}_\alpha)_{\alpha<\omega_1}$ of subsets of $[\mathbb{N}]^{<\omega}$. In that case, $\mathcal{F}_0$ and $\mathcal{F}_\alpha$ are defined for successor ordinals as in Definitions 2.9(a) and 2.9(b). And if α is a limit ordinal $\mathcal{F}_\alpha$ is defined as in Definition 2.9(c), without assuming that the approximating sequence $(\eta(\alpha,n))_{n\in\mathbb{N}}$ depends on any $\beta \ge \alpha$.

Let ξ be a countable ordinal number, and let $\xi_1 \le \omega^{\xi}$. If $\beta = \omega^{\omega^{\xi}}$ and $\gamma = \omega^{\xi_1}$, then it follows by (21) that $\eta(\gamma,n) = \theta(\gamma,n)$ for $n \in \mathbb{N}$. The choice of $(\theta(\xi_1,n))_{n\in\mathbb{N}}$ may be done so that, along with (15), we also have

$$\mathcal{F}_{\beta,\eta(\gamma,n)} = \mathcal{F}_{\beta,\theta(\xi_1,n)} \subset \mathcal{F}_{\beta,\theta(\xi_1,n+1)} = \mathcal{F}_{\beta,\eta(\gamma,n+1)}. \tag{24}$$

This can be achieved by possibly adding to $\theta'(\xi_1,n)$ a large enough natural number.

The following observation can be shown in a similar way as Proposition 2.6. We omit the proof.

PROPOSITION 2.10
Let ξ be a countable ordinal number and $\beta = \omega^{\omega^{\xi}}$. Assume that for all limit ordinals $\gamma \le \beta$ the approximating sequence $(\eta(\gamma,n))_n$ satisfies conditions (21) and (22), and for the case $\gamma = \omega^{\xi_1}$ the approximating sequence $(\theta(\xi_1,n))_n$ satisfies condition (24). Then, for all $\gamma \le \beta$ whose Cantor normal form is

$$\gamma = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \cdots + \omega^{\xi_1} m_1,$$

we have that

$$\mathcal{F}_{\beta,\eta(\gamma,n)} \subset \mathcal{F}_{\beta,\eta(\gamma,n+1)} \quad \textit{for all } n \in \mathbb{N} \textit{ if } \gamma \textit{ is a limit ordinal,} \tag{25}$$

and if, for some $1 \leq j \leq l$, $\gamma_1 = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \cdots + \omega^{\xi_j} m_j^{(1)}$ *and* $\gamma_2 = \omega^{\xi_j} m_j^{(2)} + \omega^{\xi_{j-1}} m_{j-1} + \cdots + \omega^{\xi_1} m_1$ *with* $m_j^{(1)}, m_j^{(2)} \in \mathbb{N} \cup \{0\}$, $m_j = m_j^{(1)} + m_j^{(2)}$, *then*

$$\mathcal{F}_{\beta,\gamma} = \mathcal{F}_{\beta,\gamma_2} \sqcup_< \mathcal{F}_{\beta,\gamma_1}. \tag{26}$$

COROLLARY 2.11
Let ξ *be a countable ordinal number, and let* $\gamma \leq \beta = \omega^{\omega^\xi}$ *be a limit ordinal number. Then*

$$\mathcal{F}_{\beta,\gamma} = \left\{F \in [\mathbb{N}]^{<\omega} : F \in \mathcal{F}_{\beta,\eta(\gamma,\min(F))}\right\} \cup \{\emptyset\}. \tag{27}$$

The following formula of the Cantor–Bendixson index of $\mathcal{S}_\alpha$ and $\mathcal{F}_{\beta,\alpha}$ can be easily shown by transfinite induction.

PROPOSITION 2.12
For any $\alpha, \kappa < \omega_1$, *with* $\alpha \leq \beta = \omega^{\omega^\kappa}$,

$$\mathrm{CB}(\mathcal{S}_\alpha) = \omega^\alpha + 1 \qquad \textit{and} \qquad \mathrm{CB}(\mathcal{F}_{\beta,\alpha}) = \alpha + 1.$$

Moreover, by assuming $\omega^\alpha \leq \beta$, *for every* $M \in [\mathbb{N}]^\omega$, *there is an* $M \in [N]^\omega$ *such that*

$$\mathcal{S}_\alpha^N \subset \mathcal{F}_{\beta,\omega^\alpha} \qquad \textit{and} \qquad \mathcal{F}_{\beta,\omega^\alpha}^N \subset \mathcal{S}_\alpha.$$

The main result in [14] states that if $\mathcal{F}$ and $\mathcal{G}$ are two hereditary subsets of $[\mathbb{N}]$, then for any $M \in [\mathbb{N}]^\omega$ there is an $N \in [M]^\omega$ so that $\mathcal{F} \cap [N]^{<\omega} \subset \mathcal{G}$ or $\mathcal{G} \cap [N]^{<\omega} \subset \mathcal{F}$. Together with Proposition 2.12 this yields the following.

PROPOSITION 2.13
For $\alpha, \gamma, \kappa < \omega_1$, $\beta = \omega^{\omega^\kappa}$, *and any* $M \in [N]^{<\omega}$, *there is an* $N \in [M]^{<\omega}$ *so that*

$$\mathcal{S}_\alpha^N \subset \mathcal{S}_\alpha \cap [N]^{<\omega} \subset \mathcal{F}_{\beta,\gamma} \quad \textit{if } \omega^\alpha < \gamma \leq \beta, \qquad \textit{and}$$
$$\mathcal{F}_{\beta,\gamma}^N \subset \mathcal{F}_{\beta,\gamma} \cap [N]^{<\omega} \subset \mathcal{S}_\alpha \quad \textit{if } \gamma < \omega^\alpha \textit{ and } \gamma \leq \beta.$$

2.4. *Families indexed by subsets of* $[\mathbb{N}]^{<\omega}$

We consider families of the form $(x_A : A \in \mathcal{F})$ in some set X indexed over $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$. If $\mathcal{F}$ is a tree, that is, closed under restrictions, then such a family is called an *indexed tree*. Let us also assume that $\mathcal{F}$ is spreading. The passing to a *pruning* of such an indexed tree is what corresponds to passing to subsequences for sequences.

Formally speaking we define a pruning of $(x_A : A \in \mathcal{F})$ as follows. Let $\pi : \mathcal{F} \to \mathcal{F}$ be an order isomorphism with the property that if $F \in \mathcal{F}$ is not maximal, then for any $n \in \mathbb{N}$ such that $n > \max(A)$ and $A \cup \{n\} \in \mathcal{F}$, $\pi(A \cup \{n\})$ is of the form $\pi(A) \cup \{s_n\}$, where (s_n) is a sequence which increases with n. We then call the family $(x_A : A \in \pi(\mathcal{F}))$ a *pruning* of $(x_A : A \in \mathcal{F})$. Let $\tilde{x}_A = x_{\pi(A)}$ for $A \in \mathcal{F}$. Then $(\tilde{x}_A : A \in \mathcal{F})$ is simply a *relabeling* of the family $(x_A : A \in \pi(\mathcal{F}))$, and we call it also a pruning of $(x_A : A \in \mathcal{F})$. It is important to note that the branches of a pruning of an indexed tree $(x_A : A \in \mathcal{F})$ are a subset of the branches of the original tree $(x_A : A \in \mathcal{F})$. Here a branch of $(x_A : A \in \mathcal{F})$ is a set of the form

$$\overline{x}_F = (x_{\{a_1\}}, x_{\{a_1,a_2\}}, \dots, x_{\{a_1,a_2,\dots,a_l\}}) \quad \text{for } F = \{a_1, a_2, \dots, a_l\} \in \mathcal{F}.$$

Also the nodes of a pruned tree, namely, the sequences of the form $(\tilde{x}_{A\cup\{n\}} : A \cup \{n\} \in \mathcal{F})$, with $A \in \mathcal{F}$ not maximal, are subsequences of the nodes of the original tree.

Let us finally mention how we usually choose prunings inductively. Let $\{A_n : n \in \mathbb{N}\}$ be a *consistent enumeration* of $\mathcal{F}$. By this we mean that if $\max(A_m) < \max(A_n)$, then $m < n$. Thus, we also have that if $A_m \prec A_n$, then $m < n$, and if $A_m = A \cup \{s\} \in \mathcal{F}$ and $A_n = A \cup \{t\} \in \mathcal{F}$ for some (nonmaximal) $A \in \mathcal{F}$ and $s < t$ in $\mathbb{N}$, then $m < n$. Of course, $A_1 = \emptyset$ and $\pi(\emptyset) = \emptyset$, and assuming now that $\pi(A_1), \pi(A_2), \dots, \pi(A_m)$ have been chosen, A_{m+1} must be of the form $A_m = A_l \subset \{k\}$, with $l < m = 1$. Moreover, if $k > \max(A_l) + 1$ and if $A_l \cup \{k-1\} \in \mathcal{F}$, then $A_l \cup \{k-1\} = A_j$ with $l < j < m+1$, and $\pi(A_j) = \pi(A_l) \cup \{s\}$ for some s has already been chosen. Thus, we need to choose $\pi(A_{m+1})$ to be of the form $\pi(A_l) \cup \{t\}$, where, in the case in which $A_l \cup \{k-1\} \in \mathcal{F}$, we need to choose $t > s$.

The following well-known Ramsey-type result follows from [4, Corollary 2.5, Proposition 2.6].

PROPOSITION 2.14
Assume that $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is compact. Let $r \in \mathbb{N}$ and $f : \mathrm{MAX}(\mathcal{F}) \to \{1, 2, \dots, r\}$. Then for every $M \in [\mathbb{N}]^\omega$ there exist an $N \in [M]^\omega$ and an $i \in \{1, 2, \dots, r\}$ such that

$$\mathrm{MAX}(\mathcal{F}) \cap [N]^\omega \subset \{A \in \mathrm{MAX}(\mathcal{F}) : f(A) = i\}.$$

3. Repeated averages on Schreier sets

We recall *repeated averages* defined on maximal sets of S_α, $\alpha < \omega_1$ (cf. [3]). As in our previous sections we will assume that $\mathcal{S}_\alpha$ is recursively defined using the conditions given in Section 2.2. We first need the following characterization of maximal elements of $\mathcal{S}_\alpha$, $\alpha < \omega_1$, which can be easily proven by transfinite induction using Corollary 2.7 for the limit ordinal case.

PROPOSITION 3.1
Let $\alpha < \omega_1$.

(a) $A \in \mathrm{MAX}(\mathcal{S}_{\alpha+1})$ *if and only if* $A = \bigcup_{j=1}^{n} A_j$, *with* $n = \min(A_1)$ *and* $A_1 < A_2 < \cdots < A_n$ *being in* $\mathrm{MAX}(\mathcal{S}_\alpha)$. *In this case the* A_j*'s are unique.*

(b) *If* α *is a limit ordinal, then* $A \in \mathrm{MAX}(\mathcal{S}_\alpha)$ *if and only if* $A \in \mathrm{MAX}(\mathcal{S}_{\lambda(\alpha,\min(A))})$.

For each $\alpha < \omega_1$ and each $A \in \mathrm{MAX}(\mathcal{S}_\alpha)$ we will now introduce an element $z_{(\alpha,A)} \in S_{\ell_1^+}$ with $\mathrm{supp}(z_{(\alpha,A)}) = A$, which we will call the *repeated average of complexity* α *on* $A \in \mathrm{MAX}(\mathcal{S}_\alpha)$. If $\alpha = 0$, then $\mathrm{MAX}(\mathcal{S}_0)$ consists of singletons, and for $A = \{n\} \in \mathrm{MAX}(\mathcal{S}_\alpha)$ we put $z_{(0,\{n\})} = e_n$, the nth element of the unit basis of ℓ_1. Assume that for all $\gamma < \alpha$ and all $A \in \mathrm{MAX}(\mathcal{S}_\gamma)$ we have already introduced $z_{(\gamma,A)}$, which we write as $z_{(\gamma,A)} = \sum_{a\in A} z_{(\gamma,A)}(a)e_a$, with $z_{(\gamma,A)} > 0$ for all $a \in A$. If $\alpha = \gamma + 1$ for some $\gamma < \omega_1$ and if $A \in \mathrm{MAX}(\mathcal{S}_\alpha)$, then by Proposition 3.1(a) we write A in a unique way as $A = \bigcup_{j=1}^{n} A_j$, with $n = \min A$ and $A_1 < A_2 < \cdots < A_n$ being maximal in $\mathcal{S}_\gamma$. We then define

$$z_{(\alpha,A)} = \frac{1}{n}\sum_{j=1}^{n} z_{(\gamma,A_j)} = \frac{1}{n}\sum_{j=1}^{n}\sum_{a\in A_j} z_{(\gamma,A_j)}(a)e_a, \tag{28}$$

and thus,

$$z_{(\alpha,A)}(a) = \frac{1}{n} z_{(\gamma,A_j)}(a) \quad \text{for } j = 1,2,\ldots,n \text{ and } a \in A_j. \tag{29}$$

If α is a limit ordinal and $A \in \mathrm{MAX}(\mathcal{S}_\alpha)$, then by Corollary 2.7, $A \in \mathcal{S}_{\lambda(\alpha,\min(A))}$, and we put

$$z_{(\alpha,A)} = z_{(\lambda(\alpha,\min(A)),A)} = \sum_{a\in A} z_{(\lambda(\alpha,\min(A)),A)}(a)e_a. \tag{30}$$

The following result was, with slightly different notation, proved in [3].

LEMMA 3.2 ([3, Proposition 2.15])
For all $\varepsilon > 0$, *all* $\gamma < \alpha$, *and all* $M \in [\mathbb{N}]^\omega$, *there is an* $N = N(\gamma,\alpha,M,\varepsilon) \in [M]^\omega$ *such that* $\sum_{a\in A} z_{(\alpha,B)}(a) < \varepsilon$ *for all* $B \in \mathrm{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$ *and* $A \in \mathcal{S}_\gamma$.

The following proposition will be proved by transfinite induction.

PROPOSITION 3.3
Assume $\alpha < \omega_1$ *and* $A \in \mathcal{S}_\alpha$ *(not necessarily maximal). If* B_1, B_2 *are two extensions of* A *which are both maximal in* $\mathcal{S}_\alpha$, *then it follows that*

$$z_{(\alpha,B_1)}(a) = z_{(\alpha,B_2)}(a) \quad \textit{for all } a \in A.$$

Remark

Proposition 3.3 says the following: if $\alpha < \omega_1$ and $A = \{a_1, a_2, \dots, a_n\}$ is in $\text{MAX}(\mathcal{S}_\alpha)$, then $z_{(\alpha,A)}(a_1)$ only depends on a_1, $z_{(\alpha,A)}(a_2)$ only depends on a_1 and a_2, and so on.

Proof of Proposition 3.3

Our claim is trivial for $\alpha = 0$. Assume that $\alpha = \gamma + 1$ and that our claim is true for γ, and let $A \in \mathcal{S}_{\gamma+1}$. Without loss of generality $A \neq \emptyset$; otherwise, we would be done. Using Proposition 3.1, we can find an integer $1 \le l \le \min A$, sets $A_1, A_2, \dots, A_{l-1} \in \text{MAX}(S_\gamma)$, and $A_l \in S_\gamma$ (not necessarily maximal in S_γ) so that $A_1 < A_2 < \dots < A_l$ and $A = \bigcup_{j=1}^{l} A_j$. By Proposition 3.1, any extension of A to a maximal element in S_γ will then be of the form $B = \bigcup_{j=1}^{l} A_j \cup \bigcup_{j=l}^{\min(A)} B_j$, where $A_l < B_l < B_{l+1} < \dots < B_{\min(A)}$ (B_l may be empty, in which case $A_l < B_{l+1} < \dots < B_{\min(A)}$), so that $A_l \cup B_l$ and $B_{l+1}, \dots, B_{\min(A)}$ are in $\text{MAX}(\mathcal{S}_\gamma)$. No matter how we extend A to a maximal element B in $\mathcal{S}_{\gamma+1}$, the inductive formula (28) yields

$$z_{(\gamma+1,B)}(a) = \frac{1}{\min(A)} z_{(\gamma,A_j)}(a)$$

whenever for some $j = 1, 2, \dots, l-1$ we have $a \in A_j$.

In the case in which $a \in A_l$, then, by our induction hypothesis, $z_{\gamma,A_l \cup B_l}(a)$ does not depend on the choice of B_l, and

$$z_{(\gamma+1,B)}(a) = \frac{1}{n} z_{(\gamma,A_l \cup B_l)}(a) \quad \text{whenever } a \in A_l.$$

Thus, in both cases, the value of $z_{(\gamma+1,B)}(a)$ does not depend on how we extend A to a maximal element B in $\mathcal{S}_{\gamma+1}$.

If α is a limit ordinal and $A \in \mathcal{S}_\alpha$ is not maximal, then we also can assume that $A \neq \emptyset$, and thus, it follows from (23) in Corollary (2.7) that $A \in \mathcal{S}_{\lambda(\alpha,\min(A))}$. For any two extension B of A into a maximal set of $\text{MAX}(\mathcal{S}_\alpha)$, it follows from Proposition 3.1 that B is maximal in $\mathcal{S}_{\lambda(\alpha,\min(A))}$ and that $z_{(\alpha,B)} = z_{\lambda(\alpha,\min(A),B)}$. Thus, also in this case our claim follows from the induction hypothesis. $\square$

Using Proposition 3.3 we can consistently define $z_{(\alpha,A)} \in B_{\ell_1^+}$ for any $\alpha < \omega_1$ and any $A \in S_\alpha$ by

$$z_{(\alpha,A)} = \sum_{a \in A} z_{(\alpha,B)}(a) e_a, \quad \text{where } B \text{ is any maximal extension of } A \text{ in } \text{MAX}(\mathcal{S}_\alpha).$$

In particular, this implies the following recursive definition of $z_{(\alpha,A)}$. If $A \in \mathcal{S}_{\alpha+1} \setminus \{\emptyset\}$, then we can write A in a unique way as $A = \bigcup_{j=1}^{n} A_n$, where $A_1 < A_2 < \dots < A_n$, $A_j \in \text{MAX}(\mathcal{S}_\alpha)$, for $j = 1, 2, \dots, n-1$, and $A_n \in S_\alpha \setminus \{\emptyset\}$, and note that

$$z_{(\alpha+1,A)} = \frac{1}{\min(A)} \sum_{j=1}^{d} z_{(\alpha,A_j)}, \tag{31}$$

and if α is a limit ordinal, then

$$z_{(\alpha,A)} = z_{(\lambda(\alpha,\min(A)),A)}. \tag{32}$$

For $D \in \mathcal{S}_\alpha$ define $\zeta(\alpha, D) = z_{(\alpha,D)}(\max(D))$. For $A \in \mathcal{S}_\alpha$ it therefore follows that

$$z_{(\alpha,A)} = \sum_{D \preceq A} \zeta(\alpha, D) e_{\max(D)}.$$

We also put $\zeta(\alpha, \emptyset) = 0$ and $e_{\max(\emptyset)} = 0$.

By transfinite induction we can easily show the following estimate for $1 \leq \alpha < \omega_1$:

$$\zeta(\alpha, A) \leq \frac{1}{\min A}. \tag{33}$$

From Proposition 2.6 we deduce the following formula for $z_{(\alpha,A)}$.

PROPOSITION 3.4

Assume that $\alpha < \omega_1$ and that its Cantor normal form is

$$\alpha = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \cdots + \omega^{\xi_1} m_1.$$

Let $j = 1, 2, \ldots, l$ and $m_j^{(1)}, m_j^{(2)} \in \mathbb{N} \cup \{0\}$, with $m_j^{(1)} + m_j^{(2)} = m_j$. Put

$$\gamma_1 = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \cdots + \omega^{\xi_{j-1}} m_{j-1} + \omega^{\xi_j} m_j^{(1)},$$
$$\gamma_2 = \omega^{\xi_j} m_j^{(2)} + \omega^{\xi_{j-1}} m_{j-1} + \cdots + \omega^{\xi_1} m_1.$$

For $A \in \mathrm{MAX}(\mathcal{S}_\alpha)$ we use Proposition 2.6 and write $A = \bigcup_{j=1}^{n} A_j$, where $A_j \in \mathcal{S}_{\gamma_1}$ for $j = 1, 2, \ldots, n$, $A_1 < A_2 < \cdots < A_n$, and $B = \{\min(A_j) : j = 1, 2, \ldots, n\} \in \mathcal{S}_{\gamma_2}$. Then it follows that $A_j \in \mathrm{MAX}(\mathcal{S}_{\gamma_1})$, for $j = 1, 2, \ldots, n$, $B \in \mathrm{MAX}(\mathcal{S}_{\gamma_2})$, and

$$z_{(\alpha,A)} = \sum_{j=1}^{n} z_{(\gamma_2,B)}\big(\min(A_j)\big) z_{(\gamma_1,A_j)}. \tag{34}$$

In other words, if $\emptyset \prec D \preceq A$ and, thus, $D = \bigcup_{j=1}^{i-1} A_j \cup \tilde{A}_i$, for some $0 \leq i < n$, and $\emptyset \prec \tilde{A}_i \preceq A_i$, then

$$\zeta(\alpha, D) = \zeta\big(\gamma_2, \big\{\min(A_j) : j = 1, 2, \ldots, i\big\}\big) \cdot \zeta(\gamma_1, \tilde{A}_i). \tag{35}$$

Proof
We prove by transfinite induction for all $\beta < \omega_1$, with Cantor normal form

$$\beta = \omega^{\xi_j} m_j + \omega^{\xi_{j-1}} m_{j-1} + \cdots + \omega^{\xi_1} m_1,$$

the following.

CLAIM
If $\gamma < \omega_1$ has Cantor normal form

$$\gamma = \omega^{\xi_l} \tilde{m}_l + \omega^{\xi_{l-1}} \tilde{m}_{l-1} + \cdots + \omega^{\xi_j} \tilde{m}_j,$$

where $\tilde{m}_j$ could possibly be vanishing, and if $A = \bigcup_{i=1}^n A_i \in \mathrm{MAX}(\mathcal{S}_{\gamma+\beta}) = \mathrm{MAX}(\mathcal{S}_\beta[\mathcal{S}_\gamma])$, where $A_i \in \mathcal{S}_\gamma$ for $i = 1,2,\dots,n$, $A_1 < A_2 < \cdots < A_n$, and $B = \{\min(A_i) : i = 1,2,\dots,n\} \in \mathcal{S}_\beta$, then it follows that $A_i \in \mathrm{MAX}(\mathcal{S}_\gamma)$ for $i = 1,2,\dots,n$, $B \in \mathrm{MAX}(\mathcal{S}_{\gamma_1})$, and

$$z_{(\alpha,A)} = \sum_{i=1}^n z_{(\beta,B)}\big(\min(A_i)\big) z_{(\gamma,A_i)}. \tag{36}$$

For $\beta = 0$ the claim is trivial, and for $\beta = 1$, our claim follows from Proposition 3.1 and the definition of $z_{(\gamma+1,A)}$ for $A \in \mathrm{MAX}(\mathcal{S}_{\gamma+1})$.

Assume now that the claim is true for all $\tilde{\beta} < \beta$ and that $\gamma < \omega_1$ has the above form. Let $A = \bigcup_{i=1}^n A_i \in \mathrm{MAX}(\mathcal{S}_{\gamma+\beta})$, where $A_i \in \mathcal{S}_\gamma$ for $i = 1,2,\dots,n$, $A_1 < A_2 < \cdots < A_n$, and $B = \{\min(A_i) : i = 1,2,\dots,n\} \in \mathcal{S}_\beta$.

First we note that (10) implies that the A_i's are maximal in $\mathcal{S}_\gamma$. Indeed, for some $i_0 = 1,2,\dots,n$, if A_{i_0} is not maximal in $\mathcal{S}_\gamma$, then if $i_0 = n$, it would directly follow that A cannot be maximal in $\mathcal{S}_{\gamma+\beta}$, and if $i_0 < n$, we could define $\tilde{A}_i = A_i$, for $i = 1,2,\dots,l-1$, $\tilde{A}_{i_0} = A_{i_0} \cup \{\min(A_{i_0+1})\}$, $\tilde{A}_i = (A_i \cup \{\min(A_{i+1})\}) \setminus \{\min(A_i)\}$, for $i = i_0, i_0+1, \dots, l-1$, and $\tilde{A}_l = A_l \setminus \{\min A_l\}$. Then, by (10) and the fact that the Schreier families are spreading, $A = \bigcup_{i=1}^n \tilde{A}_i$ is also a decomposition of elements of $\mathcal{S}_\gamma$ with $\tilde{B} = \{\min(\tilde{A}_i) : i = 1,2,\dots,n\} \in \mathcal{S}_\beta$. But now $\tilde{A}_n$ is not maximal in $\mathcal{S}_\gamma$ and we again get a contradiction.

It is also easy to see that B is maximal in $\mathcal{S}_\beta$. To verify (36) we first assume that β is a successor ordinal, say, $\beta = \alpha + 1$. Then we can write B as $B = \bigcup_{i=1}^m B_i$, where $m = \min(B) = \min(A)$, $B_1 < B_2 < \cdots < B_m$, and $B_i \in \mathrm{MAX}(\mathcal{S}_\alpha)$, for $i = 1,2,\dots,m$. We can write B_i as $B_i = \{\min(A_s) : s = k_{i-1}+1, k_{i-1}+2, \dots, k_i\}$, with $k_0 = 0 < k_1 < \cdots < k_m = n$. We put $\overline{A}_i = \bigcup_{s=k_{i-1}+1}^{k_i} A_s \in \mathcal{S}_{\gamma+\alpha} = \mathcal{S}_\alpha[\mathcal{S}_\gamma]$, for $i = 1,2,\dots,m$. From the definition of $z_{(\beta+1,B)}$ and from the induction hypothesis we deduce now that

$$\begin{aligned}
z_{(\gamma+\alpha+1,A)} &= \frac{1}{m}\sum_{i=1}^{m} z_{(\gamma+\alpha,\overline{A}_i)} \\
&= \frac{1}{m}\sum_{i=1}^{m}\sum_{s=k_{i-1}+1}^{k_i} z_{(\alpha,B_i)}\big(\min(A_s)\big) z_{(\gamma,A_s)} \\
&= \sum_{s=1}^{n} z_{(\beta,B)}\big(\min(A_s)\big) z_{(\gamma,A_s)},
\end{aligned}$$

which proves the claim if β is a successor ordinal.

If β is a limit ordinal, it follows from Corollary 2.7, our definition of $z_{(\beta,B)}$ and $z_{(\gamma+\beta,A)}$, and our choice of the approximating sequence $(\lambda(\gamma+\beta),n)$ that

$$\begin{aligned}
z_{(\gamma+\beta,A)} &= z_{(\lambda(\gamma+\beta,\min(A)),A)} \\
&= z_{(\gamma+\lambda(\beta,\min(B)),A)} \\
&= \sum_{j=1}^{n} z_{(\lambda(\beta,\min(B)),B)}\big(\min(A_j)\big) z_{(\gamma,A_j)} \\
&= \sum_{j=1}^{n} z_{(\beta,B)}\big(\min(A_j)\big) z_{(\gamma,A_j)},
\end{aligned}$$

which also proves our claim in the limit ordinal case. □

If $\alpha < \omega_1$ and $A \in \mathrm{MAX}(\mathcal{S}_\alpha)$, then $z_{(\alpha,A)}$ is an element of $S_{\ell_1} \cap \ell_1^+$ and can therefore be seen as a probability on A. We denote the expected value of a function f defined on A or on all of $\mathbb{N}$ as $\mathbb{E}_{(\alpha,A)}(f)$. As done in [32], we deduce the following statement from Lemma 3.2.

COROLLARY 3.5 ([32, Corollary 4.10])
For each $\alpha < \omega_1$ and $A \in \mathrm{MAX}(\mathcal{S}_\alpha)$, let $f_A : A \to [-1,1]$ have the property that $\mathbb{E}_{\alpha,A}(f_A) \geq \rho$ for some fixed number $\rho \in [-1,1]$. For $\delta > 0$ and $M \in [\mathbb{N}]^\omega$ put

$$\mathcal{A}_{\delta,M} = \left\{ A \in \mathcal{S}_\alpha \cap [M]^{<\omega} : \begin{array}{c} \exists B \in \mathrm{MAX}(S_\alpha \cap [M]^{<\omega}), \\ A \subset B, \text{ and } f_B(a) \geq \rho - \delta \text{ for all } a \in A \end{array} \right\}.$$

Then $\mathrm{CB}(\mathcal{A}_{\delta,M}) = \omega^\alpha + 1$.

We finish this section with an observation, which will be needed later.

Definition 3.6
If $A \subset \mathbb{N} \setminus \{\emptyset\}$ is finite, we can write it in a unique way as a union $A = \bigcup_{j=1}^{d} A_j$,

where $A_1 < A_2 < \cdots < A_d$, $A_j \in \mathrm{MAX}(\mathcal{S}_1)$ if $j = 1, 2, \ldots, d-1$, and $A_d \in \mathcal{S}_1 \setminus \{\emptyset\}$. We call $(A_j)_{j=1}^d$ the *optimal $\mathcal{S}_1$-decomposition of A*, and we define

$$l_1(A) = \min(A_d) - \#A_d.$$

For $A = \emptyset$ we put $l_1(A) = 0$.

The significance of this number and its connection to the repeated averages is explained in the following lemma.

LEMMA 3.7
Let $\alpha \in [1, \omega_1)$, $A \in \mathcal{S}_\alpha$, and let $(A_j)_{j=1}^d$ be its optimal $\mathcal{S}_1$-decomposition.
(a) $l_1(A) = 0$ *if and only if* $A = \emptyset$ *or* $A_d \in \mathrm{MAX}(\mathcal{S}_1)$.
(b) *If* $A \in \mathrm{MAX}(\mathcal{S}_\alpha)$, *then* $A_d \in \mathrm{MAX}(\mathcal{S}_1)$ *and, thus,* $l_1(A) = 0$.
(c) *If* $l_1(A) > 0$, *then for all* $\max(A) < k_1 < k_2 < \cdots < k_{l_1(A)}$ *it follows that* $A \cup \{k_1, k_2, \ldots, k_{l_1(A)}\} \in \mathcal{S}_\alpha$ *and*

$$\zeta\big(\alpha, A \cup \{k_1, k_2, \ldots, k_i\}\big) = \zeta(\alpha, A) \quad \text{for all } i = 1, 2, \ldots, l_1(A).$$

(d) *If* $m > l_1(A)$ *and* $\max(A) < k_1 < k_2 < \cdots < k_m$ *have the property that* $A \cup \{k_1, k_2, \ldots, k_m\} \in \mathcal{S}_\alpha$, *then*

$$\zeta\big(\alpha, A \cup \{k_1, k_2, \ldots, k_i\}\big) \le \frac{1}{k_{l_1(A)+1}}.$$

(e) *If* $A \neq \emptyset$, *then*

$$\sum_{D \preceq A, l_1(D')=0} \zeta(\alpha, D) \le \frac{1}{\min(A)} \qquad \text{and}$$

$$\sum_{D \preceq A, l_1(D)=0} \zeta(\alpha, D) \le \frac{1}{\min(A)}.$$

(Recall that $D' = D \setminus \{\max D\}$ for $D \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$ and $\emptyset' = \emptyset$.)

Proof
We prove (a)–(e) by transfinite induction for all $\alpha \in [1, \omega_1)$. For $\alpha = 1$, (a), (b), (c), and (e) follow from the definition of $\mathcal{S}_1$ and the definition of $\zeta(\alpha, A)$, for $A \in \mathcal{S}_1$, while (d) is vacuous in that case. Assume that our claim is true for some $\alpha < \omega_1$, and let $A \in \mathcal{S}_{\alpha+1}$. Without loss of generality we can assume that $A \neq \emptyset$. Indeed, if $A = \emptyset$, then (a) is clear, (b), (c), and (e) are vacuous, and (d) follows easily by induction from the fact that always $\zeta(\alpha, A) \le \frac{1}{\min(A)}$ if $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$. By the definition of $\mathcal{S}_{\alpha+1}$, A can be written in a unique way as $A = \bigcup_{j=1}^n B_j$, where $B_j \in \mathrm{MAX}(\mathcal{S}_\alpha)$,

for $j = 1, 2, \dots, n-1$, and $B_n \in \mathcal{S}_\alpha$. For $j = 1, 2, \dots, n$ let $(A_{j,i})_{i=1}^{c_j}$ be the optimal $\mathcal{S}_1$-decomposition of B_j. From the induction hypothesis for (b), it follows that $(A_{j,i})$ are maximal in $\mathcal{S}_1$, for $j < n$ or for $j = n$ and $i < c_n$. Therefore, it follows that $(A_{j,i} : j = 1, 2, \dots, n, i = 1, 2, \dots, c_j)$ (appropriately ordered) is the optimal $\mathcal{S}_1$-decomposition of A, and it follows that $l_1(A) = l_1(B_n)$ and $A_d = A_{n,c_n}$.

We can deduce (a) from the induction hypothesis. If $A \in \mathrm{MAX}(S_{\alpha+1})$, then, in particular, $B_n \in \mathrm{MAX}(S_\alpha)$ and, thus, $l_1(A) = l_1(B_n) = 0$. Conversely, if $l_1(A) = l_1(B_n) = 0$, then $A_d = A_{n,c_n} \in \mathrm{MAX}(S_1)$. This proves (b) for $\alpha + 1$.

If $l_1(A) > 0$ and $\max(A) = \max(B_n) < k_1 < k_2 < \cdots < k_{l_1(A)}$, then it follows from the fact that $l_1(A) = l_1(B_n)$ and our induction hypothesis that $B_n \cup \{k_1, k_2, \dots, k_{l_1(A)}\} \in \mathcal{S}_\alpha$ and

$$\zeta\big(\alpha, B_n \cup \{k_1, k_2, \dots, k_i\}\big) = \zeta(\alpha, B_n).$$

Therefore, $A \cup \{k_1, k_2, \dots, k_{l_1(A)}\} \in \mathcal{S}_{\alpha+1}$, and using our recursive formula, we obtain

$$\begin{aligned}\zeta\big(\alpha, A \cup \{k_1, k_2, \dots, k_i\}\big) &= \frac{1}{\min(A)}\zeta\big(\alpha, B_n \cup \{k_1, k_2, \dots, k_i\}\big)\\ &= \frac{1}{\min(A)}\zeta(\alpha, B_n) = \zeta(\alpha, A),\end{aligned}$$

which verifies (c). To show (d), let $m > l_1(A)$ and $\max(A) < k_1 < k_2 < \cdots < k_m$ be such that $A \cup \{k_1, k_2, \dots, k_m\} \in \mathcal{S}_{\alpha+1}$. We distinguish between two cases. First we consider $B_n \cup \{k_1, k_2, \dots, k_m\} \in \mathcal{S}_\alpha$. In that case we deduce from the induction hypothesis that

$$\begin{aligned}\zeta\big(\alpha, A \cup \{k_1, k_2, \dots, k_m\}\big) &= \frac{1}{\min(A)}\zeta\big(\alpha, B_n \cup \{k_1, k_2, \dots, k_m\}\big)\\ &\le \frac{1}{k_{l_1(A)+1}}.\end{aligned}$$

For the second case, we can write $A \cup \{k_1, k_2, \dots, k_m\}$ as $A \cup \{k_1, k_2, \dots, k_m\} = \bigcup_{j=1}^n B_j \cup \bigcup_{j=n}^p B'_j$, where $p > n$, $B_n < B'_n < B'_{n+1} < \cdots < B'_p$, $B_n \cup B'_n \in \mathrm{MAX}(\mathcal{S}_\alpha)$, $B'_{n+1}, \dots, B'_{p-1} \in \mathrm{MAX}(\mathcal{S}_\alpha)$, and $B'_p \in \mathcal{S}_\alpha \setminus \{\emptyset\}$. Let $s \le m$ such that $k_s = \min(B_p)$. Then $s > l_1(B_n)$, and $l_1(B_n \cup B'_n) = 0$. It follows therefore from (31) and the induction hypothesis that

$$\begin{aligned}\zeta\big(\alpha+1, A \cup \{k_1, k_2, \dots, k_m\}\big) &= \frac{1}{\min(A)}\zeta\big(\alpha, \{k_s, k_{s+1}, \dots, k_m\}\big)\\ &\le \frac{1}{k_s} \le \frac{1}{k_{l_1(A)+1}}.\end{aligned}$$

This proves (d) in both cases.

Finally, to verify (e) we observe by the induction hypothesis and (31) that

$$\begin{aligned}\sum_{D\preceq A, l_1(D')=0} \zeta(\alpha+1, D) &= \frac{1}{\min(A)} \sum_{j=1}^{n} \sum_{D\preceq B_j, l_1(D')=0} \zeta(\alpha, D)\\ &\le \frac{1}{\min A} \sum_{j=1}^{n} \frac{1}{\min(B_j)}\\ &\le \frac{n}{\min(A)} \frac{1}{\min(A)} \le \frac{1}{\min(A)},\end{aligned}$$

which proves the first part of (e), while the second follows in the same way.

If $\alpha < \omega_1$ is a limit ordinal and assuming that our claim is true for all $\gamma < \alpha$, we proceed as follows. For $A \in \mathcal{S}_\alpha$, we can again assume that $A \neq \emptyset$, and it follows from Corollary 2.7 that $A \in \mathcal{S}_{\lambda(\alpha,\min(A))}$ and, by Proposition 3.1, A is maximal in $\mathcal{S}_\alpha$ if and only if it is maximal in $\mathcal{S}_{\lambda(\alpha,\min(A))}$. Therefore, (a)–(e) follow from our claim being true for $\lambda(\alpha, \min(A))$. ☐

Remark

Recall that if $\beta = \omega^{\omega^\xi}$ is a countable ordinal number and $\gamma < \beta$, then by (17) we have $\mathcal{S}_{\beta(\gamma+1)} = \mathcal{S}_\beta[\mathcal{S}_{\beta\gamma}]$. An argument very similar to what was used in the proof of Lemma 3.7 implies the following: if $B_1 < \cdots < B_d$ are in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ so that $\bar{B} = \{\min(B_j) : 1 \le j \le d\}$ is a nonmaximal $\mathcal{S}_\beta$-set, $D = \bigcup_{j=1}^d B_j$, and $C \in \mathcal{S}_{\beta\gamma}$ with $D < C$, then

$$l_1(C) = l_1(D \cup C). \tag{37}$$

COROLLARY 3.8

Let $A = \{a_1, a_2, \ldots, a_l\}$ and $\tilde{A} = \{\tilde{a}_1, \ldots, \tilde{a}_{\tilde{l}}\}$ be two sets in $[\mathbb{N}]^{<\omega}$ whose optimal $\mathcal{S}_1$-decompositions $(A_j)_{j=1}^d$ and $(\tilde{A}_j)_{j=1}^d$, respectively, have the same length and satisfy $\min(A_j) = \min(\tilde{A}_j)$, for $j = 1, 2, \ldots, d$. Then it follows for $\alpha < \omega_1$ that $A \in \mathcal{S}_\alpha$ if and only if $\tilde{A} \in \mathcal{S}_\alpha$, and in the case in which $D \preceq A$ and $\tilde{D} \preceq \tilde{A}$, with $\#D = \#\tilde{D}$, it follows that $\zeta(\alpha, D) = \zeta(\alpha, \tilde{D})$.

Proof

We prove this lemma by transfinite induction on α. If $\alpha = 1$, then $A_1 = A$, $\tilde{A}_1 = \tilde{A}$, and $a_1 = \tilde{a}_1$, and thus $\zeta(1, D) = \zeta(1, \tilde{D})$ for all $D \preceq A$ and $\tilde{D} \prec \tilde{A}$.

Assume that the conclusion holds for some α, and let $A \in \mathcal{S}_{\alpha+1}$, $\tilde{A} \in [\mathbb{N}]^{<\omega}$ satisfy the assumption. Let $A = \bigcup_{i=1}^p C_i$, where $C_1 < \cdots < C_{p-1}$ are in $\mathrm{MAX}(\mathcal{S}_\alpha)$, whereas $C_p \in \mathcal{S}_\alpha$ and $p \le \min(A)$. Write also $\tilde{A}$ as $\tilde{A} = \bigcup_{i=1}^p \tilde{C}_i$, where $\tilde{C}_1 < \cdots < \tilde{C}_p$, and choose the $\tilde{C}_j$'s such that $\#C_j = \#\tilde{C}_j$ for $j = 1, 2, \ldots, p-1$.

From Lemma 3.7(b) it follows that for some sequence $0 = d_0 < d_1 < d_2 < \cdots < d_p = d$ the sequence $(A_j)_{j=d_{i-1}+1}^{d_i}$ is the optimal $\mathcal{S}_1$-decomposition of C_i for $i = 1, 2, \ldots, p$. Now we can first deduce that $(\tilde{A}_j)_{j=1}^{d_1}$ is the optimal $\mathcal{S}_1$-decomposition of $\tilde{C}_1$, then deduce that $(\tilde{A}_j)_{j=d_1+1}^{d_2}$ is the optimal $\mathcal{S}_1$-decomposition of $\tilde{C}_2$, and so on. We are therefore in a position to apply the induction hypothesis and deduce that, for all $i = 1, 2, \ldots, p$, $D \preceq C_i$, and $\tilde{D} \preceq \tilde{C}_i$, it follows that $\zeta(\alpha, D) = \zeta(\alpha, \tilde{D})$. Our claim follows therefore from our recursive formula (31).

As usual in the case in which α is a limit ordinal, the verification follows easily from the definition of $\mathcal{S}_\alpha$. □

LEMMA 3.9
Let X be a Banach space, α be a countable ordinal number, $B \in \mathrm{MAX}(\mathcal{S}_\alpha)$, and $(x_A)_{A \preceq B}$ be vectors in B_X. Then

$$\Big\| \sum_{A \preceq B} \zeta(\alpha, A) x_A - \sum_{A \preceq B} \zeta(\alpha, A') x_A \Big\| \le \frac{2}{\min(B)}. \tag{38}$$

Proof
Using Lemma 3.7(c) and then Lemma 3.7(e) we obtain

$$\begin{aligned}
&\Big\| \sum_{A \preceq B} \zeta(\alpha, A) x_A - \sum_{A \preceq B} \zeta(\alpha, A') x_A \Big\| \\
&\quad \le \Big\| \sum_{\substack{A \preceq B \\ l_1(A') \neq 0}} \big(\zeta(\alpha, A) - \zeta(\alpha, A')\big) x_A \Big\| \\
&\qquad + \Big\| \sum_{\substack{A \preceq B \\ l_1(A') = 0}} \zeta(\alpha, A) x_A \Big\| + \Big\| \sum_{\substack{A \preceq B \\ l_1(A') = 0}} \zeta(\alpha, A') x_A \Big\| \\
&\quad \le \sum_{\substack{A \preceq B \\ l_1(A') = 0}} \zeta(\alpha, A) + \sum_{\substack{A \preceq B \\ l_1(A') = 0}} \zeta(\alpha, A') \\
&\quad \le \frac{2}{\min(B)}.
\end{aligned}$$
□

4. Trees and their indices

Let X be an arbitrary set. We set $X^{<\omega} = \bigcup_{n=0}^\infty X^n$, the set of all finite sequences in X, which includes the sequence of length zero denoted by $\emptyset$. For $x \in X$ we shall write x instead of (x); that is, we identify X with sequences of length 1 in X. A *tree on X* is a nonempty subset $\mathcal{F}$ of $X^{<\omega}$ closed under taking initial segments: if $(x_1,\dots,x_n)\in\mathcal{F}$ and $0\le m\le n$, then $(x_1,\dots,x_m)\in\mathcal{F}$. A tree $\mathcal{F}$ on X is *hereditary* if every subsequence of every member of $\mathcal{F}$ is also in $\mathcal{F}$.

Given $\overline{x}=(x_1,\dots,x_m)$ and $\overline{y}=(y_1,\dots,y_n)$ in $X^{<\omega}$, we write $(\overline{x},\overline{y})$ for the concatenation of $\overline{x}$ and $\overline{y}$:

$$(\overline{x},\overline{y})=(x_1,\dots,x_m,y_1,\dots,y_n).$$

Given $\mathcal{F}\subset X^{<\omega}$ and $\overline{x}\in X^{<\omega}$, we let

$$\mathcal{F}(\overline{x})=\big\{\overline{y}\in X^{<\omega}:(\overline{x},\overline{y})\in\mathcal{F}\big\}.$$

Note that if $\mathcal{F}$ is a tree on X, then so is $\mathcal{F}(\overline{x})$ (unless it is empty). Moreover, if $\mathcal{F}$ is hereditary, then so is $\mathcal{F}(\overline{x})$ and $\mathcal{F}(\overline{x})\subset\mathcal{F}$.

Let X^ω denote the set of all (infinite) sequences in X. Fix $S\subset X^\omega$. For a subset $\mathcal{F}$ of $X^{<\omega}$ the *S-derivative $\mathcal{F}'_S$ of $\mathcal{F}$* consists of all $\overline{x}=(x_1,x_2,\dots,x_l)\in X^{<\omega}$ for which there is a sequence $(y_i)_{i=1}^\infty\in S$ with $(\overline{x},y_i)\in\mathcal{F}$ for all $i\in\mathbb{N}$.

Note that if $\mathcal{F}$ is a hereditary tree, then it follows that $\mathcal{F}'_S\subset\mathcal{F}$ and that $\mathcal{F}'_S$ is also a hereditary tree (unless it is empty).

We then define higher-order derivatives $\mathcal{F}_S^{(\alpha)}$ for ordinals $\alpha<\omega_1$ by recursion as follows:

$$\mathcal{F}_S^{(0)}=\mathcal{F},\qquad cF_S^{(\alpha+1)}=(\mathcal{F}_S^{(\alpha)})'_S\quad\text{for }\alpha<\omega_1,\qquad\text{and}$$
$$\mathcal{F}_S^{(\lambda)}=\bigcap_{\alpha<\lambda}\mathcal{F}_S^{(\alpha)}\quad\text{for limit ordinals }\lambda<\omega_1.$$

It is clear that $\mathcal{F}_S^{(\alpha)}\supset\mathcal{F}_S^{(\beta)}$ if $\alpha\le\beta$ and that $\mathcal{F}_S^{(\alpha)}$ is a hereditary tree (or the empty set) for all α whenever $\mathcal{F}$ is a hereditary tree. An easy induction also shows that

$$\big(\mathcal{F}(\overline{x})\big)_S^{(\alpha)}=(\mathcal{F}_S^{(\alpha)})(\overline{x})\quad\text{for all }\overline{x}\in X^{<\omega},\alpha<\omega_1.$$

We now define the *S-index $I_S(\mathcal{F})$ of $\mathcal{F}$* by

$$I_S(\mathcal{F})=\min\{\alpha<\omega_1:\mathcal{F}_S^{(\alpha)}=\emptyset\}$$

if there exists $\alpha<\omega_1$ with $\mathcal{F}_S^{(\alpha)}=\emptyset$, and by $I_S(\mathcal{F})=\omega_1$ otherwise.

Remark
If λ is a limit ordinal and $\mathcal{F}_S^{(\alpha)}\neq\emptyset$ for all $\alpha<\lambda$, then, in particular, $\emptyset\in\mathcal{F}_S^{(\alpha)}$ for all $\alpha<\lambda$, and hence $\mathcal{F}_S^{(\lambda)}\neq\emptyset$. This shows that $I_S(\mathcal{F})$ is always a successor ordinal.

Examples 4.1
(a) A family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ can be thought of as a tree on $\mathbb{N}$: a set $F = \{m_1, \dots, m_k\} \in [\mathbb{N}]^{<\omega}$ is identified with $(m_1, \dots, m_k) \in \mathbb{N}^{<\omega}$. (Recall that $m_1 < \cdots < m_k$ by our convention of always listing the elements of a subset of $\mathbb{N}$ in increasing order.)

Let S be the set of all strictly increasing sequences in $\mathbb{N}$. In this case the S-index of a compact family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is nothing else but the Cantor–Bendixson index of $\mathcal{F}$ as a compact topological space, which we will continue to denote by $\mathrm{CB}(\mathcal{F})$. We will also use the term Cantor–Bendixson derivative instead of S-derivative and use the notation $\mathcal{F}'$ and $\mathcal{F}^{(\alpha)}$.

(b) If X is an arbitrary set and $S = X^\omega$, then the S-index of a tree $\mathcal{F}$ on X is what is usually called the *order of* $\mathcal{F}$ (or the *height of* $\mathcal{F}$) and is denoted by $\mathrm{o}(\mathcal{F})$. Note that in this case the S-derivative of $\mathcal{F}$ consists of all finite sequences $\overline{x} \in X^{<\omega}$ for which there exists $y \in X$ such that $(\overline{x}, y) \in \mathcal{F}$. The function $\mathrm{o}(\cdot)$ is the largest index: for any $S \subset X^\omega$ we have $\mathrm{o}(\mathcal{F}) \geq I_S(\mathcal{F})$.

We say that $S \subset X^\omega$ *contains diagonals* if every subsequence of every member of S also belongs to S and if for any sequence $(\overline{x}_n)$ in S with $\overline{x}_n = (x_{n,i})_{i=1}^\infty$ there exist $i_1 < i_2 < \cdots$ in $\mathbb{N}$ so that $(x_{n,i_n})_{n=1}^\infty$ belongs to S. If S contains diagonals, then the S-index of a tree on X may be measured via the Cantor–Bendixson index of the fine Schreier families $(\mathcal{F}_\alpha)_{\alpha<\omega_1}$.

PROPOSITION 4.2 ([26, Proposition 5])
Let X be an arbitrary set, and let $S \subset X^\omega$. If S contains diagonals, then for a hereditary tree $\mathcal{A}$ on X and for a countable ordinal α the following are equivalent.

(a) $\alpha < I_S(\mathcal{A})$.

(b) *There is a family $(x_F)_{F\in\mathcal{F}_\alpha\setminus\{\emptyset\}} \subset \mathcal{A}$ such that for $F = (m_1, m_2, \dots, m_k) \in \mathcal{F}_\alpha$ the branch $\overline{x}_F = (x_{\{m_1\}}, x_{\{m_1,m_2\}}, \dots, x_{\{m_1,m_2,\dots,m_k\}})$ is in $\mathcal{A}$ and $(x_{F\cup\{n\}})_{n>\max F}$ is in S if F is not maximal in $\mathcal{F}_\alpha$.*

Definition 4.3
Let $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ be regular, let S be a set of sequences in the set X, and let $(x_A : A \in \mathcal{F})$ be a tree in X indexed by $\mathcal{F}$. We call $(x_A : A \in \mathcal{F})$ an *S-tree* if for every nonmaximal $A \in \mathcal{F}$ the sequence $(x_{A\cup\{n\}} : n \in \mathbb{N},\ \text{with } A \cup \{n\} \in \mathcal{F})$ is a sequence in S.

If X is a Banach space and S are the w-null sequences, we call $(x_A : A \in \mathcal{F})$ a *w-null tree*. Similarly we define w^*-null trees in X^*.

Remark 4.4
In the case of $X = \mathbb{N}$ and $S = [\mathbb{N}]^\omega$ we deduce from Proposition 4.2 that if $\mathcal{A} \subset$

$[\mathbb{N}]^{<\omega}$ is hereditary and compact, then $\mathrm{CB}(\mathcal{A}) > \alpha$ if and only if there is an order isomorphism $\pi : \mathcal{F}_\alpha \to \mathcal{A}$, so that for all $A \in \mathcal{F}_\alpha \setminus \mathrm{MAX}(\mathcal{F}_\alpha)$ and $n > \max(A)$ it follows that $\pi(A \cup \{n\}) = \pi(A) \cup \{s_n\}$, where (s_n) is an increasing sequence in $\{s \in \mathbb{N} : s > \max \pi(A)\}$.

Examples 4.5
(a) *The weak index.* Let X be a separable Banach space. Let S be the set of all weakly null sequences in B_X, the unit ball of X. We call the S-index of a tree $\mathcal{F}$ on X the *weak index of* $\mathcal{F}$, and we shall denote it by $I_w(\mathcal{F})$. We shall use the term *weak derivative* instead of S-derivative and use the notation $\mathcal{F}'_w$ and $\mathcal{F}_w^{(\alpha)}$. When the dual space X^* is separable, the weak topology on the unit ball B_X or on any bounded subset of X is metrizable. Hence, in this case the set S contains diagonals, and Proposition 4.2 applies.

(b) *The weak* index*. We can define the weak* index similarly to the weak index. If X is a separable Banach space, then the w^*-topology on B_X^* is metrizable. This implies that the set S of all w^*-null sequences in B_{X^*} is diagonalizable. We call the S-index of a tree $\mathcal{F}$ on X^* the *weak* index of* $\mathcal{F}$, and we shall denote it by $I_{w^*}(\mathcal{F})$. We shall use the term *weak* derivative* instead of S-derivative and use the notation $\mathcal{F}'_{w^*}$ and $\mathcal{F}_{w^*}^{(\alpha)}$.

5. The Szlenk index

Here we recall the definition and basic properties of the Szlenk index and prove further properties that are relevant for our purposes.

Let X be a separable Banach space, and let K be a nonempty subset of X^*. For $\varepsilon \geq 0$, set

$$K'_\varepsilon = \big\{x^* \in X^* : \exists (x_n^*) \subset K\, w^*\text{-}\lim_{n\to\infty} x_n^* = x^* \text{ and } \|x_n^* - x^*\| > \varepsilon\big\},$$

and define $K_\varepsilon^{(\alpha)}$ for each countable ordinal α by recursion as follows:

$$K_\varepsilon^{(0)} = K, \qquad K_\varepsilon^{(\alpha+1)} = (K_\varepsilon^{(\alpha)})'_\varepsilon \quad \text{for } \alpha < \omega_1, \qquad \text{and}$$

$$K_\varepsilon^{(\lambda)} = \bigcap_{\alpha<\lambda} K_\varepsilon^{(\alpha)} \quad \text{for limit ordinals } \lambda < \omega_1.$$

Next, we associate to K the following ordinal indices:

$$\eta(K,\varepsilon) = \sup\{\alpha < \omega_1 : K_\varepsilon^{(\alpha)} \neq \emptyset\} \qquad \text{and} \qquad \eta(K) = \sup_{\varepsilon>0} \eta(K,\varepsilon).$$

Finally, we define the *Szlenk index* $\mathrm{Sz}(X)$ *of* X to be $\eta(B_{X^*})$, where B_{X^*} is the unit ball of X^*.

Remark
The original definition of the derived sets K'_ε in [34] is slightly different and might lead to different values of $\mathrm{Sz}(K,\varepsilon)$. However, if X does not contain ℓ_1, then the two definitions lead to the same $\mathrm{Sz}(K)$ and, thus, to the same $\mathrm{Sz}(X)$. Nowadays the above definition is the standard one, because the later proven ℓ_1-theorem of Rosenthal guarantees that $\mathrm{Sz}(X) < \omega_1$ if and only if X^* is separable.

Szlenk used his index to show that there is no separable, reflexive space universal for the class of all separable, reflexive spaces. This result follows immediately from the following properties of the function $\mathrm{Sz}(\cdot)$.

THEOREM 5.1 ([34])
Let X and Y be separable Banach spaces.
(i) X^* *is separable if and only if* $\mathrm{Sz}(X) < \omega_1$.
(ii) *If X isomorphically embeds into Y, then* $\mathrm{Sz}(X) \le \mathrm{Sz}(Y)$.
(iii) *For all $\alpha < \omega_1$ there exists a separable, reflexive space with Szlenk index at least α.*

We also recall the following observation of [21] about the form of the Szlenk index of a Banach space with separable dual.

PROPOSITION 5.2 ([21, Proposition 5.2])
If X has a separable dual, then there is an $\alpha < \omega_1$ with $\mathrm{Sz}(X) = \omega^\alpha$.

The following theorem combines several equivalent descriptions of the Szlenk index of a separable space not containing ℓ_1.

THEOREM 5.3
Assume that X is a separable space not containing ℓ_1 and $\alpha < \omega_1$. The following conditions are equivalent.
(a) $\mathrm{Sz}(X) > \omega^\alpha$.
(b) *There are an $\varepsilon > 0$ and a tree $(z^*_A : A \in \mathcal{S}_\alpha) \subset B_{X^*}$ so that for any nonmaximal $A \in \mathcal{S}_\alpha$*

$$w^*\text{-}\lim_{n\to\infty} z^*_{A\cup\{n\}} = z^*_A \qquad \textit{and} \qquad \|z^*_A - z^*_{A\cup\{n\}}\| \quad > \varepsilon \quad \textit{for } n > \max(A). \tag{39}$$

(c) *There are an $\varepsilon > 0$, a tree $(z^*_A : A \in \mathcal{S}_\alpha) \subset B^*_X$, and a w-null tree $(z_A : A \in \mathcal{S}_\alpha) \subset B_X$, so that*

$$z_B^*(z_A) > \varepsilon \quad \textit{for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \textit{ with } A \preceq B, \tag{40}$$

$$\left|z_B^*(z_A)\right| < \varepsilon/2 \quad \textit{for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \textit{ with } A \not\preceq B, \tag{41}$$

and for all nonmaximal $A \in \mathcal{S}_\alpha$ *we have*

$$w^*\text{-}\lim_{n\to\infty} z^*_{A\cup\{n\}} = z^*_A. \tag{42}$$

(d) *There are an* $N \in [\mathbb{N}]^\omega$, *an* $\varepsilon > 0$, *and a* w*-null tree* $(x_A : A \in \mathcal{S}_\alpha \cap [N]^{<\omega})$ *so that for every maximal* B *in* $\mathcal{S}_\alpha \cap [N]^{<\omega}$ *we have*

$$\Big\| \sum_{A \preceq B} \zeta(\alpha, A) x_A \Big\| \geq \varepsilon.$$

(e) *There is an* $\varepsilon > 0$ *such that* $I_w(\mathcal{F}_\varepsilon) > \omega^\alpha$, *where*

$$\mathcal{F}_\varepsilon = \Big\{(x_1, x_2, \ldots, x_l) \subset S_X : \forall (a_j)_{j=1}^l \subset [0,1] \ \Big\| \sum_{j=1}^l a_j x_j \Big\| \geq \varepsilon \sum_{j=1}^l a_j \Big\}.$$

(f) *There is an* $\varepsilon > 0$ *such that* $I_{w^*}(\mathcal{G}_\varepsilon) > \omega^\alpha$, *where*

$$\mathcal{G}_\varepsilon = \Big\{(x_1^*, x_2^*, \ldots, x_l^*) \subset B_{X^*} : \|x_j^*\| \geq \varepsilon \text{ and } \Big\| \sum_{i=1}^j x_i^* \Big\| \leq 1, \textit{ for } j = 1, 2, \ldots, l \Big\}.$$

Proof

To show that (a)⇒(b), we first prove the following.

LEMMA 5.4

Let X *be a separable Banach space, let* $K \subset X^*$ *be* w^**-compact, let* $0 < \varepsilon < 1$, *and let* $\beta < \omega_1$. *Then for every* $x^* \in K_\varepsilon^{(\beta)}$ *there is a family* $(z^*_{(x^*,A)} : A \in \mathcal{F}_\beta) \subset K$ *such that*

$$z^*_{(x^*,A)} \in K \qquad \textit{and} \qquad z^*_{(x^*,\emptyset)} = x^*, \tag{43}$$

$$\textit{if } A \textit{ is not maximal in } \mathcal{F}_\beta, \quad \textit{then } \|z^*_{(x^*,A)} - z^*_{(x^*,A\cup\{n\})}\| > \varepsilon \quad \textit{for all } n > \max(A), \tag{44}$$

$$\textit{if } A \textit{ is not maximal in } \mathcal{F}_\beta, \quad \textit{then } z^*_{(x^*,A)} = w^*\text{-}\lim_{n\to\infty} z^*_{(x^*,A\cup\{n\})}. \tag{45}$$

Proof

We will prove our claim by transfinite induction for all $\beta < \omega_1$. Let us first assume that $\beta = 1$. For $x^* \in K'_\varepsilon$ choose a sequence (x^*_n) which w^*-converges to x^*, with $\|x^* - x^*_n\| > \varepsilon$, for $n \in \mathbb{N}$. Thus, we can choose $z_{(x^*,\emptyset)} = x^*$, and $z_{(x^*,\{n\})} = x^*_n$. This choice satisfies (43), (44), and (45), for $\beta = 1$. (Recall that $\mathcal{F}_1 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$.)

Now assume that our claim is true for all $\gamma < \beta$. First assume that β is a successor ordinal, and let $\gamma < \omega_1$ so that $\beta = \gamma + 1$. Let $x^* \in K_\varepsilon^{(\gamma+1)}$. Thus, there is a sequence $(x^*_n) \subset K_\varepsilon^{(\gamma)}$ which w^*-converges to x^*, with $\|x^*_n - x^*\| > \varepsilon$, for $n \in \mathbb{N}$. By our induction hypothesis we can choose for each $n \in \mathbb{N}$ a family $(z^*_{(x^*_n,A)} : A \in \mathcal{F}_\gamma)$ satisfying (43), (44), and (45), for γ and x^*_n instead of x^*. For every $A \in \mathcal{F}_{\gamma+1}$ it follows that $A \setminus \{\min A\} \in \mathcal{F}_\gamma$, and we define $z^*_{(x^*,\emptyset)} := x^*$ and for $A \in \mathcal{F}_{\gamma+1} \setminus \{\emptyset\}$

$$z^*_{(x^*,A)} := z^*_{(x^*_{\min A}, A \setminus \{\min(A)\})}.$$

It is then easy to see that $(z^*_{(x^*,A)} : A \in \mathcal{F}_{\gamma+1})$ satisfies (43), (44), and (45).

Assume that $\beta < \omega_1$ is a limit ordinal, and let $(\mu(\beta,n) : n \in \mathbb{N}) \subset (0,\beta)$ be the sequence of ordinals increasing to β, used to define $\mathcal{F}_\beta$. We abbreviate $\beta_n = \mu(\beta,n)$, for $n \in \mathbb{N}$. Let $x^* \in K_\varepsilon^{(\beta)} = \bigcap_{\gamma<\beta} K_\varepsilon^{(\gamma)}$. Since $\beta_n + 1 < \beta$, we can use for each $n \in \mathbb{N}$ our induction hypothesis and choose a family

$$(z^*_{(n,x^*,A)} : A \in \mathcal{F}_{\beta_n+1}) \subset X^*,$$

satisfying (43), (44), and (45), for $\beta_n + 1$. In particular, it follows that $x^* = w^*\text{-}\lim_{j\to\infty} z^*_{(n,x^*,\{j\})}$, for all $n \in \mathbb{N}$. Since the w^*-topology is metrizable on K we can find an increasing sequence $(j_n : n \in \mathbb{N})$ in $\mathbb{N}$, $j_n > n$, for $n \in \mathbb{N}$, so that $x^* = w^*\text{-}\lim_{n\to\infty} z^*_{(n,x^*,\{j_n\})}$.

Consider for $n \in \mathbb{N}$ the set

$$\begin{aligned}\mathcal{F}_{\beta_n+1}(j_n) &= \big\{A \in [\mathbb{N}]^{<\omega} : j_n < \min A \text{ and } \{j_n\} \cup A \in \mathcal{F}_{\beta_n+1}\big\}\\ &= \{A \in \mathcal{F}_{\beta_n} : j_n < \min A\}.\end{aligned}$$

Since $\mathcal{F}_{\beta_n}$ is spreading, for $n \in \mathbb{N}$, we can choose $L_n = \{l_1^{(n)}, l_2^{(n)}, \ldots\} \in [\mathbb{N}]^\omega$ so that

$$\mathcal{F}_{\beta_n}^{L_n} = \big\{\{l_{a_1}^{(n)}, l_{a_2}^{(n)}, \ldots, l_{a_m}^{(n)}\} : \{a_1, a_2, \ldots, a_m\} \in \mathcal{F}_{\beta_n}\big\} \subset \mathcal{F}_{\beta_n+1}(j_n).$$

We define the map

$$\phi_n : \mathcal{F}_{\beta_n} \to \mathcal{F}_{\beta_n+1}(j_n), \qquad \{a_1, a_2, \ldots, a_m\} \mapsto \{l_{a_1}^{(n)}, l_{a_2}^{(n)}, \ldots, l_{a_m}^{(n)}\}.$$

Then we put for $A \in \mathcal{F}_\beta$

$$z^*_{(x^*,A)} = \begin{cases} x^* & \text{if } A = \emptyset,\\ z^*_{(x^*,\{j_n\})} & \text{if } A = \{n\} \text{ for some } n \in \mathbb{N},\\ z^*_{(x^*,\{j_n\}\cup\phi_n(B))} & \text{if } A = \{n\} \cup B \text{ for some } n \in \mathbb{N} \text{ and } B \in \mathcal{F}_{\alpha_n} \setminus \{\emptyset\},\end{cases}$$

which has the desired property. □

We now continue with our proof of Theorem 5.3(a)⇒Theorem 5.3(b). By assuming that $\mathrm{Sz}(X) > \beta = \omega^\alpha$, it follows that $[B_{X^*}]_\varepsilon^{(\beta)} \neq \emptyset$ for some $\varepsilon > 0$. We choose $x^* \in [B_{X^*}]_\varepsilon^{(\beta)}$ and apply Lemma 5.4 to obtain a tree in B_{X^*} indexed by $\mathcal{F}_{\omega^\alpha}$ satisfying the conditions (43), (44), and (45). Now Proposition 2.12 and a relabeling of the tree yield (b).

(b)⇒(c). For $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$ we define $A' = A \setminus \{\max(A)\}$. Now let $(A_m)_{m\in\mathbb{N}}$ be a consistent ordering of $\mathcal{S}_\alpha$ (see Section 2.4). We write $A <_{\mathrm{lin}} B$ or $A \leq_{\mathrm{lin}} B$ if $A = A_m$ and $B = A_n$, for $m < n$ or $m \leq n$, respectively.

Let $\varepsilon > 0$ and $(z_A^* : A \in \mathcal{S}_\alpha) \subset B_{X^*}$, so that (39) is satisfied. Then choose for each $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$ an element $x_A \in S_X$ so that $(z_A^* - z_{A'}^*)(x_A) > \varepsilon$.

Let $0 < \eta < \varepsilon/8$, and let $(\eta(A) : A \in \mathcal{S}_\alpha) \subset (0,1)$ satisfy the following conditions:

$$\big(\eta(A)\big) \text{ is decreasing with respect to the linear ordering } <_{\mathrm{lin}}, \tag{46}$$

$$\sum_{A\in\mathcal{S}_\alpha} \eta(A) < \eta, \tag{47}$$

$$\sum_{B\in\mathcal{S}_\alpha, B>_{\mathrm{lin}}A} \eta(B) < \eta(A), \quad \text{for all } A \in \mathcal{S}_\alpha, \qquad \text{and} \tag{48}$$

$$\eta(A_m) < \frac{1}{2}\frac{\eta}{m+2}, \quad \text{for all } m \in \mathbb{N}. \tag{49}$$

Since X does not contain a copy of ℓ_1 we can apply Rosenthal's ℓ_1-theorem and assume, possibly after passing to a pruning, that for each nonmaximal $A \in \mathcal{S}_\alpha$ the sequence $(x_{A\cup\{n\}})_{n>\max(A)}$ is weakly Cauchy. Since $(z_{A\cup\{n\}}^* - z_A^*)_{n>\max(A)}$ is w^*-null we can assume, possibly after passing to a further pruning, that $(z_{A\cup\{n\}}^* - z_A^*)(x_{A\cup\{n-1\}}) < \eta(A\cup\{n\})$, for all nonmaximal $A \in \mathcal{S}_\alpha$ and $n > 1 + \max(A)$.

Let $z_\emptyset = 0$. For a nonmaximal element $A \in \mathcal{S}_\alpha$ and $n > 1 + \max(A)$ let

$$z_{A\cup\{n\}} = \frac{1}{2}(x_{A\cup\{n\}} - x_{A\cup\{n-1\}})$$

and

$$z_{A\cup\{\max(A)+1\}} = x_{A\cup\{\max(A)+1\}}.$$

Then the families $(z_A : A \in \mathcal{S}_\alpha)$ and $(z_A^* : A \in \mathcal{S}_\alpha)$ are in B_X and B_{X^*}, respectively, $(z_A : A \in \mathcal{S}_\alpha)$ is weakly null, and $(z_A^* : A \in \mathcal{S}_\alpha)$ satisfies (42). Moreover, it follows that

$$(z_A^* - z_{A'}^*)(z_A) \geq \frac{\varepsilon}{2} - \eta(A), \quad \text{for all } A \in \mathcal{S}_\alpha \setminus \{\emptyset\}. \tag{50}$$

Since $w\text{-}\lim_{n\to\infty} z_{B\cup\{n\}} = 0$ and $w^*\text{-}\lim_{n\to\infty} z_{B\cup\{n\}}^* = z_B^*$, for every nonmaximal $B \in \mathcal{S}_\alpha$ we can, after passing again to a pruning, assume that

$$\begin{aligned} &\big|(z^*_B - z^*_{B'})(z_A)\big| < \eta(B) \qquad \text{and} \\ &\qquad\qquad \big|z^*_A(z_B)\big| < \eta(B) \quad \text{for all } A, B \in \mathcal{S}_\alpha, \text{ with } A <_{\text{lin}} B. \end{aligned} \tag{51}$$

We are left with verifying (40) and (41) for $\varepsilon/4$ instead of ε. To show (40) let $A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}$, with $A \preceq B$. We choose $l \in \mathbb{N}$ and $A = B_0 \prec B_1 \prec B_2 \prec \cdots \prec B_l = B$ so that $B'_j = B_{j-1}$, for $j = 1, 2, \ldots, l$, and deduce from (50) and (51)

$$\begin{aligned} z^*_B(z_A) &= \sum_{j=1}^{l} (z^*_{B_j} - z^*_{B'_j})(z_A) + (z^*_A - z^*_{A'})(z_A) + z^*_{A'}(z_A) \\ &\geq \frac{\varepsilon}{2} - \sum_{j=1}^{l} \eta(B_j) - 2\eta(A) > \frac{\varepsilon}{4}. \end{aligned}$$

To show (41), let $A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}$, with $A \not\preceq B$. We choose $l \in \mathbb{N}$ and $\emptyset = B_0 \prec B_1 \prec B_2 \prec \cdots \prec B_l = B$ so that $B'_j = B_{j-1}$, for $j = 1, 2, \ldots, l$, and since for every $j = 1, 2, \ldots, l$ either $A <_{\text{lin}} B_j$ or $B_j <_{\text{lin}} A$ we deduce from (51) and the conditions (47) and (49) on $\eta(\cdot)$ that

$$\begin{aligned} \big|z^*_B(z_A)\big| &\leq \Big|\sum_{j=1}^{l} (z^*_{B_j} - z^*_{B'_j})(z_A)\Big| + \big|z^*_\emptyset(z_A)\big| \\ &\leq \sum_{A <_{\text{lin}} B_j} \big|(z^*_{B_j} - z^*_{B'_j})(z_A)\big| \\ &\quad + \sum_{A >_{\text{lin}} B_j} \big(\big|z^*_{B_j}(z_A)\big| + \big|z^*_{B_{j-1}}(z_A)\big|\big) + \eta(A) \\ &\leq \sum_{A <_{\text{lin}} B_j} \eta(B_j) + 2 \sum_{B_j <_{\text{lin}} A} \eta(A) + \eta(A) \\ &\leq \big(2\#\{j \leq l : B_j <_{\text{lin}} A\} + 2\big)\eta(A) < \frac{\varepsilon}{4}, \end{aligned}$$

which verifies (41) and finishes the proof of our claims.

(c)$\Rightarrow$(d). Let $\varepsilon > 0$, $(z^*_A : A \in \mathcal{S}_\alpha)$, and $(z_A : A \in \mathcal{S}_\alpha)$ satisfy the condition in (c). Then it follows for a maximal $B \in \mathcal{S}_\alpha$ that

$$\Big\| \sum_{A \preceq B} \zeta(\alpha, A) z_A \Big\| \geq \sum_{A \preceq B} \zeta(\alpha, A) z^*_B(z_A) \geq \varepsilon \sum_{A \preceq B} \zeta(\alpha, A) = \varepsilon,$$

which proves our claim.

(d)$\Rightarrow$(e). Assume that $N \in [\mathbb{N}]^\omega$, $\varepsilon > 0$, and $(x_A : A \in \mathcal{S}_\alpha \cap [N]^{<\omega}) \subset B_X$ satisfy (d). For $B \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$ put $y_B = \sum_{A \prec B} \zeta(\alpha, A) x_A$, and choose $y^*_B \in S_{X^*}$ so that $y^*_B(y_B) = \|y_B\| > \varepsilon$.

For $B \in \mathrm{MAX}(\mathcal{S}_\alpha) \cap [N]^{<\omega}$, we define $f_B : B \to [-1,1]$, $b \mapsto y_B^*(x_{\{a\in B, a\le b\}})$. From Corollary 3.5 it follows now for $\delta = \varepsilon/2$ that $\mathrm{CB}(\mathcal{A}_{\delta,N}) = \omega^\alpha + 1$, where

$$\mathcal{A}_{\delta,N} = \left\{ A \in \mathcal{S}_\alpha \cap [N]^{<\omega} : \begin{array}{c} \exists B \in \mathrm{MAX}(S_\alpha \cap [N]^{<\omega}), \\ A \subset B, \text{ and } f_B(a) \ge \delta \text{ for all } a \in A \end{array} \right\}.$$

But from Proposition 4.2 and the remark thereafter we deduce that there is an order isomorphism $\pi : \mathcal{F}_{\omega^\alpha} \to \mathcal{A}_{\delta,N}$ such that for every nonmaximal $A \in \mathcal{F}_{\omega^\alpha}$ and any $n > \max(A)$ it follows that $\pi(A \cup \{n\}) = \pi(A) \cup \{s_n\}$, for some increasing sequence $(s_n) \subset \mathbb{N}$. By putting $z_A = x_{\pi(A)}$ it follows that $(z_A)_{A\in\mathcal{F}_{\omega^\alpha}}$ is a weakly null tree and for every $A = \{a_1, a_2, \ldots, a_l\}$ it follows that $(z_{\{a_1,a_2,\ldots,a_i\}})_{i=1}^l \in \mathcal{F}_\delta$. Again applying Proposition 4.2 yields (e).

(e) $\Longleftrightarrow$ (a). This follows from [2, Theorem 4.2], where it was shown that $\mathrm{Sz}(X) = \sup_{\varepsilon>0} I_w(\mathcal{F}_\varepsilon)$ if ℓ_1 does not embed into X.

(b) $\Longleftrightarrow$ (f). This follows from Proposition 2.13, an application of Proposition 4.2 to the tree $\mathcal{G}_\varepsilon$ on B_{X^*}, and $S = \{(x_n^*) \subset B_X^* : w^*\text{-}\lim_{n\to\infty} = 0\}$. □

Remark

We note that in the implication (a)⇒(b) the assumption that ℓ_1 does not embed into X was not needed. In fact, (a) is equivalent to (b) for all separable Banach spaces.

We will also need the following *dual version* of Theorem 5.3.

PROPOSITION 5.5

Assume that X is a Banach space whose dual X^* is separable, with $\mathrm{Sz}(X^*) > \omega^\alpha$. *Then there are an $\varepsilon > 0$, a tree $(z_A : A \in \mathcal{S}_\alpha) \subset B_X$, and a w^*-null tree $(z_A^* : A \in \mathcal{S}_\alpha) \subset B_X^*$ such that*

$$z_A^*(z_B) > \varepsilon \quad \textit{for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \textit{ with } A \preceq B, \tag{52}$$

$$\left|z_A^*(z_B)\right| < \frac{\varepsilon}{2} \quad \textit{for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \textit{ with } A \not\preceq B. \tag{53}$$

Proof

Recall that, as stated above, the implication (a)⇒(b) of Theorem 5.3 holds even if the space to which the theorem is applied contains ℓ_1. Applying this implication to X^*, we find $\varepsilon > 0$ and $\{z_A^{**} : A \in \mathcal{S}_\alpha\} \subset B_{X^{**}}$ so that (39) is satisfied. Then choose for each $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$ an element $x_A^* \in S_X$ so that $(z_A^{**} - z_{A'}^{**})(x_A^*) > \varepsilon$. Again let (A_n) be a consistent enumeration of $\mathcal{S}_\alpha$, and write $A_m <_{\mathrm{lin}} A_n$ if $m < n$. We also assume that $(\eta(A) : A \in \mathcal{S}_\alpha) \subset (0,1)$ has the property that

$$\sum_{A\in\mathcal{S}_\alpha} \eta(A) < \frac{\varepsilon}{32}. \tag{54}$$

After passing to a first pruning we can assume that for all nonmaximal $A \in \mathcal{S}_\alpha$ the sequence $(x^*_{A\cup\{n\}})$ w^*-converges and that for any $B \in \mathcal{S}_\alpha$ the sequence $z^{**}_B(x^*_{A\cup\{n\}})$ converges to some number $r_{A,B}$. (For fixed $A, B \in \mathcal{S}_\alpha$ we only need to pass to a subsequence of $(A \cup \{n\} : n \in \mathbb{N}, A \cup \{n\} >_{\text{lin}} B)$.) Since $(z^{**}_{A\cup\{n\}} - z^{**}_A)_{n>\max(A)}$ is w^*-null, we can assume, after passing to a second pruning, that we have

$$\left|(z^{**}_B - z^{**}_{B'})(x^*_A)\right| < \eta(B) \quad \text{for all } A, B \in \mathcal{S}_\alpha, \text{ with } A <_{\text{lin}} B. \tag{55}$$

We put $z^*_\emptyset = 0$ and for any nonmaximal element $A \in \mathcal{S}_\alpha$

$$z^*_{A\cup\{\max(A)+1\}} = x^*_{A\cup\{\max(A)+1\}} \qquad \text{and}$$
$$z^*_{A\cup\{n\}} = \frac{1}{2}(x^*_{A\cup\{n\}} - x^*_{A\cup\{n-1\}}) \quad \text{if } n > \max(A) + 1.$$

It follows that $(z^*_A : A \in \mathcal{S}_\alpha)$ is a w^*-null tree in B_{X^*} and that for any $A \in \mathcal{S}_\alpha$

$$(z^{**}_A - z^{**}_{A'})(z^*_A) > \frac{\varepsilon}{2} - \frac{\eta(A)}{2}.$$

Since $z^{**}_B(x^*_{A\cup\{n\}})$ converges to $r_{A,B}$ we can assume, after passing to a third pruning, that

$$\left|z^{**}_A(z^*_B)\right| < \frac{\eta(B)}{2} \quad \text{whenever } A <_{\text{lin}} B, \tag{56}$$

and hence,

$$z^{**}_A(z^*_A) > \frac{\varepsilon}{2} - \eta(A) \quad \text{for all } A \in \mathcal{S}_\alpha. \tag{57}$$

Since $w^*\text{-}\lim_n z^{**}_{A\cup\{n\}} = z^{**}_A$ we can assume, by passing to a further pruning, that

$$\left|(z^{**}_B - z^{**}_{B'})(z^*_A)\right| < \eta(B) \quad \text{for all } A, B \in \mathcal{S}_\alpha, \text{ with } A <_{\text{lin}} B. \tag{58}$$

Since B_X is w^*-dense in $B_{X^{**}}$ we can choose, for every $A \in \mathcal{S}_\alpha$, a vector $z_A \in B_X$ so that

$$\left|z^*_A(z_B) - z^{**}_B(z^*_A)\right| < \eta(B), \quad \text{for all } A, B \in \mathcal{S}_\alpha, \text{ with } A \le_{\text{lin}} B. \tag{59}$$

Combining (58) and (59) we obtain that for all A and B in $\mathcal{S}_\alpha$ with $A <_{\text{lin}} B$ we have

$$\left|z^*_A(z_B) - z^{**}_{B'}(z^*_A)\right| \le \left|z^*_A(z_B) - z^{**}_B(z^*_A)\right| + \left|(z^{**}_B - z^{**}_{B'})(z^*_A)\right| < 2\eta(B). \tag{60}$$

Using that $(z^*_A : A \in \mathcal{S}_\alpha)$ is a w^*-null tree, we can pass to a further pruning, so that

$$\left|z^*_B(z_A)\right| < \eta(B), \quad \text{for all } A, B \in \mathcal{S}_\alpha \text{ with } A <_{\text{lin}} B. \tag{61}$$

We deduce from (58) and (59), for $A, B \in \mathcal{S}_\alpha$ with $A \le_{\mathrm{lin}} B'$ (resp., $A <_{\mathrm{lin}} B$), that

$$\begin{aligned}\big|z_A^*(z_B - z_{B'})\big| &\le \big|z_A^*(z_B) - z_B^{**}(z_A^*)\big| + \big|z_A^*(z_{B'}) - z_{B'}^{**}(z_A^*)\big| \\ &\quad + \big|(z_B^{**} - z_{B'}^{**})(z_A^*)\big| \\ &\le 2\eta(B) + \eta(B'). \end{aligned} \tag{62}$$

By (60), (56), and (61) for $A, B \in \mathcal{S}_\alpha$ with $B' <_{\mathrm{lin}} A < B$ we obtain

$$\begin{aligned}\big|z_A^*(z_B - z_{B'})\big| &\le \big|z_A^*(z_B) - z_{B'}^{**}(z_A^*)\big| + \big|z_{B'}^{**}(z_A^*)\big| + \big|z_A^*(z_{B'})\big| \\ &\le 2\eta(B) + 2\eta(A). \end{aligned} \tag{63}$$

We now claim that the families $\{z_A : A \in \mathcal{S}_\alpha\}$ and $\{z_A^* : A \in \mathcal{S}_\alpha\}$ satisfy (52) and (53). To verify (52), let $A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}$, with $A \preceq B$. Then let $k \in \mathbb{N}$ and $B_j \in \mathcal{S}_\alpha$, for $j = 0, 1, 2, \dots, k$, be such that $A = B_0 \prec B_1 \prec B_2 \prec \cdots \prec B_k = B$ and $B_j' = B_j$, for $j = 1, 2, \dots, k$. We have that

$$\begin{aligned} z_A^*(z_B) &= \sum_{j=1}^k z_A^*(z_{B_j} - z_{B_j'}) + z_A^*(z_A) \\ &\ge z_A^{**}(z_A^*) - \big|z_A^{**}(z_A^*) - z_A^*(z_A)\big| - \big|z_A^*(z_{B_1} - z_A)\big| - \sum_{j=2}^k \big|z_A^*(z_{B_j} - z_{B_j'})\big| \\ &> \frac{\varepsilon}{2} - 3\eta(A) - 2\sum_{j=1}^k \eta(B_j) \ge \frac{\varepsilon}{4} \quad \text{(by (57), (59), (62), and (60))}, \end{aligned}$$

which yields (52) if we replace ε by $\varepsilon/4$.

To verify (53), let $A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}$, with $A \not\preceq B$. If $A >_{\mathrm{lin}} B$, we deduce our claim from (61). If $A \le_{\mathrm{lin}} B$ and, thus, $A <_{\mathrm{lin}} B$, we choose $k \in \mathbb{N}$ and $B_0 \prec B_1 \prec B_2 \prec \cdots \prec B_k = B$, with $B_j' = B_{j-1}$, for $j = 1, 2, \dots, k$, and $B_0 <_{\mathrm{lin}} A <_{\mathrm{lin}} B_1$. Applying (63), (61), (62), and finally (54), we obtain

$$\begin{aligned} \big|z_A^*(z_B)\big| &\le \big|z_A^*(z_{B_1} - z_{B_0})\big| + \big|z_A^*(z_{B_0})\big| + \Big|\sum_{j=2}^k z_A^*(z_{B_j} - z_{B_j'})\Big| \\ &\le 2\eta(B_1) + \eta(B_0) + 3\eta(A) + \sum_{j=2}^k \big(2\eta(B_j) + \eta(B_{j-1})\big) \le \frac{\varepsilon}{8}, \end{aligned}$$

which proves our claim. □

Example 5.6
Let us construct an example of families $(z_A : A \in [\mathbb{N}]^{<\omega}) \subset B_{c_0}$ and $(z^*_A : A \in [\mathbb{N}]^{<\omega}) \subset B_{\ell_1}$ satisfying Proposition 5.5. Let $<_{\mathrm{lin}}$ again be a linear consistent ordering of $[\mathbb{N}]^{<\omega}$. We first choose a family $(\tilde{A} : A \in [\mathbb{N}]^{<\omega}) \subset [\mathbb{N}]^{<\omega}$ with the following properties:

$$\tilde{A} \text{ is a spread of } A, \quad \text{for each } A \in [\mathbb{N}]^{<\omega}, \tag{64}$$

$$A \prec B \quad \text{if and only if} \quad \tilde{A} \prec \tilde{B}, \tag{65}$$

$$\begin{aligned} &\text{if } A, B \in [\mathbb{N}]^{<\omega}, \emptyset \neq A <_{\mathrm{lin}} B, \qquad \text{and} \\ &C \in [\mathbb{N}]^{<\omega} \text{ is the maximal element in } [\mathbb{N}]^{<\omega} \\ &\text{such that } C \preceq A \text{ and } C \preceq B, \text{ then } (\tilde{A} \setminus \tilde{C}) \cap (\tilde{B} \setminus \tilde{C}) = \emptyset. \end{aligned} \tag{66}$$

We define for $A \in [\mathbb{N}]^{<\omega}$

$$z_A = \sum_{a \in \tilde{A}} e_a \qquad \text{and} \qquad z^*_A = e^*_{\max(\tilde{A})},$$

where (e_j) and (e^*_j) denote the unit vector bases in c_0 and ℓ_1, respectively. It is now easy to verify that the tree (z^*_A) is w^*-null and that (52) is satisfied for any $\varepsilon \in (0, 1)$. To verify (53), let $A, B \in [\mathbb{N}]^{<\omega}$ with $A \not\preceq B$. If $A >_{\mathrm{lin}} B$, then $\max(\tilde{A}) \notin \tilde{B}$, and our claim follows. If $A <_{\mathrm{lin}} B$, let $C \in [\mathbb{N}]^{<\omega}$ be the maximal element for which $C \preceq A$ and $C \preceq B$. It follows that $C \prec A$, but also that $C \prec B$, which implies by (66) that $\max(\tilde{A}) \notin \tilde{B}$ and, thus, our claim.

6. Estimating certain convex combinations of blocks by using the Szlenk index

In this section we will assume that X has an FDD (F_j). This means that $F_j \subset X$ is a finite-dimensional subspace of X, for $j \in \mathbb{N}$, and that every x has a unique representation as the sum $x = \sum_{j=1}^{\infty} x_j$, with $x_j \in F_j$, for $j \in \mathbb{N}$. For $x = \sum_{j=1}^{\infty} x_j \in X$ we call $\mathrm{supp}(x) = \{j \in \mathbb{N} : x_j \neq 0\}$ the *support of* x (with respect to (F_j)), and the smallest interval in $\mathbb{N}$ containing $\mathrm{supp}(x)$ is called the *range of* x (with respect to (F_j)) and is denoted by $\mathrm{ran}(x)$. A (finite or infinite) sequence $(x_n) \subset X$ is called a *block (with respect to* (F_j)*)* if $x_n \neq 0$, for all $n \in \mathbb{N}$, and $\mathrm{supp}(x_n) < \mathrm{supp}(x_{n+1})$, for all $n \in \mathbb{N}$ for which x_{n+1} is defined.

We call an FDD *shrinking* if every bounded block $(x_n)_{n=1}^{\infty}$ is weakly null. As in the case of bases, X^* is separable, and thus, $\mathrm{Sz}(X) < \omega_1$ if X has a shrinking FDD.

THEOREM 6.1
Let X be a Banach space with a shrinking FDD, and let α be a countable ordinal number with $\mathrm{Sz}(X) \leq \omega^\alpha$. Then for every $\varepsilon > 0$ and $M \in [\mathbb{N}]^\omega$, there exists $N \in [M]^\omega$

satisfying the following: for every $B = \{b_1, \dots, b_d\}$ *in* $\mathcal{S}_\alpha \cap [N]^{<\omega}$ *and sequence* $(x_i)_{i=1}^d$ *in* B_X*, with* $\mathrm{ran}(x_j) \subset (b_{j-1}, b_{j+1})$ *for* $j = 1, \dots, d$ *(where* $b_0 = 0$ *and* $b_{d+1} = \infty$*), we have*

$$\Big\| \sum_{j=1}^{d} \zeta(\alpha, B_j) x_j \Big\| < \varepsilon, \tag{67}$$

where $B_j = \{b_1, \dots, b_j\}$ *for* $j = 1, \dots, d$.

Proof
It is enough to find N in $[M]^\omega$ so that (67) holds whenever $B \in \mathrm{MAX}(\mathcal{S}_\alpha) \cap [N]^{<\omega}$. Indeed, if (67) holds for all B in $\mathrm{MAX}(S_\alpha) \cap [N]^{<\omega}$, then for any $A \in \mathcal{S}_\alpha \cap [N]^{<\omega}$ and $(x_i)_{i=1}^{\#A}$ satisfying the assumption of Theorem 6.1, one may extend A to any maximal set B and extend the sequence $(x_k)_{k=1}^{\#A}$ by concatenating the zero vector $\#B - \#A$ times. Toward a contradiction, we assume that such a set N does not exist. Applying Proposition 2.14 to the partition $(\mathcal{F}, \mathcal{S}_\alpha \setminus \mathcal{F})$ of $\mathcal{S}_\alpha$, where

$$\mathcal{F} = \left\{ B = \{b_1, b_2, \dots, b_n\} \in \mathrm{MAX}(\mathcal{S}_\alpha) : \begin{array}{c} \exists (x_j) \subset B_X, \mathrm{ran}(x_j) \subset (b_{j-1}, b_{j+1}), \\ \text{for } j = 1, 2, \dots, n, \\ \| \sum_{j=1}^n \zeta(\alpha, \{b_1, b_2, \dots, b_j\}) x_j \| > \varepsilon \end{array} \right\},$$

yields that there is L in $[M]^\omega$ such that, for all $B = \{b_1^B, \dots, b_{d_B}^B\}$ in $\mathrm{MAX}(\mathcal{S}_\alpha) \cap [L]^{<\omega}$, there exists a sequence $(x_i^B)_{i=1}^{d_B}$ in B_X with $\mathrm{ran}(x_j) \subset (b_{j-1}^B, b_{j+1}^B)$ for $j = 1, \dots, d_B$ such that

$$\Big\| \sum_{j=1}^{d_B} \zeta(\alpha, B_j^B) x_j^B \Big\| \geq \varepsilon, \tag{68}$$

where $B_j^B = \{b_1^B, \dots, b_j^B\}$ for $j = 1, \dots, d_B$. For $A \preceq B$, if $A = B_j^B$, we use the notation $x_j^B = x_A^B$. Note that, under this notation, (68) takes the more convenient form

$$\Big\| \sum_{A \preceq B} \zeta(\alpha, A) x_A^B \Big\| \geq \varepsilon \tag{69}$$

and that

$$\mathrm{ran}(x_{A'}^B) \subset \big(\max(A''), \max(A)\big), \quad \text{for all } A \preceq B \text{ with } A' \neq \varnothing, \tag{70}$$

where $A'' = (A')'$ and $\max(\emptyset) = 0$.

We will now apply several stabilization and perturbation arguments to show that we may assume that for $B \in \mathrm{MAX}(\mathcal{S}_\alpha)$ and $A \preceq B$ the vector $x_{A'}^B$ only depends on

A and will then be renamed x_A. By using a compactness argument, Proposition 2.14 yields the following: if $A \in \mathcal{S}_\alpha \cap [L]^{<\omega}$ is nonmaximal with $A' \neq \emptyset$, then for $\delta > 0$, there is $L' \in [L]^\omega$ such that, for all D_1, D_2 in $\mathrm{MAX}(\mathcal{S}_\alpha(A)) \cap [L']^{<\omega}$, we have $\|x_{A'}^{A\cup D_1} - x_{A'}^{A\cup D_2}\| < \delta$. Combining the above with a standard diagonalization argument we may pass to a further infinite subset of L and a perturbation of the block vectors $x_{A'}^B$ with $B \in \mathrm{MAX}(\mathcal{S}_\alpha) \cap [L]^{<\omega}$ and $\{\min(B)\} \prec A \preceq B$ (and perhaps pass to a smaller ε in (69)), so that, for every B_1, B_2 in $\mathrm{MAX}(\mathcal{S}_\alpha) \cap [L]^{<\omega}$ and A with $A' \neq \emptyset$ such that $A \preceq B_1$ and $A \preceq B_2$, we have $x_{A'}^{B_1} = x_{A'}^{B_2}$. For every $A \in \mathcal{S}_\alpha \cap [L]^{<\omega}$, we call this common vector x_A. Note that x_A indeed depends on A and not only on A'. For A such that $A' = \emptyset$, that is, for those sets A that are of the form $A = \{n\}$ for some $n \in L$, choose any normalized vector x_A with $\mathrm{supp}(x_A) = \{n\}$. Note that, using (70), we have

$$\mathrm{ran}(x_A) \subset \big(\max(A''), \max(A)\big) \quad \text{for all } A \in \mathcal{S}_\alpha \cap [L]^{<\omega} \text{ with } A' \neq \varnothing, \tag{71}$$

where $\max(\emptyset) = 0$, and if $A' = 0$, that is, $A = \{n\}$ for some $n \in \mathbb{N}$, then $\mathrm{ran}(x_A) = \{n\}$. Furthermore, by fixing $0 < \delta < \varepsilon/12$ and passing to an infinite subset of L, again denoted by L, satisfying $\min(L) \geq 1/\delta$, (69) and (38) yield that for all $B \in \mathrm{MAX}(\mathcal{S}_\alpha) \cap [L]^{<\omega}$

$$\begin{aligned}\Big\|\sum_{A\preceq B} \zeta(\alpha, A)x_A\Big\| &\geq \Big\|\sum_{A\preceq B} \zeta(\alpha, A')x_A\Big\| - 2\delta \\ &= \Big\|0x_{\{\min(B)\}} + \sum_{\{\min(B)\}\prec A\preceq B} \zeta(\alpha, A')x_{A'}^B\Big\| - 2\delta \\ &= \Big\|\sum_{A\prec B} \zeta(\alpha, A)x_A^B\Big\| - 2\delta \geq \varepsilon - 3\delta.\end{aligned} \tag{72}$$

For $B \in \mathrm{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$ and $i = 0, 1, 2$ define

$$B^{(i)} = \{A \preceq B : \#A \bmod 3 = i\}.$$

By the triangle inequality, for some $0 \leq i(B) \leq 2$, we have $\|\sum_{A\in B^{(i(B))}} \zeta(\alpha, A)x_A\| \geq \varepsilon/3 - \delta$. By Proposition 2.14, we may pass to some infinite subset of L, again denoted by L, so that for all $B \in \mathrm{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$ we have $i(B) = i_0$ for some common $i_0 \in \{0, 1, 2\}$. We shall assume that $i_0 = 0$, as the other cases are treated similarly. Therefore, for all $B \in \mathrm{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$ we have

$$\Big\|\sum_{A\in B^{(0)}} \zeta(\alpha, A)x_A\Big\| \geq \frac{\varepsilon}{3} - \delta. \tag{73}$$

Lemmas 3.7(c) and 3.7(e) also imply the following. If $B \in \mathrm{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$, then

$$\sum_{\substack{A\in B^{(0)}:\\ l_1(A')=0}} \zeta(\alpha,A) + \sum_{\substack{A\in B^{(0)}:\\ l_1(A'')=0}} \zeta(\alpha,A) + \sum_{\substack{A\in B^{(0)}\\ l_1(A''')=0}} \zeta(\alpha,A) \le \frac{3}{\min(B)} \le 3\delta. \tag{74}$$

Hence, if for $B \in \mathrm{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$ we set

$$\hat{B}^{(0)} = B^{(0)} \setminus \{A \preceq B : l_1(A') = 0,\ \text{or}\ l_1(A'') = 0,\ \text{or}\ l_1(A''') = 0\},$$

then

$$\Big\| \sum_{A\in \hat{B}^{(0)}} \zeta(\alpha,A) x_A \Big\| \ge \frac{\varepsilon}{3} - 4\delta. \tag{75}$$

If $L = \{\ell_1, \ell_2, \ldots, \ell_k, \ldots\}$, define $N = \{\ell_3, \ell_6, \ldots, \ell_{3k}, \ell_{3(k+1)}, \ldots\}$. For each $A \in \mathcal{S}_\alpha \cap [N]^{<\omega}$, with $(\#A) \mod 3 = 0$, we define $\tilde{A} \in [L]^{<\omega}$ as described below. If $A = \{a_1, \ldots, a_d\}$, where $a_j = \ell_{3b_j}$ and $A_j = \{a_1, \ldots, a_j\}$ for $1 \le j \le d$, we define the elements of a set $\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_d\}$ in groups of three as follows. If $j \mod 3 = 0$, we put

$$(\tilde{a}_{j-2}, \tilde{a}_{j-1}, \tilde{a}_j) = \begin{cases} (a_{j-2}, a_{j-1}, a_j) & \text{if } l_1(A'_j) = 0 \text{ or } l_1(A'_{j-1}) = 0 \text{ or } l_1(A'_{j-2}) = 0, \\ (\ell_{3b_j-2}\ell_{3b_j-1}, a_j) & \text{if } l_1(A'_j), l_1(A'_{j-1}), l_1(A'_{j-2}) \ne 0. \end{cases}$$

It is not hard to see that A and $\tilde{A}$ satisfy the assumptions of Corollary 3.8; hence, $\tilde{A} \in \mathcal{S}_\alpha \cap [L]^{<\omega}$ and $\zeta(\alpha, A) = \zeta(\alpha, \tilde{A})$. Observe the following.

(a) If $B \in \mathrm{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$ and $A^{(1)} \prec A^{(2)}$ are in $\hat{B}^{(0)}$, then $\tilde{A}^{(1)} \prec \tilde{A}^{(2)}$.

(b) If $B \in \mathrm{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$ and $A \in \hat{B}^{(0)}$ and

$$\text{if } \max(A) = \ell_{3n}, \quad \text{then we have } \max(\tilde{A}'') = \ell_{3n-2}. \tag{76}$$

Statement (a) is clear, while (b) follows from the fact that $A \in \hat{B}^{(0)}$ implies that d is divisible by 3 and $l_1(A'') \ne 0$.

We define a weakly null tree $(z_A)_{A\in\mathcal{S}_\alpha\cap[N]^{<\omega}}$ such that for all $B \in \mathrm{MAX}(\mathcal{S}_\alpha \cap [N]^\infty)$ we have

$$\Big\| \sum_{A\preceq B} \zeta(\alpha,A) z_A \Big\| \ge \frac{\varepsilon}{3} - 4\delta. \tag{77}$$

The choice of δ and Theorem 5.3(d) will yield a contradiction.

For $A \in \mathcal{S}_\alpha \cap [N]^{<\omega}$ define

$$z_A = \begin{cases} x_{\tilde{A}} & \text{if } \#A \bmod 3 = 0 \text{ and } l_1(A'), l_1(A''), l_1(A''') \ne 0, \\ 0 & \text{else.} \end{cases}$$

Let $C \in \mathrm{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$, and by (a), we can find B in $\mathrm{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$ such that $\tilde{A} \preceq B$, for all $A \preceq C$, with $\#A = 0 \mod 3$. Then, one can verify that

$$\begin{aligned}\Big\| \sum_{A \preceq C} \zeta(\alpha, A) z_A \Big\| &= \Big\| \sum_{A \preceq C} \zeta(\alpha, \tilde{A}) x_{\tilde{A}} \Big\| \\ &= \Big\| \sum_{A \in \hat{C}^{(0)}} \zeta(\alpha, \tilde{A}) x_{\tilde{A}} \Big\| \\ &= \Big\| \sum_{A \in \hat{B}^{(0)}} \zeta(\alpha, A) x_A \Big\| \\ &\geq \frac{\varepsilon}{3} - 4\delta.\end{aligned}$$

Now let $A \in \mathcal{S}_\alpha \cap [N]^{<\omega}$ be nonmaximal. We will show that w-$\lim_{n \in N} z_{A \cup \{n\}} = 0$. By the definition of the vectors z_A, we need only treat the case in which $(\#A + 1) \bmod 3 = 0$, that is, when $z_{A \cup \{n\}} = x_{\widetilde{A \cup \{n\}}}$ for all $n \in N$ with $n > \max(A)$. In this case, by (71), we deduce that if $\ell_{3n} \in N$, then

$$\min\operatorname{supp}(z_{A \cup \{\ell_{3n}\}}) = \min\operatorname{supp}(x_{\widetilde{A \cup \{\ell_{3n}\}}}) > \max\big((\widetilde{A \cup \{\ell_{3n}\}})''\big) = \ell_{3n-2},$$

where the last equality follows by (b). Hence, $\lim_{n \in N} \min\operatorname{supp}(z_{A \cup \{n\}}) = \infty$. The fact that the FDD of X is shrinking completes the proof. □

7. Two metrics on $\mathcal{S}_\alpha$, $\alpha < \omega_1$

Since $[\mathbb{N}]^{<\omega}$ with $\prec$ is a tree, with a unique root $\emptyset$, we could consider on $[\mathbb{N}]^{<\omega}$ the usual *tree distance* which we denote by d. For $A = \{a_1, a_2, \dots, a_l\} \in [\mathbb{N}]^{<\omega}$ or $B = \{b_1, b_2, \dots, b_m\}$, we let $n = \max\{j \geq 0 : a_i = b_i \text{ for } i = 1, 2, \dots, j\}$ and then let $d(A, B) = l + m - 2n$. But this distance will not lead to the results we are seeking. Indeed, it was shown in [7, Theorem 1.2] that for any reflexive space X the tree $[\mathbb{N}]^{<\omega}$ with the graph metric embeds bi-Lipschitzly into X if and only if $\max(\mathrm{Sz}(X), \mathrm{Sz}(X^*)) > \omega$. We will need *weighted graph metrics* on $\mathcal{S}_\alpha$.

(a) The *weighted tree distance on* $\mathcal{S}_\alpha$. For A, B in S_α let C be the largest element in $\mathcal{S}_\alpha$ (with respect to $\prec$) such that $C \preceq A$ and $C \preceq B$ (i.e., C is the common initial segment of A and B), and then let

$$\begin{aligned} d_{1,\alpha}(A, B) &= \sum_{a \in A \setminus C} z_{(\alpha, A)}(a) + \sum_{b \in B \setminus C} z_{(\alpha, B)}(b) \\ &= \sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D).\end{aligned}$$

(b) The *weighted interlacing distance on* $\mathcal{S}_\alpha$ can be defined as follows. For $A, B \in \mathcal{S}_\alpha$, say, $A = \{a_1, a_2, \dots, a_l\}$ and $B = \{b_1, b_2, \dots, b_m\}$, with $a_1 < a_2 < \cdots <$

a_l and $b_1 < b_2 < \cdots < b_m$, we put $a_0 = b_0 = 0$ and $a_{l+1} = b_{m+1} = \infty$ and define

$$d_{\infty,\alpha}(A,B) = \max_{i=1,\dots,m+1} \sum_{a\in A, b_{i-1}<a<b_i} z_{(\alpha,A)}(a) + \max_{i=1,\dots,l+1} \sum_{b\in B, a_{i-1}<b<a_i} z_{(\alpha,B)}(b).$$

Remark
To explain $d_{\infty,\alpha}$ let us take some sets $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_m\}$ in $\mathcal{S}_\alpha$ and fix some $i \in \{0, 1, 2, \dots, m\}$. Now we measure how large the part of B is which lies between a_i and a_{i+1} (as before $a_0 = 0$ and $a_{m+1} = \infty$) by putting

$$m_i(B) := \sum_{j, b_j \in (a_i, a_{i+1})} \zeta\big(\alpha, \{b_1, b_2, \dots, b_j\}\big).$$

Then we define $m_j(A)$ for $j = 1, 2, \dots, n$ similarly and put

$$d_{\infty,\alpha}(A,B) = \max_{1\le i\le m} m_i(B) + \max_{1\le j\le n} m_j(A).$$

We note that if C is maximal such that $C \preceq A$ and $C \preceq B$ and if $A \setminus C < B \setminus C$, then $d_{1,\alpha}(A,B) = d_{\infty,\alpha}(A,B)$.

The following observations on the stability of the metrics $d_{1,\alpha}$ and $d_{\infty,\alpha}$, $\alpha < \omega_1$ are easy to show.

PROPOSITION 7.1
The metric space $(\mathcal{S}_\alpha, d_{1,\alpha,})$ is stable*, that is, for any sequences (A_n) and (B_n) in $\mathcal{S}_\alpha$ and any ultrafilter $\mathcal{U}$ on $\mathbb{N}$ it follows that*

$$\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} d_{1,\alpha}(A_m, B_n) = \lim_{n\in\mathcal{U}} \lim_{m\in\mathcal{U}} d_{1,\alpha}(A_m, B_n),$$

while $(\mathcal{S}_\alpha, d_{\infty,\alpha})$ is not stable.

We can now conclude one direction of Theorem A from James's characterization of reflexive spaces.

PROPOSITION 7.2
If X is a nonreflexive Banach space, then for any $0 < c < 1/4$ and every $\alpha > 0$ there is a map $\Phi_\alpha : \mathcal{S}_\alpha \to X$ such that

$$c d_{\infty,\alpha}(A,B) \le \big\|\Phi(A) - \Phi(B)\big\| \le d_{1,\alpha}(A,B) \quad \textit{for all } A, B \in \mathcal{S}_\alpha. \tag{78}$$

Remark

Our argument will show that if X is nonreflexive, then there is a sequence $(x_n) \subset B_X$ such that for all $\alpha < \omega_1$ the map

$$\Phi_\alpha : \mathcal{S}_\alpha \to X, \qquad A \mapsto \sum_{D \preceq A} \zeta(\alpha, D) x_{\max(D)}$$

satisfies (78). The fact that $c > 0$ in Proposition 7.2 can be chosen arbitrarily close to $\frac{1}{4}$ will be irrelevant for the rest of our arguments; therefore, at the beginning of the proof, one could also cite [15, Theorem 8] (weakening accordingly the statement of the proposition).

Proof

Let Θ be any number in $(0,1)$. Then by [16] there is a normalized basic sequence in X whose basic constant is at most $\frac{2}{\Theta}$ satisfying

$$\Big\| \sum_{j=1}^{\infty} a_j x_j \Big\| \geq \Theta \sum_{j=1}^{\infty} a_j \quad \text{for all } (a_j) \in c_{00} \qquad a_j \geq 0, \quad \text{for all } j \in \mathbb{N}. \tag{79}$$

Thus, its bimonotonicity constant is at most $\frac{4}{\Theta}$, which means that for $m \leq n$ the projection

$$P_{[m,n]} : \overline{\operatorname{span}(x_j)} \to \overline{\operatorname{span}(x_j)}, \qquad \sum_{j=1}^{\infty} a_j x_j \mapsto \sum_{j=m}^{n} a_j x_j$$

has norm at most $\frac{4}{\Theta}$.

We define

$$\Phi : \mathcal{S}_\alpha \to X, \qquad A \mapsto \sum_{D \preceq A} \zeta(\alpha, D) x_{\max(D)}.$$

For $A, B \in \mathcal{S}_\alpha$ we let C be the maximal element in $\mathcal{S}_\alpha$ for which $C \preceq A$ and $C \preceq B$. Then

$$\begin{aligned} \big\| \Phi(A) - \Phi(B) \big\| &= \Big\| \sum_{C \prec D \preceq A} \zeta(\alpha, D) x_{\max D} - \sum_{C \prec D \prec B} \zeta(\alpha, D) x_{\max(D)} \Big\| \\ &\leq \sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) = d_{1,\alpha}(A, B). \end{aligned}$$

On the other hand, if we write $A = \{a_1, a_2, \ldots, a_l\}$ and put $a_0 = 0$ and $a_{l+1} = \infty$, it follows for all $i = 1, 2, \ldots, l+1$ that

$$\big\|\Phi(A)-\Phi(B)\big\| \geq \frac{\Theta}{4}\big\|P_{(a_{i-1},a_i)}\big(\Phi(A)-\Phi(B)\big)\big\| \geq \frac{\Theta^2}{4}\sum_{a_{i-1}<b<a_i} z_{(\alpha,B)}(b).$$

Similarly, if we write $B=\{b_1,b_2,\dots,b_m\}$ and put $b_0=0$ and $b_{m+1}=\infty$, then it follows for all $j=1,2,\dots,m+1$ that

$$\big\|\Phi(A)-\Phi(B)\big\| \geq \frac{\Theta^2}{4}\sum_{b_{j-1}<a<b_j} z_{(\alpha,B)}(a).$$

Thus, for any $i=1,2,\dots,l$ and any $j=1,2,\dots,m$

$$\big\|\Phi(A)-\Phi(B)\big\| \geq \frac{\Theta^2}{8}\Big[\sum_{a_{i-1}<b<a_i} z_{(\alpha,B)}(b) + \sum_{b_{j-1}<a<b_j} z_{(\alpha,B)}(a)\Big],$$

which implies our claim. □

We finish this section with an observation which we will need later.

LEMMA 7.3
Let ξ and γ be countable ordinal numbers with $\gamma<\beta=\omega^\xi$. Let $B_1<\dots<B_d$ be in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ such that $\bar{B}=\{\min(B_j):1\leq j\leq d\}$ is a nonmaximal $\mathcal{S}_\beta$-set with $l_1(\bar{B})>0$ ($l_1(A)$ for $A\in[\mathbb{N}]^{<\omega}$ was defined before Lemma 3.7), and set $D=\bigcup_{j=1}^d B_j\in\mathcal{S}_{\beta(\gamma+1)}$. Then for every A, B in $\mathcal{S}_{\beta\gamma}$ with $D<A$ and $D<B$ we have

$$d_{1,\beta\gamma}(A,B)=\frac{1}{\zeta(\beta,\bar{B})}d_{1,\beta(\gamma+1)}(D\cup A,D\cup B) \qquad \textit{and} \tag{80}$$

$$d_{\infty,\beta\gamma}(A,B)=\frac{1}{\zeta(\beta,\bar{B})}d_{\infty,\beta(\gamma+1)}(D\cup A,D\cup B). \tag{81}$$

Proof
We will only prove (80), as the proof of (81) uses the same argument. Let C be the maximal element in $\mathcal{S}_{\beta\gamma}$ such that $C\preceq A$ and $C\preceq B$. Note that $\tilde{C}=D\cup C$ is the largest element of $\mathcal{S}_{\beta(\gamma+1)}$ such that $\tilde{C}\preceq D\cup A$ and $\tilde{C}\preceq D\cup B$. Define $\tilde{B}_1=\bar{B}\cup\{\min(A)\}$ and $\tilde{B}_2=\bar{B}\cup\{\min(B)\}$, and observe that, since $l_1(\bar{B})>0$, $\zeta(\beta,\tilde{B}_1)=\zeta(\beta,\tilde{B}_2)=\zeta(\beta,\bar{B})$. Using (34) in Proposition 3.4 we conclude the following:

$$\begin{aligned}\sum_{a\in(D\cup A)\setminus\tilde{C}} z_{(\beta(\gamma+1),D\cup A)}(a) &= \zeta(\beta,\tilde{B}_1)\sum_{a\in A\setminus C} z_{(\beta\gamma,A)}(a)\\ &= \zeta(\beta,\bar{B})\sum_{a\in A\setminus C} z_{(\beta\gamma,A)}(a),\end{aligned} \tag{82}$$

and similarly we obtain

$$\sum_{a\in(D\cup B)\setminus\tilde{C}} z_{(\beta(\gamma+1),D\cup B)}(a) = \zeta(\beta,\bar{B}) \sum_{a\in B\setminus C} z_{(\beta\gamma,B)}(a). \tag{83}$$

By applying (82) and (83) to the definition of the $d_{1,\alpha}$-metrics, the result easily follows. □

8. The Szlenk index and embeddings of $(\mathcal{S}_\alpha, d_{1,\alpha})$ into X

In this section we show Theorems 8.1 and 8.3, which establish a proof of Theorem B.

THEOREM 8.1

Let X be a separable Banach space, and let α be a countable ordinal. Assume that $\mathrm{Sz}(X) > \omega^\alpha$. *Then $(\mathcal{S}_\alpha, d_{1,\alpha})$ bi-Lipschitzly embeds into X and X^*.*

Before proving Theorem 8.1 we first cover the case in which ℓ_1 embeds into X.

Example 8.2

For each $\alpha < \omega_1$ we want to define a bi-Lipschitz embedding of $(\mathcal{S}_\alpha, d_{1,\alpha})$ into a Banach space X and its dual X^* under the assumption that ℓ_1 embeds into X. We first choose for every $A \in [\mathbb{N}]^{<\omega}$ a spread $\tilde{A}$ of A as in Example 5.6. Then we define for $\alpha < \omega_1$

$$\Phi : \mathcal{S}_\alpha \to \ell_1, \qquad A \mapsto \sum_{D\preceq A} \zeta(\alpha, D) e_{\max(\tilde{D})}.$$

Since for $A, B \in \mathcal{S}_\alpha$ it follows that

$$\begin{aligned}\|\Phi(A) - \Phi(B)\| &= \Big\| \sum_{C\prec D\preceq A} \zeta(\alpha, D) e_{\max(\tilde{D})} - \sum_{C\prec D\preceq B} \zeta(\alpha, D) e_{\max(\tilde{D})} \Big\| \\ &= \sum_{C\prec D\preceq A} \zeta(\alpha, D) + \sum_{C\prec D\preceq B} \zeta(\alpha, D) = d_{1,\alpha}(A, B),\end{aligned}$$

where $C \in \mathcal{S}_\alpha$ is the maximal element for which $C \preceq A$ and $C \preceq B$. It follows that Φ is an isometric embedding of $(\mathcal{S}_\alpha, d_{1,\alpha})$ into ℓ_1.

Thus, if ℓ_1 embeds into X, then $(\mathcal{S}_\alpha, d_{1,\alpha})$ bi-Lipschitzly embeds into X. Additionally, ℓ_∞ is a quotient of X^* in that case, and since ℓ_1 embeds into ℓ_∞, it follows easily that ℓ_1 embeds into X^* and, thus, that $(\mathcal{S}_\alpha, d_{1,\alpha})$ also bi-Lipschitzly embeds into X^*.

Proof of Theorem 8.1

For the case in which X contains a copy of ℓ_1 our claim follows from Example 8.2.

Thus, we may assume that ℓ_1 does not embed into X. Thus, we can apply Theorem 5.3, (a) $\iff$ (c) and obtain $\varepsilon > 0$, a tree $(z^*_A : A \in \mathcal{S}_\alpha) \subset B_{X^*}$, and a w-null tree $(z_A : A \in \mathcal{S}_\alpha) \subset B_X$, so that

$$z^*_B(z_A) > \varepsilon \quad \text{for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \text{ with } A \preceq B, \tag{84}$$

$$\left|z^*_B(z_A)\right| \leq \frac{\varepsilon}{2} \quad \text{for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \text{ with } A \not\preceq B. \tag{85}$$

Then we define

$$\Phi : \mathcal{S}_A \to X, \qquad A \mapsto \sum_{D \preceq A} \zeta(\alpha, D) z_D.$$

If $A, B \in \mathcal{S}_\alpha$ and $C \in \mathcal{S}_\alpha$ is the maximal element of $\mathcal{S}_\alpha$ for which $C \preceq A$ and $C \preceq B$, we note that

$$\begin{aligned}\left\|\Phi(A) - \Phi(B)\right\| &= \Big\| \sum_{C \prec D \preceq A} \zeta(\alpha, D) z_D - \sum_{C \prec D \preceq B} \zeta(\alpha, D) z_D \Big\| \\ &\leq \sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) = d_{1,\alpha}(A, B).\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\left\|\Phi(A) - \Phi(B)\right\| &= \Big\| \sum_{C \prec D \preceq A} \zeta(\alpha, D) z_D - \sum_{C \prec D \preceq B} \zeta(\alpha, D) z_D \Big\| \\ &\geq z^*_A\Big(\sum_{C \prec D \preceq A} \zeta(\alpha, D) z_D - \sum_{C \prec D \preceq B} \zeta(\alpha, D) z_D \Big) \\ &\geq \varepsilon \sum_{C \prec D \preceq A} \zeta(\alpha, D) - \frac{\varepsilon}{2} \sum_{C \prec D \preceq B} \zeta(\alpha, D).\end{aligned}$$

Similarly we can show that

$$\left\|\Phi(A) - \Phi(B)\right\| \geq \varepsilon \sum_{C \prec D \preceq B} \zeta(\alpha, D) - \frac{\varepsilon}{2} \sum_{C \prec D \preceq A} \zeta(\alpha, D),$$

and thus,

$$\left\|\Phi(A) - \Phi(B)\right\| \geq \frac{\varepsilon}{4}\Big(\sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) \Big) = \frac{\varepsilon}{4} d_{1,\alpha}(A, B).$$

To define a Lipschitz embedding from $(\mathcal{S}_\alpha, d_{1,\alpha})$ into X^*, we let

$$\Psi : \mathcal{S}_\alpha \to X^*, \qquad A \mapsto \sum_{D \preceq A} \zeta(\alpha, D) z^*_D.$$

As in the case of Φ it is easy to see that Ψ is a Lipschitz function with constant not exceeding the value 1. Again if $A, B \in \mathcal{S}_\alpha$, let $C \in \mathcal{S}_\alpha$ be the maximal element of $\mathcal{S}_\alpha$ for which $C \preceq A$ and $C \preceq B$. In the case in which $C \prec A$, we let $C^+ \in \mathcal{S}_\alpha$ be the minimal element for which $C \prec C^+ \preceq A$. We note that $C^+ \not\preceq D$ for any $D \in \mathcal{S}_\alpha$ with $C \prec D \preceq B$, and it therefore follows that

$$\begin{aligned}\left\|\Psi(A)-\Psi(B)\right\| &= \Big\| \sum_{C\prec D\preceq A} \zeta(\alpha, D) z_D^* - \sum_{C\prec D\preceq B} \zeta(\alpha, D) z_D^* \Big\| \\ &\geq \Big(\sum_{C\prec D\preceq A} \zeta(\alpha, D) z_D^* - \sum_{C\prec D\preceq B} \zeta(\alpha, D) z_D^* \Big)(z_{C^+}) \\ &\geq \varepsilon \sum_{C\prec D\preceq A} \zeta(\alpha, D) - \frac{\varepsilon}{2} \sum_{C\prec D\preceq B} \zeta(\alpha, D).\end{aligned}$$

If $C = A$, we arrive trivially to the same inequality. Similarly we obtain that

$$\left\|\Psi(A)-\Psi(B)\right\| \geq \varepsilon \sum_{C\prec D\preceq B} \zeta(\alpha, D) - \frac{\varepsilon}{2} \sum_{C\prec D\preceq A} \zeta(\alpha, D).$$

This yields

$$\begin{aligned}\left\|\Psi(A)-\Psi(B)\right\| &\geq \frac{\varepsilon}{4}\Big(\sum_{C\prec D\preceq A} \zeta(\alpha, D) + \sum_{C\prec D\preceq B} \zeta(\alpha, D)\Big) \\ &= \frac{\varepsilon}{4} d_{1,\alpha}(A, B),\end{aligned}$$

which finishes the proof of our claim. □

The following dual result can be deduced from Proposition 5.5 in the same way as we deduced Theorem 8.1 from Theorem 5.3.

THEOREM 8.3
Assume that X is a Banach space having a separable dual X^* with $\mathrm{Sz}(X^*) > \omega^\alpha$. *Then $(\mathcal{S}_\alpha, d_{1,\alpha})$ can be bi-Lipschitzly embedded into X.*

9. Refinement argument

Before providing a proof of Theorem C and, thus, the still missing implication of Theorem A, we will introduce in this and the next section some more notation and make some preliminary observations. Then we will consider maps $\Phi : \mathcal{S}_\alpha \to X$ satisfying weaker conditions compared to the ones required by Theorems A and C. On the one hand it will make an argument using transfinite induction possible; on the other hand it is sufficient to arrive at the desired conclusions.

Definition 9.1
Let $\alpha < \omega_1$. For $r \in (0, 1]$ we define

$$\mathcal{S}_\alpha^{(r)} = \Big\{A \in \mathcal{S}_\alpha : \sum_{D \preceq A} \zeta(\alpha, D) \le r\Big\}.$$

It is not hard to see that $\mathcal{S}_\alpha^{(r)}$ is a closed subset of $\mathcal{S}_\alpha$ and, hence, compact and closed under restrictions. We also put

$$\mathcal{M}_\alpha^{(r)} = \mathrm{MAX}(\mathcal{S}_\alpha^{(r)}) = \{A \in \mathcal{S}_\alpha^{(r)} : A \text{ is maximal in } \mathcal{S}_\alpha^{(r)} \text{ with respect to } \prec\}$$

and, for $A \in \mathcal{S}_\alpha$,

$$\mathcal{M}_\alpha^{(r)}(A) = \big\{B \in \mathcal{S}_\alpha(A) : A \cup B \in \mathcal{M}_\alpha^{(r)}\big\}.$$

Definition 9.2
Let X be a Banach space, α be a countable ordinal number, L be an infinite subset of $\mathbb{N}$, and A_0 be a set in $\mathcal{S}_\alpha$ that is either empty or a singleton. A map $\Phi : \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega} \to X$ is called a *semiembedding of* $\mathcal{S}_\alpha \cap [L]^{<\omega}$ *into* X *starting at* A_0 if there is a number $c > 0$ such that

$$\begin{aligned}&\big\|\Phi(A) - \Phi(B)\big\| \\ &\quad \le d_{1,\alpha}(A_0 \cup A, A_0 \cup B) \quad \text{for all } A, B \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega} \qquad \text{and} \end{aligned} \tag{86}$$

for all $A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}$, with $l_1(A_0 \cup A) > 0$, for all $r \in (0, 1]$, and for all B_1, B_2 in $\mathcal{M}^{(r)}(A_0 \cup A) \cap [L]^{<\omega}$ with $B_1 < B_2$

$$\big\|\Phi(A \cup B_1) - \Phi(A \cup B_2)\big\| \ge c\, d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2). \tag{87}$$

(Note that $l_1(A)$ for $A \in [\mathbb{N}]^{<\omega}$ was introduced in Definition 3.6.) We call the supremum of all numbers $c > 0$ such that (87) holds for all $A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}$ and B_1, $B_2 \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [L]^{<\omega}$, with $B_1 < B_2$, the *semiembedding constant of* Φ and denote it by $c(\Phi)$.

Remark
If $\Phi : \mathcal{S}_\alpha \to X$ is for some $0 < c < C$ a c-lower-$d_{\infty,\alpha}$ and C-upper-$d_{1,\alpha}$ embedding, we can, after rescaling Φ if necessary, assume that $C = 1$, and from the definition of $d_{1,\alpha}$ and $d_{\infty,\alpha}$ we can easily see that, for every A_0 that is either empty or a singleton and for L in $[\mathbb{N}]^\omega$, the restriction $\Phi|_{\mathcal{S}_\alpha(A_0)\cap[L]^{<\omega}} X$ is a semiembedding.

Assume that $\Phi : \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega} \to X$ is a semiembedding of $\mathcal{S}_\alpha \cap [L]^{<\omega}$ into X starting at A_0. For $A \in \mathcal{S}_\alpha(A_0)$, with $A \ne \emptyset$, we put $A' = A \setminus \{\max(A)\}$ and define

$$x_{A_0\cup A} = \frac{1}{\zeta(\alpha, A_0 \cup A)}\big(\Phi(A) - \Phi(A')\big).$$

If $A_0 = \emptyset$, put $x_\emptyset = \Phi(\emptyset)$, whereas if A_0 is a singleton, define $x_\emptyset = 0$ and $x_{A_0} = (1/\zeta(\alpha, A_0))\Phi(\emptyset)$. Note that $\{x_\emptyset\} \cup \{x_{A_0\cup A} : A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}\} \subset B_X$. Recall that $\zeta(\alpha, \emptyset) = 0$, and hence, for $A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}$, we have

$$\Phi(A) = x_\emptyset + \sum_{\emptyset \preceq D \preceq A_0\cup A} \zeta(\alpha, D)x_D.$$

We say in that case that the *family* $\{x_\emptyset\} \cup \{x_{A_0\cup A} : A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}\}$ *generates* Φ. In that case the map $\Phi_0 : \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega} \to X$ with $\Phi_0 = \Phi - x_\emptyset$, that is, for $A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}$

$$\Phi_0(A) = \sum_{\emptyset \preceq D \preceq A_0\cup A} \zeta(\alpha, D)x_D, \tag{88}$$

is also a semiembedding of $\mathcal{S}_\alpha \cap [L]^{<\omega}$ into X starting at A_0, with $c(\Phi_0) = c(\Phi)$.

LEMMA 9.3

Let $\gamma, \xi < \omega_1$, with $\gamma < \beta = \omega^\xi$, and let $B_1 < \cdots < B_d$ be in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ such that $\bar{B} = \{\min(B_j) : 1 \le j \le d\}$ is a nonmaximal $\mathcal{S}_\beta$-set with $l_1(\bar{B}) > 0$. Set $D = \bigcup_{j=1}^d B_j$, let $r \in (0, 1]$, and also let $A \in \mathcal{M}^{(r)}_{\beta\gamma}$ with $D < A$. Then, if $r_0 = \sum_{C \preceq D} \zeta(\beta(\gamma+1), C) + \zeta(\beta, \bar{B})r$, we have that $A \in \mathcal{M}^{(r_0)}_{\beta(\gamma+1)}(D)$.

Proof

From Proposition 3.4 and Lemma 3.7(c) we obtain that for $C \preceq A$ we have

$$\zeta\big(\beta(\gamma+1), D\cup C\big) = \zeta\big(\beta, \bar{B} \cup \{\min(A)\}\big)\zeta(\beta\gamma, C) = \zeta(\beta, \bar{B})\zeta(\beta\gamma, C),$$

which implies that $D \cup A \in \mathcal{S}^{(r_0)}_{\beta(\gamma+1)}$. If we assume that $D \cup A$ is not in $\mathcal{M}^{(r_0)}_{\beta(\gamma+1)}$, then there is $B \in \mathcal{S}^{(r_0)}_{\beta(\gamma+1)}$ with $D \cup A \prec B$. Possibly after trimming B, we may assume that $B' = D \cup A$. Define $B_0 = B \setminus D$. Evidently, $A \prec B_0$ and $B_0' = A$. We claim that $B_0 \in \mathcal{S}_{\beta\gamma}$. If we assume that this is not the case, then A is a maximal $\mathcal{S}_{\beta\gamma}$-set. This yields that $\sum_{C \preceq A} \zeta(\beta\gamma, C) = 1$, and hence, $r = 1$ and $r_0 = \sum_{C \preceq D} \zeta(\beta(\gamma+1)) + \zeta(\beta, \bar{B}) = \sum_{C \preceq D\cup A} \zeta(\beta(\gamma+1), C)$, that is, $D \cup A \in \mathcal{M}^{(r_0)}_{\beta(\gamma+1)}$, which we assumed to be false. Thus, we conclude that $B_0 \in \mathcal{S}_{\beta\gamma}$, and thus, by using Proposition 3.4 and the definition of r_0, $B_0 \in \mathcal{S}^{(r)}_{\beta\gamma}$, which is a contradiction, as $A \in \mathcal{M}^{(r)}_{\beta\gamma}$ and $A \prec B_0$. □

LEMMA 9.4

Let $\alpha < \omega_1$, let $N \in [\mathbb{N}]^{<\omega}$, let A_0 be a subset of $\mathbb{N}$ that is either empty or a singleton, and let $c \in (0, 1]$. Let $\Psi : \mathcal{S}_\alpha(A_0) \cap [N]^{<\omega} \to X$ be a semiembedding of $\mathcal{S}_\alpha \cap [N]^{<\omega}$

into X starting at A_0, generated by a family of vectors $\{z_\emptyset\}\cup\{z_{A\cup A_0} : A\in \mathcal{S}_\alpha(A_0)\cap [N]^{<\omega}\}$, such that $c(\Psi)>c$. Let $\varepsilon<c(\Psi)-c$, and let $\{\tilde z_\emptyset\}\cup\{\tilde z_{A\cup A_0} : A\in \mathcal{S}_\alpha(A_0)\cap [N]^{<\omega}\}\subset B_X$, with $\|\tilde z_{A_0\cup A}-z_{A_0\cup A}\|<\varepsilon$, for all $A\in\mathcal{S}_\alpha(A_0)\cap[N]^{<\omega}$ with $A_0\cup A\neq\emptyset$. Then the map $\tilde\Psi:\mathcal{S}_\alpha(A_0)\cap[N]^{<\omega}\to X$ defined by

$$\tilde\Psi(A)=\sum_{D\preceq A_0\cup A}\zeta(\alpha,D)\tilde z_D,\quad \textit{for } A\in\mathcal{S}_\alpha\cap[N]^{<\omega},$$

is a semiembedding of $\mathcal{S}_\alpha\cap[N]^{<\omega}$ into X starting at A_0 with $c(\tilde\Psi)>c$.

Proof
For any $r\in(0,1]$, any $A\in\mathcal{S}_\alpha(A_0)\cap[L]^{<\omega}$, and $B_1,\ B_2\in\mathcal{M}_\alpha^{(r)}(A_0\cup A)\cap[N]^{<\omega}$ with $B_1<B_2$, we obtain

$$\begin{aligned}
&\big\|\tilde\Psi(A\cup B_1)-\tilde\Psi(A\cup B_2)\big\|\\
&\quad=\Big\|\sum_{A_0\cup A\prec D\preceq A_0\cup A\cup B_1}\zeta(\alpha,D)\tilde z_D-\sum_{A_0\cup A\prec D\preceq A_0\cup A\cup B_2}\zeta(\alpha,D)\tilde z_D\Big\|\\
&\quad\geq\Big\|\sum_{A_0\cup A\prec D\preceq A_0\cup A\cup B_1}\zeta(\alpha,D)z_D-\sum_{A_0\cup A\prec D\preceq A_0\cup A\cup B_2}\zeta(\alpha,D)z_D\Big\|\\
&\qquad-\varepsilon\Big(\sum_{A_0\cup A\prec D\preceq A_0\cup A\cup B_1}\zeta(\alpha,D)+\sum_{A_0\cup A\prec D\preceq A_0\cup A\cup B_2}\zeta(\alpha,D)\Big)\\
&\quad\geq\big(c(\Psi)-\varepsilon\big)d_{1,\alpha}(A_0\cup A\cup B_1,A_0\cup A\cup B_2),
\end{aligned}$$

which implies (87). Then (86) follows from the fact that $\tilde z_{A_0\cup A}\in B_X$ for all $A\in\mathcal{S}_\alpha(A_0)\cap[N]^{<\omega}$ with $A_0\cup A\neq\emptyset$. □

For the rest of the section we will assume that X has a bimonotone FDD (E_n). For finite or cofinite sets $A\subset\mathbb{N}$, we denote the canonical projections from X onto $\overline{\operatorname{span}(E_j:j\in A)}$ by P_A, that is,

$$P_A:X\to X,\qquad \sum_{j=1}^\infty x_j\mapsto\sum_{j\in A}x_j,\quad \text{for } x=\sum_{j=1}^\infty x_j\in X,\text{ with } x_j\in E_j,\text{ for } j\in\mathbb{N},$$

and we write P_j instead of $P_{\{j\}}$, for $j\in\mathbb{N}$. We denote the linear span of the E_j's by $c_{00}(E_j:j\in\mathbb{N})$, that is,

$$c_{00}(E_j:j\in\mathbb{N})=\Big\{\sum_{j=1}^\infty x_j : x_j\in F_j,\ \text{for } j\in\mathbb{N}\text{ and }\#\{j:x_j\neq 0\}<\infty\Big\}.$$

Definition 9.5
Let α be a countable ordinal number, let $M \in [\mathbb{N}]^\omega$, and let A_0 be a subset of $\mathbb{N}$ that is either empty or a singleton. A semiembedding $\Phi : \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega} \to X$ of $\mathcal{S}_\alpha \cap [M]^{<\omega}$ into X starting at A_0 is said to be *c-refined*, for some $c \leq c(\Phi)$, if the following conditions are satisfied:

(a) the family $\{x_\emptyset\} \cup \{x_{A_0 \cup A} : A \in \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega}\}$ generating Φ is contained in $B_X \cap c_{00}(E_j : j \in \mathbb{N})$;

(b) for all $A \in \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega}$ with $A_0 \cup A \neq \emptyset$ we have
$$\max(A_0 \cup A) \leq \max \operatorname{supp}(x_{A_0 \cup A}) < \min\{m \in M : m > \max(A_0 \cup A)\};$$

(c) for all $r \in (0,1]$, $A \in \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega}$, with $l_1(A_0 \cup A) > 0$, and B_1, B_2 in $\mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [M]^{<\omega}$, with $B_1 < B_2$, we have
$$\begin{aligned}&\big\| P_{(\max \operatorname{supp}(x_{A_0 \cup A}),\infty)}\big(\Phi(A \cup B_1) - \Phi(A \cup B_2)\big)\big\| \\ &\qquad \geq c d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2);\end{aligned}$$

(d) for all $r \in (0,1]$, $A \in \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega}$, and B in $\mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [M]^{<\omega}$ we have
$$\big\| P_{(\max \operatorname{supp}(x_{A_0 \cup A}),\infty)}\big(\Phi(A \cup B)\big)\big\| \geq \frac{c}{2} \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C).$$

Remark 9.6
Let $\xi < \omega_1$, let $\gamma \leq \beta = \omega^{\omega^\xi}$ be a limit ordinal, let $0 < c \leq 1$, and let $M \in [\mathbb{N}]^\omega$. If $a_0 \in \mathbb{N}$, we note that $\mathcal{S}_{\beta\gamma}(\{a_0\}) \cap [M]^{<\omega} = \mathcal{S}_{\beta\eta(\gamma,a_0)}(\{a_0\}) \cap [M]^{<\omega}$, and $\zeta(\beta\gamma, \{a_0\} \cup D) = \zeta(\beta\eta(\gamma, a_0), \{a_0\} \cup D)$ for $D \in S_{\beta\gamma}(\{a_0\}) \cap [M]^{<\omega}$, where $(\eta(\gamma, n))_{n \in \mathbb{N}}$ is the sequence provided by Proposition 2.6. It follows that a semiembedding of $\mathcal{S}_{\beta\gamma}(\{a_0\}) \cap [M]^{<\omega}$ into X, starting at $\{a_0\}$, that is c-refined is a c-refined semiembedding of $S_{\beta\eta(\gamma,n)}(\{a_0\}) \cap [M]^{<\omega}$ into X.

Additionally, if $\Phi : \mathcal{S}_{\beta\gamma} \cap [M]^{<\omega} \to X$ is a semiembedding of $\mathcal{S}_{\beta\gamma} \cap [M]^{<\omega}$ into X, starting at $\emptyset$, that is c-refined, then for every $a_0 \in M$ and $N = M \cap [a_0, \infty)$ the map $\Psi = \Phi|_{\mathcal{S}_{\beta\eta(\gamma,a_0)}(\{a_0\}) \cap [N]^{<\omega}}$ is a semiembedding of $\mathcal{S}_{\beta\eta(\gamma,a_0)} \cap [N]^{<\omega}$ into X, starting at $\{a_0\}$, that is c-refined. Furthermore, Ψ is generated by the family $\{x_{A_0 \cup A} : A \in \mathcal{S}_{\beta\eta(\gamma,a_0)}(\{a_0\}) \cap [N]^{<\omega}\}$, where $\{x_A : A \in \mathcal{S}_{\beta\gamma} \cap [M]^{<\omega}\}$ is the family generating Φ.

LEMMA 9.7
Let $\xi, \gamma < \omega_1$, with $\gamma < \beta = \omega^{\omega^\xi}$, let $M \in [\mathbb{N}]^\omega$, and let A_0 be a subset of $\mathbb{N}$ that is either empty or a singleton. Also let $\Phi : \mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [M]^{<\omega} \to X$ be a semiembedding of $\mathcal{S}_{\beta(\gamma+1)} \cap [M]^{<\omega}$ into X, starting at A_0, that is c-refined. The family generating Φ is denoted by $\{x_\emptyset\} \cup \{x_{A_0 \cup A} : A \in \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega}\}$. Extend the set A_0 to a set

$A_0 \cup A_1$, $A_0 < A_1$, which can be written as $A_0 \cup A_1 = \bigcup_{j=1}^k B_j \in \mathcal{S}_{\beta(\gamma+1)} \cap [A_0 \cup M]^{<\omega}$*, where* $B_1 < \cdots < B_k$ *are in* $\mathrm{MAX}(S_{\beta\gamma})$ *and* $\bar{B} = \{\min(B_j) : 1 \le j \le k\}$ *is a nonmaximal* $\mathcal{S}_\beta$*-set with* $l_1(\bar{B}) > 0$.

Then, for $N = M \cap (\max(A_0 \cup A_1), \infty)$ *and* $n_0 = \max \mathrm{supp}(x_{A_0 \cup A_1})$*, the map*

$$\Psi : \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega} \to X, \qquad A \mapsto \frac{1}{\zeta(\beta, \bar{B})} P_{(n_0,\infty)}\big(\Phi(A_1 \cup A)\big)$$

is a semiembedding of $\mathcal{S}_{\beta\gamma} \cap [N]^\omega$ *into* X*, starting at* $\emptyset$*, that is* c*-refined. Moreover,* Ψ *is generated by the family* $\{z_A : A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}\}$*, where*

$$z_\emptyset = 0 \qquad \text{and} \qquad z_A = P_{(n_0,\infty)}(x_{A_0 \cup A_1 \cup A}) \quad \text{for } A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega} \setminus \{\emptyset\}. \quad (89)$$

Proof

By Lemma 7.3 we easily obtain that, for $A, B \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}$,

$$\begin{aligned} \big\|\Psi(A) - \Psi(B)\big\| &\le \frac{1}{\zeta(\beta, \bar{B})} d_{1,\beta(\gamma+1)}(A_0 \cup A_1 \cup A, A_0 \cup A_1 \cup B) \\ &= d_{1,(\beta\gamma)}(A, B), \end{aligned} \quad (90)$$

that is, (86) from Definition 9.2 is satisfied for Ψ. We will show that (86) from Definition 9.2 is satisfied for Ψ as well. Let $r \in (0, 1]$, let A be in $\mathcal{S}_{\beta\gamma}$, with $l_1(A) > 0$, and let $B_1 < B_2$ in $\mathcal{M}^{(r)}_{\beta\gamma}(A) \cap [N]^{<\omega}$ (i.e., $A \cup B_1, A \cup B_2 \in \mathcal{M}^{(r)}_{\beta\gamma}$). Note that we have $l_1(A_0 \cup A_1 \cup A) > 0$. If we set $r_0 = \sum_{C \preceq A_0 \cup A_1} \zeta(\beta(\gamma+1), C) + \zeta(\beta, \bar{B})r$, by Lemma 9.3 we deduce that $A \cup B_1$ and $A \cup B_2$ are in $\mathcal{M}^{(r_0)}_{\beta(\gamma+1)}(A_0 \cup A_1) \cap [N]^{<\omega}$, that is, $B_1, B_2 \in \mathcal{M}^{(r_0)}_{\beta(\gamma+1)}(A_0 \cup (A_1 \cup A)) \cap [N]^{<\omega}$. Definition 9.5(b) implies $n_0 \le \max \mathrm{supp}(x_{A_0 \cup A_1 \cup A})$, and thus, by Definition 9.5(c) for Φ we deduce

$$\begin{aligned} &\big\|\Psi(A \cup B_1) - \Psi(A \cup B_2)\big\| \\ &\quad = \frac{1}{\zeta(\beta, \bar{B})} \big\| P_{(n_0,\infty)}\big(\Phi(A_1 \cup A \cup B_1) - \Phi(A_1 \cup A \cup B_2)\big)\big\| \\ &\quad \ge \frac{1}{\zeta(\beta, \bar{B})} \big\| P_{(\max \mathrm{supp}(x_{A_0 \cup A_1 \cup A}),\infty)}\big(\Phi(A_1 \cup A \cup B_1) - \Phi(A_1 \cup A \cup B_2)\big)\big\| \\ &\quad \ge \frac{c}{\zeta(\beta, \bar{B})} d_{1,\beta(\gamma+1)}(A_0 \cup A_1 \cup A \cup B_1, A_0 \cup A_1 \cup A \cup B_2) \\ &\quad = c d_{1,\beta\gamma}(A \cup B_1, A \cup B_2), \end{aligned} \quad (91)$$

where the last equality follows from Lemma 7.3. In particular, (90) and (91) yield that $\Psi : \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega} \to X$ is a semiembedding of $\mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}$ into X starting at $\emptyset$ with $c(\Psi) \ge c$.

It remains to show that Ψ satisfies Definitions 9.5(a)–9.5(d). Observe that Definition 9.5(b) implies that, for $C \preceq A_0 \cup A_1$, we have $P_{(n_0,\infty)}(x_C) = 0$. We combine this with (34) of Proposition 3.4 to obtain that, for $A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}$,

$$\begin{aligned}\Psi(A) &= \frac{1}{\zeta(\beta,\bar{B})} P_{(n_0,\infty)}\big(\Phi(A_1 \cup A)\big)\\ &= \frac{1}{\zeta(\beta,\bar{B})} \sum_{C \preceq A_0 \cup A_1 \cup A} \zeta\big(\beta(\gamma+1), C\big) P_{(n_0,\infty)}(x_C)\\ &= \frac{1}{\zeta(\beta,\bar{B})} \sum_{A_0 \cup A_1 \prec C \preceq A_0 \cup A_1 \cup A} \zeta\big(\beta(\gamma+1), C\big) P_{(n_0,\infty)}(x_C)\\ &= \sum_{C \preceq A} \zeta(\beta\gamma, C) P_{(n_0,\infty)}(x_{A_0 \cup A_1 \cup C}). \end{aligned} \tag{92}$$

For $A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}$ define $z_A = P_{(n_0,\infty)}(x_{A_0 \cup A_1 \cup A})$ and $z_\emptyset = 0$. It then easily follows by (92) that Ψ is generated by the family $\{z_A : A \in \mathcal{S}_\alpha \cap [N]^{<\omega}\}$. Moreover, as $\max\operatorname{supp}(z_A) = \max\operatorname{supp}(x_{A_0 \cup A_1 \cup A})$, it is straightforward to check that Definitions 9.5(a) and 9.5(b) are satisfied for Ψ. Observing that for all $A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}$, with $A \neq \emptyset$ (which is the case when $l_1(A) > 0$), we have $\max\operatorname{supp}(z_A) = \max\operatorname{supp}(x_{A_0 \cup A_1 \cup A}) \geq n_0$. An argument similar to the one used to obtain (91) also yields that Ψ satisfies Definitions 9.5(c) and 9.5(d). □

The main result of this section is the following refinement argument.

LEMMA 9.8
Let $\alpha < \omega_1$, $M \in [\mathbb{N}]^\omega$, and A_0 be a subset of $\mathbb{N}$ that is either empty or a singleton. Also let $\Phi : \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega} \to X$ be a semiembedding of $\mathcal{S}_\alpha \cap [M]^{<\omega}$ into X starting at A_0. Then, for every $c < c(\Phi)$, there exist $N \in [M]^\omega$ and a semiembedding $\tilde{\Phi} : \mathcal{S}_\alpha(A_0) \cap [N]^{<\omega} \to X$ of $\mathcal{S}_\alpha \cap [N]^{<\omega}$ into X, starting at A_0, that is c-refined.

Proof
Put $\tilde{c} = (c(\Phi) + c)/2$. Let $\{x_\emptyset\} \cup \{x_{A_0 \cup A} : A \in \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega}\}$ be the vectors generating Φ, and choose $\eta > 0$ with $\eta < c(\Phi) - \tilde{c}$. After shifting we can assume without loss of generality that $x_\emptyset = 0$. Set $\tilde{x}_\emptyset = 0$, and choose for each $A \in \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega}$, with $A_0 \cup A \neq \emptyset$, a vector $\tilde{x}_{A_0 \cup A} \in B_X \cap c_{00}(E_j : j \in \mathbb{N})$ such that
(a) $\|\tilde{x}_{A_0 \cup A} - x_{A_0 \cup A}\| < \eta/2$, and $\max(A_0 \cup A) \leq \max\operatorname{supp}(\tilde{x}_{A_0 \cup A})$.
Moreover, recursively choose $\tilde{m}_1 < \cdots < \tilde{m}_k < \cdots$ in M so that we have

$$\tilde{m}_{k+1} > \max\big\{\max\operatorname{supp}(\tilde{x}_{A_0 \cup A}) : A \in \mathcal{S}_\alpha(A_0) \cap \big[\{\tilde{m}_1, \ldots, \tilde{m}_k\}\big]\big\} \quad \text{for all } k.$$

Define $\tilde{M} = \{\tilde{m}_k : k \in \mathbb{N}\}$ and $\tilde{\Phi} : \mathcal{S}_\alpha(A_0) \cap [\tilde{M}]^{<\omega} \to X$ so that for all $A \in \mathcal{S}_\alpha(A_0) \cap [\tilde{M}]^{<\omega}$ we have

$$\tilde{\Phi}(A) = \sum_{C \preceq A_0 \cup A} \zeta(\alpha, C)\tilde{x}_C.$$

By Lemma 9.4, $\tilde{\Phi}$ is a semiembedding from $\mathcal{S}_\alpha \cap [\tilde{M}]^{<\omega}$ into X, starting at A_0, for which $c(\tilde{\Phi}) > \tilde{c} > c$ and for which Definitions 9.5(a) and 9.5(b) are satisfied.

The goal is to find $N \in [\tilde{M}]^\omega$ so that, by restricting $\tilde{\Phi}$ to $\mathcal{S}_\alpha(A_0) \cap [N]^{<\omega}$, Definitions 9.5(c) and 9.5(d) are satisfied as well. Put $M_0 = \tilde{M}$. Recursively, we will choose for every $k \in \mathbb{N}$ an infinite set $M_k \subset M_{k-1}$ so that for each $k \in \mathbb{N}$ the following conditions are met:

(b) $\min(M_{k-1}) < \min(M_k)$.

(c) Putting $m_j = \min(M_j)$ for $j = 1, \ldots, k-1$, for every $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \ldots, m_{k-1}\}]$ with $l_1(A_0 \cup A) > 0$, $r \in (0, 1]$, and for $B_1, B_2 \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [\{m_1, \ldots, m_{k-1}\} \cup M_k]^{<\omega}$ with $B_1 < B_2$, we have

$$\begin{aligned}\big\| P_{(\max \operatorname{supp}(\tilde{x}_{A_0 \cup A}), \infty)}\big(\tilde{\Phi}(A \cup B_1) - \tilde{\Phi}(A \cup B_2)\big)\big\| \\ \geq \tilde{c} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2).\end{aligned}$$

(d) For every $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \ldots, m_{k-1}\}]$ with $l_1(A_0 \cup A) > 0$, $r \in (0, 1]$, $B \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [\{m_1, \ldots, m_{k-1}\} \cup M_k]^{<\omega}$, we have

$$\big\| P_{(\max \operatorname{supp}(\tilde{x}_{A_0 \cup A}), \infty)}\big(\tilde{\Phi}(A \cup B)\big)\big\| \geq \frac{c}{2} \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C).$$

(If $k = 1$, then $[\{m_1, \ldots, m_{k-1}\}] = \{\emptyset\}$.)

If we assume that such a sequence $(M_k)_k$ has been chosen, it is straightforward to check that $N = \{m_k : k \in \mathbb{N}\}$ is the desired set. In the case in which $k = 1$, for $A \in \mathcal{S}_\alpha(A_0) \cap \{\emptyset\}$ we have $A = \emptyset$. Hence, if $A_0 = \emptyset$, then for all $A \in \mathcal{S}_\alpha(A_0) \cap \{\emptyset\}$ we have $A_0 \cup A = \emptyset$, that is, $l_1(A_0 \cup A) = 0$, and hence, (c) and (d) are always satisfied. Choosing M_1 satisfying (b) completes the first inductive step. If, on the other hand, A_0 is a singleton, then for all $A \in \mathcal{S}_\alpha(A_0) \cap \{\emptyset\}$ we have $A_0 \cup A = A_0$, that is, $l_1(A_0 \cup A) > 0$. This means that conditions (c) and (d) are reduced to the case in which $A = \emptyset$. The choice of M_1 is done in the same manner as in the general inductive step, and we omit it.

Assume that we have chosen, for some $k \geq 1$, infinite sets $M_k \subset M_{k-1} \subset \cdots \subset M_1 \subset M_0$ such that (b), (c), and (d) are satisfied for all $1 \leq k' \leq k$. Observe that the inductive assumption implies that it is enough to choose $M_{k+1} \in [M_k]^\omega$ satisfying (b) and the conditions (c) and (d) for sets $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \ldots, m_{k-1}\}]$ with $l_1(A_0 \cup A) > 0$ and $\max(A) = m_k = \min(M_k)$ (or $A = \emptyset$, in the case in which $k = 1$ and A_0 is a singleton). We set

$$\delta = \min\big\{\zeta(\alpha, A_0 \cup A) : A \in \mathcal{S}_\alpha(A_0) \cap \big[\{m_1, m_2, \ldots, m_k\}\big] \text{ and } A_0 \cup A \neq \emptyset\big\},$$

$$\varepsilon = \frac{\delta}{30}(\tilde{c} - c),$$

$$d = \max\big\{\max \operatorname{supp}(\tilde{x}_{A_0 \cup A}) : A \in \mathcal{S}_\alpha(A_0) \cap \big[\{m_1, \ldots, m_k\}\big]\big\},$$

and we choose a finite ε-net R of the interval $(0,1]$, with $1 \in R$, which also has the property that, for all $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \ldots, m_k\}]$ and all $j = 0, 1, 2, \ldots, l_1(A_0 \cup A)$,

$$j \cdot \zeta(\alpha, A_0 \cup A) + \sum_{C \preceq A_0 \cup A} \zeta(\alpha, C) \in R. \tag{93}$$

Fix a finite $\frac{\varepsilon}{2}$-net K of the unit ball of the finite-dimensional space $\operatorname{span}(E_j : 1 \leq j \leq d)$. For every $r \in R$ and $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \ldots, m_k\}]$ we apply Proposition 2.14 to $\mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [M_k]^{<\omega}$ and find an infinite subset $\tilde{M}_{k+1}$ of M_k such that, for all $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \ldots, m_k\}]$ and $r \in R$, there exists $y_A^{(r)}$ in K such that

$$\big\| y_A^{(r)} - P_{[1,d]}\big(\tilde{\Phi}(A \cup B)\big)\big\| < \frac{\varepsilon}{2}, \quad \text{for all } B \text{ in } \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [\tilde{M}_{k+1}]^{<\omega}.$$

In particular, note that, for all $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \ldots, m_k\}]$, for all $r \in R$, and for any B_1, B_2 in $\mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [\tilde{M}_{k+1}]^{<\omega}$, we have

$$\big\| P_{[1,d]}\big(\tilde{\Phi}(A \cup B_1) - \tilde{\Phi}(A \cup B_2)\big)\big\| < \varepsilon. \tag{94}$$

Using Lemma 3.7(d) for $l_1(\cdot)$, we can pass to an infinite subset $\widehat{M}_{k+1}$ of $\tilde{M}_{k+1}$, so that (b) is satisfied, and moreover,

$$\begin{aligned} \zeta(\alpha, A_0 \cup A \cup B) < \varepsilon \quad &\text{if } A \in \mathcal{S}_\alpha(A_0) \cap \big[\{m_1, \ldots, m_k\}\big] \qquad \text{and} \\ &B \in \mathcal{S}_\alpha(A_0 \cup A) \cap [M_{k+1}]^{<\omega} \quad \text{with } \#B > l_1(A_0 \cup A) > 0. \end{aligned} \tag{95}$$

We will show that (c) is satisfied. To that end, fix $0 < r \leq 1$, $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \ldots, m_k\}]$ with $\max(A_0 \cup A) = m_k$ and $l_1(A_0 \cup A) > 0$, and $B_1, B_2 \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [\widehat{M}_{k+1}]^{<\omega}$ with $B_1 < B_2$. If both sets B_1 and B_2 are empty, then (c) trivially holds, as the right-hand side of the inequality has to be zero. Otherwise, $B_s \neq \emptyset$, where $s = 1$ or $s = 2$. Note that $\max(A) = m_k$, that is, $A_0 \cup A \neq \emptyset$, and hence, since $l_1(A_0 \cup A) > 0$, putting $\tilde{C} = A_0 \cup A \cup \{\min(B_s)\}$, by the definition of δ we obtain $\zeta(\alpha, \tilde{C}) = \zeta(\alpha, A_0 \cup A) \geq \delta$. This easily yields

$$\begin{aligned} &d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2) \\ &\quad = \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B_1} \zeta(\alpha, C) + \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B_2} \zeta(\alpha, C) \\ &\quad \geq \delta = \frac{30\varepsilon}{\tilde{c} - c}. \end{aligned}$$

Hence,

$$\varepsilon \leq \frac{\tilde{c}-c}{30} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2). \tag{96}$$

Arguing similarly, we obtain

$$r \geq \sum_{C \preceq A_0 \cup A \cup B_s} \zeta(\alpha, C) \geq \sum_{C \preceq A_0 \cup A} \zeta(\alpha, C) + \zeta(\alpha, \tilde{C}) \geq \delta > \min(R). \tag{97}$$

Choose r_0 to be the maximal element of R with $r_0 \leq r$. Since $r_0 \leq r$, we can find $\tilde{B}_1$ and $\tilde{B}_2$ in $\mathcal{M}_\alpha^{(r_0)}(A_0 \cap A) \cap [\widehat{M}_{k+1}]^{<\omega}$ so that $\tilde{B}_1 \preceq B_1$ and $\tilde{B}_2 \preceq B_2$. We will show that

$$\begin{aligned} &d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup \tilde{B}_1) < 3\varepsilon \qquad \text{and} \\ &d_{1,\alpha}(A_0 \cup A \cup B_2, A_0 \cup A \cup \tilde{B}_2) < 3\varepsilon. \end{aligned} \tag{98}$$

We shall only show that this is the case for B_1; for B_2 the proof is identical. If $\tilde{B}_1 = B_1$, then there is nothing to prove, so we may therefore assume that $\tilde{B}_1 \prec B_1$. Define $C_1 = \tilde{B}_1 \cup \{\min(B_1 \setminus \tilde{B}_1)\}$, $r_1 = \sum_{C \preceq A_0 \cup A \cup B_1} \zeta(\alpha, C)$, $\tilde{r}_1 = \sum_{C \preceq A_0 \cup A \cup \tilde{B}_1} \zeta(\alpha, C)$, and $r' = \sum_{C \preceq A_0 \cup A \cup C_1} \zeta(\alpha, C)$. The maximality of $\tilde{B}_1$ in $\mathcal{S}_\alpha^{(r_0)}(A_0 \cup A)$ implies

$$\tilde{r}_1 \leq r_0 < r' \leq r_1. \tag{99}$$

We first assume that $\#C_1 \leq l_1(A_0 \cup A)$. In this case, however, by (93), we obtain that $r' = \sum_{C \preceq A_0 \cup A} \zeta(\alpha, C) + (\#C_1)\zeta(\alpha, A_0 \cup A)$ is in R. This contradicts the maximality of r_0. We conclude that $\#C_1 > l_1(A_0 \cup A)$, which by (95) yields $r' - \tilde{r}_1 = \zeta(\alpha, A_0 \cup A \cup C_1) < \varepsilon$. Hence,

$$\begin{aligned} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup \tilde{B}_1) &= r_1 - \tilde{r}_1 \\ &= (r_1 - r') + (r' - \tilde{r}_1) \\ &< (r_1 - r_0) + \varepsilon < 3\varepsilon. \end{aligned} \tag{100}$$

We now verify (c) as follows:

$$\begin{aligned} &\big\| P_{(d,\infty)}\big(\tilde{\Phi}(A \cup B_1) - \tilde{\Phi}(A \cup B_2)\big)\big\| \\ &\quad \geq \big\| P_{(d,\infty)}\big(\tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2)\big)\big\| \\ &\qquad - \big(\big\| P_{(d,\infty)}\big(\tilde{\Phi}(A \cup B_1) - \tilde{\Phi}(A \cup \tilde{B}_1)\big)\big\| \\ &\qquad + \big\| P_{(d,\infty)}\big(\tilde{\Phi}(A \cup B_2) - \tilde{\Phi}(A \cup \tilde{B}_2)\big)\big\|\big) \\ &\quad \geq \big\| P_{(d,\infty)}\big(\tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2)\big)\big\| \end{aligned}$$

$$
\begin{aligned}
&- \big(d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup \tilde{B}_1) \\
&\quad + d_{1,\alpha}(A_0 \cup A \cup B_2, A_0 \cup A \cup \tilde{B}_2)\big) \\
&> \big\| P_{(d,\infty)}\big(\tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2)\big)\big\| - 6\varepsilon \\
&\geq \big\| \tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2)\big\| - \big\| P_{[1,d]}\big(\tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2)\big)\big\| - 6\varepsilon \\
&\geq \big\| \tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2)\big\| - 7\varepsilon \\
&\geq c(\tilde{\Phi}) d_{1,\alpha}(A_0 \cup A \cup \tilde{B}_1, A_0 \cup A \cup \tilde{B}_2) - 7\varepsilon \\
&> \tilde{c} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2).
\end{aligned}
$$

Here the second inequality follows from (86), the third from (98), the fifth from (94), the sixth from (87), and the last one from (96).

We will need to pass to a further subset of $\widehat{M_{k+1}}$ to obtain (d). An application of the triangle inequality and (c) (for $k+1$) yield that for every $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \ldots, m_k\}]$ with $l_1(A_0 \cup A)$, $r \in (0,1]$, and $B_1, B_2 \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A)$, $B_1 < B_2$, it follows that

$$\big\| P_{(\max\operatorname{supp}(\tilde{x}_{A_0 \cup A}),\infty)}\big(\tilde{\Phi}(A \cup B_1)\big)\big\| \geq \frac{\tilde{c}}{2} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2)$$

or

$$\big\| P_{(\max\operatorname{supp}(\tilde{x}_{A_0 \cup A}),\infty)}\big(\tilde{\Phi}(A \cup B_2)\big)\big\| \geq \frac{\tilde{c}}{2} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2).$$

We may therefore pass to a further infinite subset M_{k+1} of $\widehat{M_{k+1}}$ such that, for any $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \ldots, m_k\}]$, with $l_1(A_0 \cup A) > 0$ and $\max(A_0 \cup A) = m_k$, for any $r \in R$, and for any B in $\mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [M_{k+1}]^{<\omega}$, we have

$$\big\| P_{(d,\infty)}\big(\tilde{\Phi}(A \cup B)\big)\big\| \geq \frac{\tilde{c}}{2} \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C). \tag{101}$$

Now the verification of (d) can be done along the same lines as the proof of (c), and we therefore omit it. □

10. Some further observations on the Schreier families

In this section β will be a fixed ordinal of the form $\beta = \omega^{\omega^\xi}$, with $1 \leq \xi < \omega_1$.

10.1. Analysis of a maximal set B in $\mathcal{S}_{\beta\gamma}$

Recall that by Proposition 2.6 for every $\gamma \leq \beta$ there exists a sequence $\eta(\gamma, n)$ of ordinal numbers increasing to γ, so that $\lambda(\beta\gamma, n) = \beta\eta(\gamma, n)$. (Recall that $\eta(\gamma, n)$ may also depend on β.)

For every $\gamma \le \beta$ and $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ we define a family of subsets of B, which we shall call the *β-analysis of B* and denote by $\mathcal{A}_{\beta,\gamma}(B)$. The definition is done recursively on γ. For $\gamma = 1$, set

$$\mathcal{A}_{\beta,\gamma}(B) = \{B\}. \tag{102a}$$

Let $\gamma < \beta$, and assume that $\mathcal{A}_{\beta,\gamma}(B)$ has been defined for all $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma})$. For $B \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)}) = \mathrm{MAX}(\mathcal{S}_\beta[\mathcal{S}_{\beta\gamma}])$ there are (uniquely defined) $B_1 < \cdots < B_\ell$ in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ with $\{\min B_j : j = 1, \ldots, \ell\}$ in $\mathrm{MAX}(\mathcal{S}_\beta)$ so that $B = \bigcup_{j=1}^{\ell} B_j$. Set

$$\mathcal{A}_{\beta,\gamma+1}(B) = \{B\} \cup \Big(\bigcup_{j=1}^{\ell} \mathcal{A}_{\beta,\gamma}(B_j)\Big). \tag{102b}$$

Let $\gamma \le \beta$ be a limit ordinal, and assume that $\mathcal{A}_{\beta,\gamma'}(B)$ has been defined for all $\gamma' < \gamma$ and $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma'})$. If now $B \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma)})$, then $B \in \mathrm{MAX}(\mathcal{S}_{\beta\eta(\gamma,\min(B))})$. Set

$$\mathcal{A}_{\beta,\gamma}(B) = \mathcal{A}_{\beta,\eta(\gamma,\min(B))}(B). \tag{102c}$$

Remark 10.1
Let $\gamma \le \beta$ and $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma})$. The following properties are straightforward consequences of the definition of $\mathcal{A}_{\beta,\gamma}(B)$ and a transfinite induction.

(a) The set $\mathcal{A}_{\beta,\gamma}(B)$ is a tree when endowed with $\supset$.
(b) For C, D in $\mathcal{A}_{\beta,\gamma}(B)$ that are incomparable with respect to inclusion, we have either $C < D$ or $D < C$.
(c) The minimal elements (with respect to inclusion) of $\mathcal{A}_{\beta,\gamma}(B)$ are in $\mathcal{S}_\beta$.
(d) If $D \in \mathcal{A}_{\beta,\gamma}(B)$ is a nonminimal element, then its direct successors $(D_j)_{j=1}^{\ell}$ in $\mathcal{A}_{\beta,\gamma}(B)$ can be enumerated so that $D_1 < \cdots < D_\ell$ and $D = \bigcup_{j=1}^{\ell} D_j$.

10.2. Components of a set A in $\mathcal{S}_{\beta\gamma}$
We recursively define, for all nonempty sets $A \in \mathcal{S}_{\beta\gamma}$ and $\gamma \le \beta$, a natural number $s(\beta,\gamma,A)$ and subsets $\mathrm{Cp}_{\beta,\gamma}(A,1), \ldots, \mathrm{Cp}_{\beta,\gamma}(A, s(\beta,\gamma,A))$ of A. We will call $(\mathrm{Cp}_{\beta,\gamma}(A,i))_{i=1}^{s(\beta,\gamma,A)}$ the *components of A in $\mathcal{S}_{\beta\gamma}$ with respect to $\mathcal{S}_\beta$*.

If $\gamma = 1$, that is, A is a nonempty set in $\mathcal{S}_\beta$, define

$$s(\beta,\gamma,A) = 1 \qquad \text{and} \qquad \mathrm{Cp}_{\beta,\gamma}(A,1) = A. \tag{103a}$$

Let $\gamma \le \beta$, and assume that $(\mathrm{Cp}_{\beta,\gamma}(A,i))_{i=1}^{s(\beta,\gamma,A)}$ has been defined for all nonempty sets A in $\mathcal{S}_{\beta\gamma}$. If now A is a nonempty set in $\mathcal{S}_{\beta(\gamma+1)} = \mathcal{S}_\beta[\mathcal{S}_{\beta\gamma}]$, then there are nonempty sets $A_1 < A_2 < \cdots < A_d$ in $\mathcal{S}_{\beta\gamma}$ such that

(a) $A = \bigcup_{i=1}^{d} A_i$,

(b) $\{\min A_i : i = 1, \dots, d\}$ is in $\mathcal{S}_\beta$, and
(c) the sets $A_1, \dots, A_{d-1}$ are in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$.

Note that the set A_d may or may not be in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$. It may also be the case that $d = 1$, which in particular happens when $A \in \mathcal{S}_{\beta\gamma}$. Set $\bar{A} = A_d$, which is always nonempty, and we define

$$\begin{aligned} s(\beta, \gamma + 1, A) &= s(\beta, \gamma, \bar{A}) + 1, \\ \mathrm{Cp}_{\beta,\gamma+1}(A, 1) &= \bigcup_{i<d} A_i, \qquad \text{and} \\ \mathrm{Cp}_{\beta,\gamma+1}(A, i) &= \mathrm{Cp}_{\beta,\gamma}(\bar{A}, i-1) \quad \text{if } 2 \le i \le s(\beta, \gamma+1, A). \end{aligned} \tag{103b}$$

Note that, in the case $d = 1$, $\mathrm{Cp}_{\beta,\gamma+1}(A, 1)$ is the empty set.

Let $\gamma \le \beta$ be a limit ordinal, and assume that $(\mathrm{Cp}_{\beta,\gamma'}(A, i))_{i=1}^{s(\beta,\gamma',A)}$ has been defined for all $\gamma' < \gamma$ and nonempty sets A in $\mathcal{S}_{\beta\gamma'}$. If$A$ is a nonempty set in $\mathcal{S}_{\beta\gamma}$, then $A \in \mathcal{S}_{\beta\eta(\gamma,\min(A))}$, and we define

$$\begin{aligned} s(\beta, \gamma, A) &= s\big(\beta, \eta\big(\gamma, \min(A)\big), A\big) \qquad \text{and} \\ \mathrm{Cp}_{\beta,\gamma}(A, i) &= \mathrm{Cp}_{\beta,\eta(\gamma,\min(A))}(A, i) \quad \text{for } i = 1, 2, \dots, s(\beta, \gamma, A). \end{aligned} \tag{103c}$$

Remark

Let $\gamma \le \beta$ and $A \in \mathcal{S}_{\beta\gamma} \setminus \{\emptyset\}$. The following properties follow easily from the definition of $(\mathrm{Cp}_{\beta,\gamma}(A, i))_{i=1}^{s(\beta,\gamma,A)}$ and a transfinite induction on γ.

(a) $A = \bigcup_{i=1}^{s(\beta,\gamma,A)} \mathrm{Cp}_{\beta,\gamma}(A, i)$.
(b) For $1 \le i < j \le s(\beta, \gamma, A)$ such that both $\mathrm{Cp}_{\beta,\gamma}(A, i)$ and $\mathrm{Cp}_{\beta,\gamma}(A, j)$ are nonempty, we have $\mathrm{Cp}_{\beta,\gamma}(A, i) < \mathrm{Cp}_{\beta,\gamma}(A, j)$.
(c) $\mathrm{Cp}(A, s(\beta, \gamma(A))) \ne \emptyset$.

LEMMA 10.2

Let ξ and γ be countable ordinal numbers with $\gamma \le \beta = \omega^{\omega^\xi}$. Also let B be a set in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ and $\emptyset \prec A \preceq B$. If $\mathcal{A}_{\beta,\gamma}(B)$ is the β-analysis of B and $(\mathrm{Cp}_{\beta,\gamma}(A, i))_{i=1}^{s(\beta,\gamma,A)}$ are the components of A in $\mathcal{S}_{\beta\gamma}$ with respect to $\mathcal{S}_\beta$, then there exists a maximal chain $B = D_1(A) \supsetneq D_2(A) \supsetneq \cdots \supsetneq D_{s(\beta,\gamma,A)}(A)$ in $\mathcal{A}_{\beta,\gamma}(B)$ satisfying the following:

(a) *$\mathrm{Cp}_{\beta,\gamma}(A, i) \preceq D_i(A)$ for $1 \le i \le s(\beta, \gamma, A)$, and*
(b) *if $1 \le i < s(\beta, \gamma, A)$, then $\mathrm{Cp}_{\beta,\gamma}(A, i) \subsetneq D_i(A)$.*

Proof

We prove the statement by transfinite induction on $1 \le \gamma \le \beta$. If $\gamma = 1$, then $\mathcal{A}_{\beta,\gamma}(B) = \{B\}$ and $(\mathrm{Cp}_{\beta,\gamma}(A, i))_{i=1}^{s(\beta,\gamma,A)} = \{A\}$, and our claim follows trivially.

Let $\gamma < \beta$, and assume that the statement holds for all nonempty $A \in \mathcal{S}_{\beta\gamma}$ and $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ with $A \preceq B$. Now let A be a nonempty set in $\mathcal{S}_{\beta(\gamma+1)}$, and let $B \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)})$. If $B = \bigcup_{i=1}^{\ell} B_j$, where $B_1 < \cdots < B_\ell$ are in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ with $\{\min(B_j) : 1 \le j \le \ell\} \in \mathrm{MAX}(\mathcal{S}_\beta)$, then by (102b) we obtain $\mathcal{A}_{\beta,\gamma+1}(B) = \{B\} \cup (\bigcup_{j=1}^{\ell} \mathcal{A}_{\beta,\gamma}(B_j))$. Define

$$d = \max\{1 \le j \le \ell : B_j \cap A \ne \emptyset\},$$

$$A_i = B_i \quad \text{for } 1 \le i < d, \qquad \text{and} \qquad A_d = A \cap B_d.$$

Then, by (103b), letting $\bar{A} = A_d$, we obtain $s(\beta,\gamma+1,A) = s(\beta,\gamma,\bar{A}) + 1$, $\mathrm{Cp}_{\beta,\gamma+1}(A,1) = \bigcup_{i<d} A_i$, and $\mathrm{Cp}_{\beta,\gamma+1}(A,i) = \mathrm{Cp}_{\beta,\gamma}(\bar{A},i-1)$, for $2 \le i \le s(\beta, \gamma+1, A)$. Apply the inductive assumption to $A_d = \bar{A} \preceq B_d$ to find a maximal chain $B_d = D_1(\bar{A}) \supsetneq \cdots \supsetneq D_{s(\beta,\gamma,\bar{A})}(\bar{A})$ in $\mathcal{A}_{\beta\gamma}(B_d)$ satisfying (a) and (b) with respect to $(\mathrm{Cp}_{\beta,\gamma}(\bar{A},i))_{i=1}^{s(\beta,\gamma,\bar{A})}$. Define

$$D_1(A) = B \qquad \text{and} \qquad D_i(A) = D_{i-1}(\bar{A}) \quad \text{for } 2 \le i \le s(\beta,\gamma,A). \tag{104}$$

Clearly, $(D_i(A))_{i=1}^{s(\beta,\gamma,\bar{A})}$ is a maximal chain in $\mathcal{A}_{\beta,\gamma+1}(B)$. It remains to verify that (a) and (b) are satisfied with respect to $(\mathrm{Cp}_{\beta,\gamma+1}(A,i))_{i=1}^{s(\beta,\gamma+1,A)}$. Assertions (a) and (b), in the case in which $i = 1$, follow trivially from $\mathrm{Cp}_{\beta,\gamma+1}(A,1) = \bigcup_{j<d} A_j = \bigcup_{j<d} B_j \prec \bigcup_{j<d} B_j \preceq B$. Assertions (a) and (b), in the case in which $i \ne 1$, follow easily from the inductive assumption and $\mathrm{Cp}_{\beta,\gamma+1}(A,i) = \mathrm{Cp}_{\beta,\gamma}(A',i-1)$ for $2 \le i \le s(\beta,\gamma+1,A)$.

To conclude the proof, if $\gamma \le \beta$ is a limit ordinal number such that the conclusion is satisfied for all $\gamma' < \gamma$, we just observe that the result is an immediate consequence of (102c) and (103c). ☐

For the next result recall the definition of the doubly indexed fine Schreier families $\mathcal{F}_{\beta,\gamma}$ introduced in Section 2.3.

LEMMA 10.3
Let $\gamma \le \beta$. Then for all $A \in \mathcal{S}_{\beta\gamma}$ with $\mathrm{Cp}_{\beta\gamma}(A,i) \ne \emptyset$ for $1 \le i \le s(\beta,\gamma,A)$, we have

$$\big\{\min\big(\mathrm{Cp}_{\beta,\gamma}(A,i)\big) : 1 \le i \le s(\beta,\gamma,A)\big\} \in \mathrm{MAX}(\mathcal{F}_{\beta,\gamma}). \tag{105}$$

Proof
We prove this statement by induction on γ. If $\gamma = 1$ and $A \in \mathcal{S}_\beta$ satisfy the assumptions of this lemma, then $A = \mathrm{Cp}_{\beta,1}(A,1) \ne \emptyset$, and hence, the result easily follows from $\mathrm{MAX}(\mathcal{F}_{\beta,1}) = \{\{n\} : n \in \mathbb{N}\}$.

Assume that the result holds for some $\gamma < \beta$, and let $A \in \mathcal{S}_{\beta(\gamma+1)}$, with $\mathrm{Cp}_{\beta\gamma}(A, i) \ne \emptyset$ for $1 \le i \le s(\beta,\gamma+1,A)$. By the inductive assumption and (103b) we obtain that $B = \{\min(\mathrm{Cp}_{\beta,\gamma}(A,i)) : 2 \le i \le s(\beta,\gamma+1,A)\} \in \mathrm{MAX}(\mathcal{F}_{\beta,\gamma})$. We claim that $\tilde{A} = \{\min(\mathrm{Cp}_{\beta,\gamma+1}(A,1))\} \cup B \in \mathrm{MAX}(\mathcal{F}_{\beta,\gamma+1})$. Indeed, if this is not the case, then by the spreading property of $\mathcal{F}_{\beta,\gamma+1}$ there is $C \in \mathcal{F}_{\beta,\gamma+1}$ with $\tilde{A} \prec C$. Then, $B \prec \tilde{C} = C \setminus \{\min(\mathrm{Cp}_{\beta,\gamma+1}(A,1))\}$. It follows by $\mathcal{F}_{\beta,\gamma+1} = \mathcal{F}_{\beta,1} \sqcup_{<} \mathcal{F}_{\beta,\gamma}$ that $\tilde{C} \in \mathcal{F}_{\beta,\gamma}$. The maximality of B yields a contradiction.

Assume now that $\gamma \le \beta$ is a limit ordinal number so that the conclusion holds for all $\tilde{\gamma} < \gamma$. Let $A \in \mathcal{S}_{\beta\gamma}$ so that $\mathrm{Cp}_{\beta,\gamma}(A,i) \ne \emptyset$ for $1 \le i \le s(\beta,\gamma,A)$. Note that $A \in \mathcal{S}_{\beta\eta(\gamma,\min(A))}$, and by (103c) and the inductive assumption we have $\tilde{A} = \{\min(\mathrm{Cp}_{\beta,\gamma}(A,i)) : 1 \le i \le s(\beta,\gamma,A)\} \in \mathrm{MAX}(\mathcal{F}_{\beta,\eta(\gamma,\min(A))})$. By (27) we obtain $\tilde{A} \in \mathrm{MAX}(\mathcal{F}_{\beta,\gamma})$. □

For $\gamma \le \beta$ and a set B in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ we define

$$\mathcal{E}_{\beta,\gamma}(B) = \big\{\emptyset \prec A \preceq B : \mathrm{Cp}_{\beta,\gamma}(A,i) \ne \emptyset \text{ for } 1 \le i \le s(\beta,\gamma,A)\big\}. \tag{106}$$

LEMMA 10.4
Let $\gamma < \beta$. If B is in $\mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)}) = \mathcal{S}_{\beta}[\mathcal{S}_{\beta\gamma}]$ and $B = \bigcup_{j=1}^{\ell} B_j$, where $B_1 < \cdots < B_\ell$ are in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ and $\{\min(B_j) : 1 \le j \le \ell\} \in \mathrm{MAX}(\mathcal{S}_\beta)$, then

$$\begin{aligned}&\big\{A : A \preceq B \text{ and } A \notin \mathcal{E}_{\beta(\gamma+1)}(B)\big\}\\ &\quad = \{A : A \preceq B_1\} \cup \Big(\bigcup_{m=2}^{\ell} \Big\{\Big(\bigcup_{j=1}^{m-1} B_j\Big) \cup A : A \preceq B_m \text{ and } A \notin \mathcal{E}_{\beta,\gamma}(B_m)\Big\}\Big).\end{aligned}$$

Proof
Let $\emptyset \ne D \preceq B$. Define $m = \max\{1 \le j \le \ell : D \cap B_j \ne \emptyset\}$ and $A = B_m \cap D$. Note that $D = (\bigcup_{j<m} B_j) \cup A$, where $\bigcup_{j<m} B_j = \emptyset$ if $m = 1$. By (103b) we obtain $s(\beta,\gamma+1,D) = s(\beta,\gamma,A)+1$, $\mathrm{Cp}_{\beta,\gamma+1}(D,1) = \bigcup_{j<m} B_j$, and $\mathrm{Cp}_{\beta,\gamma+1}(D,i) = \mathrm{Cp}_{\beta,\gamma}(A,i-1)$ for $2 \le i \le s(\beta,\gamma+1,D)$.

Observe that $\mathrm{Cp}_{\beta,\gamma+1}(D,1) = \emptyset$ if and only if $m = 1$, that is, $A = D \preceq B_1$. On the other hand, if $\mathrm{Cp}_{\beta,\gamma+1}(D,1) \ne \emptyset$, then for some $2 \le i \le s(\beta,\gamma+1,D)$, we have $\mathrm{Cp}_{\beta,\gamma+1}(D,i) = \emptyset$ if and only if $\mathrm{Cp}_{\beta,\gamma}(A,i-1) = \emptyset$. These observations yield our claim. □

Remark
Under the assumptions of Lemma 10.4, if for $1 \le j \le \ell - 1$ we define

$$\mathcal{E}^{(j)}_{\beta,\gamma+1}(B) = \Big\{A \in \mathcal{E}_{\beta,\gamma+1}(B) : \mathrm{Cp}_{\beta,\gamma}(A,1) = \bigcup_{i \le j} B_i\Big\},$$

then, using a similar argument to the one used in the proof of Lemma 10.4, we obtain

$$\mathcal{E}^{(j)}_{\beta,\gamma+1}(B) = \Big\{\Big(\bigcup_{i=1}^{j} B_i\Big) \cup C : C \in \mathcal{E}_{\beta,\gamma}(B_{j+1})\Big\} \qquad \text{and}$$

$$\mathcal{E}_{\beta(\gamma+1)}(B) = \bigcup_{j=1}^{\ell-1} \mathcal{E}^{(j)}_{\beta,\gamma+1}.$$

Remark

By using the fact that $\mathcal{S}_1 \subset \mathcal{S}_\alpha$ for all countable ordinal numbers α and that $\mathrm{MAX}(\mathcal{S}_1) = \{F \subset \mathbb{N} : \min(F) = \#F\}$, it is easy to verify that for all $F \in \mathrm{MAX}(\mathcal{S}_\alpha)$ we have $\max(F) \geq 2\min(F) - 1$. In particular, if $B_1 < B_2$ are both in $\mathrm{MAX}(\mathcal{S}_\alpha)$, then

$$2\min(B_1) \leq \min(B_2). \tag{107}$$

LEMMA 10.5

Let $\gamma \leq \beta$, and let B be a set in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$. *Then*

$$\sum_{\substack{A \preceq B \\ A \notin \mathcal{E}_{\beta,\gamma}(B)}} \zeta(\beta\gamma, A) < \frac{2}{\min(B)}. \tag{108}$$

Proof

We prove the statement by transfinite induction for all $1 \leq \gamma \leq \beta$. If $\gamma = 1$, then the complement of $\mathcal{E}_{\beta,1}(B)$ only contains the empty set, and the result trivially holds.

Let $\gamma < \beta$, and assume that the statement holds for all $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma})$. Let $B \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)})$. Let $B_1 < \cdots < B_\ell$ in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ such that $\{\min(B_j) : 1 \leq j \leq \ell\} \in \mathrm{MAX}(\mathcal{S}_\beta)$ and $B = \bigcup_{j=1}^{\ell} B_j$. For $m = 1, \ldots, \ell$, define $D_m = \{\min(B_j) : 1 \leq j \leq m\}$. Proposition 3.4 implies the following: if $C \preceq B$, $m = \max\{1 \leq j \leq \ell : C \cap B_j \neq \emptyset\}$, and $A = C \cap B_m$, then $\zeta(\beta(\gamma+1), C) = \zeta(\beta, D_m)\zeta(\beta\gamma, A)$. We combine this fact with Lemma 10.4, (107), and (33) to obtain that

$$\begin{aligned}\sum_{\substack{A \preceq B \\ A \notin \mathcal{E}_{\beta,\gamma}(B)}} \zeta(\beta\gamma, A) &= \zeta(\beta, D_1) \sum_{A \preceq B_1} \zeta(\beta\gamma, A) \\ &\quad + \sum_{j=2}^{\ell} \zeta(\beta, D_j) \sum_{\left\{\substack{A \preceq B_j \\ A \notin \mathcal{E}_{\beta\gamma}(B_j)}\right\}} \zeta(\beta\gamma, A) \\ &< \zeta(\beta, D_1) + \sum_{j=2}^{\ell} \zeta(\beta, D_j) \frac{2}{\min(B_j)}\end{aligned}$$

$$\leq \frac{1}{\min(B_1)} + \sum_{j=2}^{\ell} \zeta(\beta, D_j) \frac{2}{\min(B_j)}$$
$$\leq \frac{1}{\min(B_1)} + \sum_{j=2}^{\ell} \zeta(\beta, D_j) \frac{1}{\min(B_1)}$$
$$\leq \frac{2}{\min(B_1)}$$
$$= \frac{2}{\min(B)}.$$

If $\gamma \leq \beta$ is a limit ordinal number such that the conclusion is satisfied for all $\gamma' < \gamma$, we just observe that the result is an immediate consequence of (103c) and (30). □

LEMMA 10.6
Let $\gamma \leq \beta$ and $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma})$. If $A^{(1)}, A^{(2)} \in \mathcal{E}_{\beta,\gamma}(B)$, $(D_k(A^{(1)}))_{k=1}^{s(\beta,\gamma,A^{(1)})}$, and $(D_k(A^{(2)}))_{i=1}^{s(\beta,\gamma,A^{(2)})}$ are the maximal chains of $\mathcal{A}_{\beta,\gamma}(B)$ given by Lemma 10.2, and we assume that $1 \leq i \leq \min\{s(\beta,\gamma,A^{(1)}), s(\beta,\gamma,A^{(2)})\}$ is such that $D_i(A^{(1)}) = D_i(A^{(2)})$, then we have $D_j(A^{(1)}) = D_j(A^{(2)})$, for all $1 \leq j < i$.

Proof
As $(D_k(A^{(1)}))_{k=1}^{s(\beta,\gamma,A)}$ and $(D_k(A^{(2)}))_{i=1}^{s(\beta,\gamma,A^{(2)})}$ are both maximal chains of $\mathcal{A}_{\beta,\gamma}(B)$ such that $D_i(A^{(1)}) = D_i(A^{(2)})$, the result follows from Remark 10.1(a). □

LEMMA 10.7
Let $\gamma \leq \beta$, $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma})$, $A^{(1)}, A^{(2)} \in \mathcal{E}_{\beta,\gamma}(B)$, and $1 \leq i \leq \min\{s(\beta,\gamma,A^{(1)}), s(\beta,\gamma,A^{(2)})\}$. Then $D_i(A^{(1)}) = D_i(A^{(2)})$ if and only if $\min(\mathrm{Cp}_{\beta,\gamma}(A^{(1)}, i)) = \min(\mathrm{Cp}_{\beta,\gamma}(A^{(2)}, i))$, where $(D_k(A^{(1)}))_{k=1}^{s(\beta,\gamma,A)}$ and $(D_k(A^{(2)}))_{i=1}^{s(\beta,\gamma,A^{(2)})}$ are the maximal chains of $\mathcal{A}_{\beta,\gamma}(B)$ provided by Lemma 10.2.

Proof
Assume that $D_i(A^{(1)}) = D_i(A^{(2)})$. Lemma 10.2(a) and the assumptions $\mathrm{Cp}(A^{(1)}, i) \neq \emptyset$ and $\mathrm{Cp}(A^{(2)}, i) \neq \emptyset$ yield $\min(\mathrm{Cp}_{\beta,\gamma}(A^{(1)}, i)) = \min(\mathrm{Cp}_{\beta,\gamma}(A^{(2)}, i))$. For the converse let $A^{(1)}, A^{(2)} \in \mathcal{E}_{\beta,\gamma}(B)$ with $\min(\mathrm{Cp}_{\beta,\gamma}(A^{(1)}, i)) = \min(\mathrm{Cp}_{\beta,\gamma}(A^{(2)}, i))$. Since all elements of $\mathcal{A}_{\beta,\gamma}(B)$ either compare with respect to $\subset$ or are disjoint, Lemma 10.2(a) and $\min(\mathrm{Cp}_{\beta,\gamma}(A^{(1)}, i)) = \min(\mathrm{Cp}_{\beta,\gamma}(A^{(2)}, i))$ imply that either $D_i(A^{(1)}) \subset D_i(A^{(2)})$ or $D_i(A^{(2)}) \subset D_i(A^{(1)})$. We assume the first, and toward a contradiction assume that $D_i(A^{(1)}) \subsetneq D_i(A^{(2)})$. The maximality of $(D_j(A^{(2)}))_{j=1}^{s(\beta,\gamma,A^{(2)})}$

in $\mathcal{A}_{\beta,\gamma}(B)$ implies that there is $1 \le j < i$ such that $D_j(A^{(2)}) = D_i(A^{(1)})$. As $A^{(2)}_{\beta,\gamma}(j) \ne \emptyset$, we obtain by Lemma 10.2(a) $\min(\mathrm{Cp}_{\beta,\gamma}(A^{(1)}, i)) = \min(D_i(A^{(1)})) = \min(D_j(A^{(2)})) = \min(\mathrm{Cp}_{\beta,\gamma}(A^{(2)}, j)) < \min(\mathrm{Cp}_{\beta,\gamma}(A^{(2)}, i))$, which is a contradiction. □

10.3. Special families of convex combinations

Definition 10.8
Let $\gamma \le \beta$ and $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma})$. A family of nonnegative numbers $\{r(A,k) : A \in \mathcal{E}_{\beta,\gamma}(B), 1 \le k \le s(\beta,\gamma,A)\}$ is called a (β,γ)-*special family of convex combinations for* B if the following are satisfied.

(a) $\sum_{k=1}^{s(\beta,\gamma,A)} r(A,k) = 1$ for all $A \in \mathcal{E}_{\beta,\gamma}(B)$.

(b) If $A^{(1)}, A^{(2)}$ are both in $\mathcal{E}_{\beta,\gamma}(B)$, $(D_k(A^{(1)}))_{k=1}^{s(\beta,\gamma,A)}$ and $(D_k(A^{(2)}))_{i=1}^{s(\beta,\gamma,A^{(2)})}$ are the maximal chains in $\mathcal{A}_{\beta,\gamma}(B)$ provided by Lemma 10.2, and for some k we have $D_k(A^{(1)}) = D_k(A^{(2)})$, then $r(A^{(1)},k) = r(A^{(2)},k)$.

Remark
Let $\gamma \le \beta$ and $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma})$, and let $\{r(A,k) : A \in \mathcal{E}_{\beta,\gamma}(B),\ k = 1,2,\dots,s(\beta,\gamma, A)\}$ be a family of (β,γ)-special convex combinations for B. For $A \in \mathcal{E}_{\beta,\gamma}(B)$, let $(D_k(A))_{i=1}^{s(\beta,\gamma,A)}$ be the maximal chain in $\mathcal{A}_{\beta,\gamma}(B)$ provided by Lemma 10.2, and let $(A(i))_{i=1}^{s(\beta,\gamma,A)}$ be the components of A in $\mathcal{S}_{\beta\gamma}$.

By construction, $D_1(A) = B$ for all $A \in \mathcal{E}_{\beta,\gamma}(B)$, and thus, $r(A,1)$ does not depend on A. Additionally, $D_2(A)$ only depends on $A(1)$; thus, $r(A^{(1)},2) = r(A^{(2)}, 2)$ if $\mathrm{Cp}_{\beta,\gamma}(A^{(1)},1) = \mathrm{Cp}_{\beta,\gamma}(A^{(2)},1)$, for any $A^{(1)}, A^{(2)} \in \mathcal{E}_{\beta,\gamma}(B)$. We can continue, and inductively we observe that for all $k \le \min(s(\beta,\gamma,A^{(1)}), s(\beta,\gamma,A^{(2)}))$ if $\mathrm{Cp}_{\beta,\gamma}(A^{(1)},i) = \mathrm{Cp}_{\beta,\gamma}(A^{(2)},i)$, for all $i = 1,2,\dots,k-1$, then $r(A^{(1)},k) = r(A^{(2)},k)$.

Let $\gamma \le \beta$ and $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma})$. If γ is a limit ordinal number, then $B \in \mathrm{MAX}(\mathcal{S}_{\beta\eta(\gamma,\min(B))})$ and any (β,γ)-special family of convex combinations $\{r(A,k) : A \in \mathcal{E}_{\beta,\gamma}(B), 1 \le k \le s(\beta,\gamma,A)\}$ is also a $(\beta,\eta(\gamma,\min(B)))$-special family of convex combinations, as $\mathcal{E}_{\beta,\gamma}(B) = \mathcal{E}_{\beta,\eta(\gamma,\min(B))}(B)$ and for $A \in \mathcal{E}_{\beta,\eta(\gamma,\min(B))}(B)$ we have $s(\beta,\eta(\gamma,\min(B)),A) = s(\beta,\gamma,A)$.

LEMMA 10.9
We are given $\gamma < \beta$, $B \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)})$, and a $(\beta,\gamma+1)$-special family of convex combinations $\{r(A,k) : A \in \mathcal{E}_{\beta,\gamma+1}(B), 1 \le k \le s(\beta,\gamma+1,A)\}$. Assume that for some $D \in \mathcal{E}_{\beta,\gamma+1}(B)$ (and hence for all of them) we have $r(D,1) < 1$. Let $B = \bigcup_{j=1}^d B_j$, where $B_1 < \cdots < B_d$ are the immediate predecessors of B in $\mathcal{A}_{\beta\gamma}(B)$. For every $1 \le j < d$ consider the family $\{r^{(j)}(C,k) : C \in \mathcal{E}_{\beta,\gamma}(B_{j+1})\}$, with

$$r^{(j)}(C,k) = \frac{1}{1-r(D,1)} r\Big(\Big(\bigcup_{i=1}^{j} B_i\Big) \cup C, k+1\Big)$$

for $k = 1, \dots, s(\beta,\gamma,C) = s(\beta,\gamma,(\bigcup_{i=1}^{j} B_i) \cup C) - 1$. *Then* $\{r^{(j)}(C,k) : C \in \mathcal{E}_{\beta,\gamma}(B_{j+1})\}$ *is a* (β,γ)*-special family of convex combinations.*

Proof
By (103b), if $C \in \mathcal{E}_{\beta,\gamma}(B_{j+1})$, then $A = (\bigcup_{i=1}^{j} B_i) \cup C \in \mathcal{E}_{\beta,\gamma+1}(B)$ and $s(\beta,\gamma+1,A) = s(\beta,\gamma,C)+1$, which implies that Definition 10.8(a) is satisfied. To see that (b) holds, let $C^{(1)}, C^{(2)}$ be in $\mathcal{E}_{\beta,\gamma}(B_{j+1})$ such that for some k we have $D_k(C^{(1)}) = D_k(C^{(2)})$. Then by Lemma 10.7 we have $\min(\mathrm{Cp}_{\beta,\gamma}(C^{(1)},k)) = \min(\mathrm{Cp}_{\beta,\gamma}(C^{(2)},k))$. Setting $A^{(1)} = (\bigcup_{i=1}^{j} B_t) \cup C^{(1)}$ and $A^{(2)} = (\bigcup_{i=1}^{j} B_t) \cup C^{(2)}$, by (103b) we obtain $\mathrm{Cp}_{\beta,\gamma+1}(A^{(1)},k+1) = \mathrm{Cp}_{\beta,\gamma}(C^{(1)},k)$ and $\mathrm{Cp}_{\beta,\gamma}(A^{(2)},k+1) = \mathrm{Cp}_{\beta,\gamma}(C^{(2)},k)$, that is, $\min(\mathrm{Cp}_{\beta,\gamma}(A^{(1)},k+1)) = \min(\mathrm{Cp}_{\beta,\gamma}(A^{(2)},k+1))$. By Lemma 10.7 we obtain $D_{k+1}(A^{(1)}) = D_{k+1}(A^{(2)})$ and therefore $r(A^{(1)},k+1) = r(A^{(2)},k+1)$, which yields that $r^{(j)}(C^{(1)},k) = r^{(j)}(C^{(2)},k)$. $\square$

11. Conclusion of the proofs of Theorems A and C

Again, we fix $\xi < \omega_1$ and put $\beta = \omega^{\omega^\xi}$. We additionally assume that X is a Banach space X with a bimonotone FDD (F_j). By [32, Main Theorem] every reflexive Banach space X embeds into a reflexive Banach space Z with basis, so that $\mathrm{Sz}(Z) = \mathrm{Sz}(X)$ and $\mathrm{Sz}(Z^*) = \mathrm{Sz}(X^*)$. The coordinate projections on finitely or cofinitely many coordinates are denoted by P_A (see Section 9 after Definition 9.5).

Definition 11.1
Let $\gamma \le \beta$, let $M \in [N]^\omega$, and let A_0 be a subset of $\mathbb{N}$ that is either empty or a singleton. Also let $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [M]^{<\omega} \to X$ be a semiembedding of $\mathcal{S}_{\beta\gamma} \cap [M]^{<\omega}$ into X, starting after A_0, that is c-refined, for some $0 < c \le 1$. Let $\{x_\emptyset\} \cup \{x_{A_0 \cup A} : A \in \mathcal{S}_{\beta\gamma}(A_0) \cap [M]^{<\omega}\}$ be the family generating Φ. (Recall that notation from the remark after Definition 9.2.)

Let $E \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma}(A_0) \cap [M]^{<\omega})$. For $A \preceq E$ recall the definition of $s(\beta,\gamma,A)$ and of $(\mathrm{Cp}_{\beta,\gamma}(A,i))_{i=1}^{s(\beta,\gamma,A)}$. Recall also from (106) $\mathcal{E}_{\beta,\gamma}(A_0 \cup E) = \{\emptyset \prec A \preceq A_0 \cup E : \mathrm{Cp}_{\beta,\gamma}(A,i) \ne \emptyset, \text{ for } i = 1,2,\dots,s(\beta,\gamma,A)\}$.

For each $A \in \mathcal{E}_{\beta,\gamma}(A_0 \cup E)$ we will write x_A as a sum of a block sequence

$$x_A = \sum_{k=1}^{s(\beta,\gamma,A)} x_{\Phi,A}^{(k)}, \tag{109}$$

with $x^{(k)}_{\Phi,A} = P_{I(A,k)}(x_A)$, for $k = 1, 2, \dots, s(\beta,\gamma,A)$, where $I(A,1) = [1, \max\operatorname{supp}(x_{\mathrm{Cp}_{\beta,\gamma}(A,1)})]$ and $I(A,k) = (\max\operatorname{supp}(x_{\bigcup_{i=1}^{k-1}\mathrm{Cp}_{\beta,\gamma}(A,i)}), \max\operatorname{supp}(x_{\bigcup_{i=1}^{k}\mathrm{Cp}_{\beta,\gamma}(A,i)})]$ for $1 < k < s(\beta,\gamma,A)$. We call the family $((x^{(k)}_{\Phi,A})_{k=1}^{s(\beta,\gamma,A)})_{A\in\mathcal{E}_{\beta,\gamma}(A_0\cup E)}$ the *block step decomposition of E with respect to Φ*.

Remark

Let $M \in [\mathbb{N}]^\omega$, let $\gamma \le \beta$ be a limit ordinal number, and let $\eta(\gamma,n)$ be the sequence provided by Proposition 2.6. Assume that A_0 is a singleton or the empty set and that $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [M]^{<\omega} \to X$ is a semiembedding of $\mathcal{S}_{\beta\gamma} \cap [M]^{<\omega}$ into X, starting at A_0, that is c-refined.

If A_0 is a singleton, say, $A_0 = \{a_0\}$, let $\Psi : \mathcal{S}_{\beta\eta(\gamma,a_0)}(A_0) \cap [M]^{<\omega} \to X$, with $\Psi(A) = \Phi(A)$, be the semiembedding of $\mathcal{S}_{\beta\eta(\gamma,a_0)} \cap [M]^{<\omega}$ into X, starting at A_0, that is c-refined and given by Remark 9.6. Then, for every $E \in \mathrm{MAX}(\mathcal{S}_{\beta\eta(\gamma,a_0)}(A_0) \cap [N]^{<\omega})$, we have

$$\big((x^{(k)}_{\Psi,A})_{k=1}^{s(\beta,\eta(\gamma,a_0),A)}\big)_{A\in\mathcal{E}_{\beta,\eta(\gamma,a_0)}(A_0\cup E)} = \big((x^{(k)}_{\Phi,A})_{k=1}^{s(\beta,\gamma,A)}\big)_{A\in\mathcal{E}_{\beta,\gamma}(A_0\cup E)}. \tag{110}$$

If $A_0 = \emptyset$, let $a_0 \in M$, set $A_0 = \{a_0\}$, set $N = M \cap [a_0,\infty)$, and set $\Psi = \Phi|_{\mathcal{S}_{\beta\eta(\gamma,a_0)}(A_0)\cap[N]^{<\omega}}$, which is, by Remark 9.6, a semiembedding of $\mathcal{S}_{\beta\eta(\gamma,a_0)} \cap [N]^{<\omega}$ into X, starting at A_0, that is c-refined. Then, again for every $E \in \mathrm{MAX}(\mathcal{S}_{\beta\eta(\gamma,a_0)}(A_0) \cap [N]^{<\omega})$,

$$\big((x^{(k)}_{\Psi,A})_{k=1}^{s(\beta,\eta(\gamma,a_0),A)}\big)_{A\in\mathcal{E}_{\beta,\eta(\gamma,a_0)}(A_0\cup E)} = \big((x^{(k)}_{\Phi,A})_{k=1}^{s(\beta,\gamma,A)}\big)_{A\in\mathcal{E}_{\beta,\gamma}(A_0\cup E)} \tag{111}$$

is the step block decomposition of E with respect to Ψ.

Before formulating and proving the missing parts from Theorems A and B (see upcoming Theorem 11.6) we present the argument which is the main inductive step.

Let $\gamma < \beta$. For $B \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)}) = \mathcal{S}_\beta[\mathcal{S}_{\beta\gamma}]$ (see Proposition 2.6), we let $B_1 < B_2 < \cdots < B_d$ be the (unique) elements of $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$ for which $B = \bigcup_{j=1}^d B_j$. We also define $\bar{B} = \{\min(B_j) : j = 1, 2, \dots, d\} \in \mathrm{MAX}(\mathcal{S}_\beta)$ and for $i = 1, \dots, d$ $\bar{B}_i = \{\min(B_j) : j = 1, 2, \dots, i\}$.

If $\emptyset \prec A \preceq B$, we can write A as $A = \bigcup_{i=1}^{j-1} B_i \cup C$, for some $j = 1, 2, \dots, d$ and some $\emptyset \prec C \preceq B_j$, and thus, by Proposition 3.4, $\zeta(\beta(\gamma+1), A) = \zeta(\beta, \bar{B}_j)\zeta(\beta\gamma, C)$. We define

$$\tilde{\mathcal{E}}_{\beta,\gamma+1}(B) = \Big\{\Big(\bigcup_{i=1}^{j} B_j\Big) \cup C \in \mathcal{E}_{\beta,\gamma+1}(B) : 1 \le j < d,\ l_1(\bar{B}_j) > 0, \text{ and } C \preceq B_{j+1}\Big\}. \tag{112}$$

Let $M \in [N]^\omega$, let A_0 be a subset of $\mathbb{N}$ that is either empty or a singleton, and let $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [M]^{<\omega} \to X$ be a semiembedding of $\mathcal{S}_{\beta\gamma} \cap [M]^{<\omega}$ into X, starting after A_0, that is c-refined, for some $0 < c \le 1$. Also let $E \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)})(A_0) \cap [M]^{<\omega}$, and put $B = A_0 \cup E$. For $1 \le j < d$, with $l_1(\bar{B}_j) > 0$, put $M^{(j)} = M \cap (\max(B_j), \infty)$ and $\Phi_E^{(j)} : \mathcal{S}_{\beta\gamma} \cap [M^{(j)}]^{<\omega} \to X$ with

$$\Phi_E^{(j)}(C) = \frac{1}{\zeta(\beta, \bar{B}_{j+1})} \Phi\Big(\Big(\Big(\bigcup_{1 \le i \le j} B_i\Big) \setminus A_0\Big) \cup C\Big). \tag{113}$$

Recall that by Lemma 9.7, $\Phi_E^{(j)}$ is a semiembedding of $\mathcal{S}_{\beta\gamma} \cap [M^{(j)}]^{<\omega}$ into X, starting at $\emptyset$, that is c-refined. Recall that for $1 \le j < d$,

$$\mathcal{E}^{(j)}_{\beta,\gamma+1}(A_0 \cup E) = \Big\{A \in \mathcal{E}_{\beta,\gamma+1}(A_0 \cup E) : \mathrm{Cp}_{\beta,\gamma}(A, 1) = \bigcup_{i \le j} B_i\Big\},$$

and moreover, if $l_1(\bar{B}_j) > 0$, define

$$y^{(j)}_{\Phi,E} = \sum_{A \in \mathcal{E}^{(j)}_{\beta,\gamma+1}(A_0 \cup E)} \frac{\zeta(\beta(\gamma+1), A)}{\zeta(\beta, \bar{B}_j)} \sum_{k=2}^{s(\beta,\gamma+1,A)} x^{(k)}_{\Phi,A}. \tag{114}$$

Remark

(a) By Lemma 9.7, each $\Phi_E^{(j)}$ is a semiembedding of $\mathcal{S}_{\beta\gamma} \cap [M^{(j)}]^{<\omega}$ into X, starting at $\emptyset$, that is c-refined.

(b) We note for later use that $(y^{(j)}_{\Phi,E})_{j=1}^d$ is a sequence in X which satisfies the conditions of the sequence $(x_j)_{j=1}^d$ in Theorem 6.1 with $\alpha = \beta$ and thus, assuming that $\mathrm{Sz}(X) \le \omega^\beta$, also its conclusion.

(c) By the definition of the components of a set A, we conclude that (for $j = d$, we have $l_1(\bar{B}_d) = 0$)

$$\tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0 \cup E) = \bigcup_{\substack{1 \le j < d \\ l_1(\bar{B}_j) > 0}} \mathcal{E}^{(j)}_{\beta,\gamma+1}(A_0 \cup E), \tag{115}$$

which yields

$$\sum_{A \in \tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0 \cup E)} \zeta\big(\beta(\gamma+1), A\big) \sum_{k=1}^{s(\beta,\gamma+1,A)} x^{(k)}_{\Phi,A}$$
$$= \sum_{\substack{1 \le j < d \\ l_1(\bar{B}_j) > 0}} \zeta(\beta, \bar{B}_j) y^{(j)}_{\Phi,E} + \sum_{A \in \tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0 \cup E)} \zeta\big(\beta(\gamma+1), A\big) x^{(1)}_{\Phi,A}. \tag{116}$$

LEMMA 11.2
Let $\gamma < \beta$, let $M \in [N]^\omega$, let A_0 be either empty or a singleton in $\mathbb{N}$, and let $\Phi : \mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [M]^{<\omega} \to X$ be a semiembedding of $\mathcal{S}_{\beta(\gamma+1)} \cap [M]^{<\omega}$ into X, starting after A_0, that is c-refined, for some $0 < c \leq 1$. Then, for every $E \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [M]^{<\omega})$,

$$\Big\| \Phi(E) - \sum_{A \in \tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0 \cup E)} \zeta\big(\beta(\gamma+1), A\big) \sum_{k=1}^{s(\beta,\gamma+1,A)} x_{\Phi,A}^{(k)} \Big\| < \frac{3}{\min(A_0 \cup E)}. \quad (117)$$

Proof
Recall that, for $A \in \mathcal{E}_{\beta,\gamma+1}(A_0 \cup E)$, we have $x_A = \sum_{k=1}^{s(\beta,\gamma+1,A)} x_{\Phi,A}^{(k)}$ and that $\Phi(E) = \sum_{A \preceq A_0 \cup E} \zeta(\beta(\gamma+1), A) x_A$. Hence,

$$\begin{aligned} &\Big\| \Phi(E) - \sum_{A \in \tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0 \cup E)} \zeta\big(\beta(\gamma+1), A\big) \sum_{k=1}^{s(\beta,\gamma,A)} x_{\Phi,A}^{(k)} \Big\| \\ &\quad = \Big\| \sum_{\substack{A \preceq A_0 \cup E: \\ A \notin \tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0 \cup E)}} \zeta\big(\beta(\gamma+1), A\big) x_A \Big\|. \end{aligned}$$

We calculate

$$\begin{aligned} &\sum_{\substack{A \preceq A_0 \cup E \\ A \notin \tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0 \cup E)}} \zeta\big(\beta(\gamma+1), A\big) \\ &= \sum_{\substack{A \preceq A_0 \cup E \\ A \notin \mathcal{E}_{\beta,\gamma+1}(A_0 \cup E)}} \zeta\big(\beta(\gamma+1), A\big) \\ &\quad + \sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j) = 0}} \sum_{A \in \mathcal{E}^{(j)}_{\beta,\gamma+1}(A_0 \cup E)} \zeta\big(\beta(\gamma+1), A\big) \\ &= \sum_{\substack{A \preceq A_0 \cup E \\ A \notin \mathcal{E}_{\beta,\gamma+1}(A_0 \cup E)}} \zeta\big(\beta(\gamma+1), A\big) \\ &\quad + \sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j) = 0}} \zeta(\beta, \bar{B}_{j+1}) \sum_{A \in \mathcal{E}^{(j)}_{\beta,\gamma+1}(A_0 \cup E)} \zeta\Big(\beta\gamma, A \setminus \Big(\bigcup_{i \leq j} B_j\Big)\Big) \\ &< \frac{2}{\min(A_0 \cup E)} + \frac{1}{\min(\bar{B})} = \frac{3}{\min(A_0 \cup E)}, \end{aligned} \quad (118)$$

where the last inequality follows from Lemmas 3.7(e) and 10.5. □

LEMMA 11.3

Let γ, $M \in [N]^\omega$, A_0, Φ, and c be as in the statement of Lemma 11.2. If $E \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [M]^{<\omega})$ and $B_1 < \cdots < B_d$ are in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma})$, with $A_0 \cup E = \bigcup_{j=1}^d B_j$ and $\bar{B} = \{\min B_j : 1 \le j \le d\} \in \mathrm{MAX}(\mathcal{S}_\beta)$, and if $(y^{(j)}_{E,\Phi})_{j=1}^d$ is defined as in (114), then for $j = 1, \dots, d-1$ with $l_1(\bar{B}_j) > 0$, where $\bar{B}_j = \{\min B_i : 1 \le i \le j\}$, we have $\|y^{(j)}_{\Phi,E}\| \le 1$ and $\mathrm{ran}(y^{(j)}_{\Phi,E}) \subset (\max(\bar{B}_j), \max(\bar{B}_{j+2}))$. Put $\max(\bar{B}_{d+1}) = \infty$. Then

$$\sum_{A \in \tilde{\mathcal{E}}_{\beta(\gamma+1)}(A_0 \cup E)} \zeta\big(\beta(\gamma+1), A\big) \sum_{k=1}^{s(\beta,\gamma+1,A)} x^{(k)}_{\Phi,A}$$
$$= \sum_{A \in \tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0 \cup E)} \zeta\big(\beta(\gamma+1), A\big) x^{(1)}_{\Phi,A} + \sum_{\substack{1 \le j < d \\ l_1(\bar{B}_j) > 0}} \zeta(\beta, \bar{B}_{j+1}) y^{(j)}_{\Phi,E}. \tag{119}$$

Proof

Observe that (119) immediately follows from (116) and the fact that, for $1 \le j < d$ with $l_1(\bar{B}_j) > 0$, we have $\zeta(\beta, \bar{B}_j) = \zeta(\beta, \bar{B}_{j+1})$. For $1 \le j < d$ with $l_1(\bar{B}_j) > 0$ and $A \in \mathcal{E}^{(j)}_{\beta,\gamma+1}(A_0 \cup E)$, note that $\bigcup_{i=1}^j B_i \prec A \preceq \bigcup_{i=1}^{j+1} B_i$ and

$$\begin{aligned} u_A &= \sum_{k=2}^{s(\beta,\gamma+1,A)} x^{(k)}_{\Phi,A} \\ &= P_{(\max\mathrm{supp}(x_{\mathrm{Cp}_{\beta,\gamma}(A,1)}), \max\mathrm{supp}(x_A)]}(x_A) \\ &= P_{(\max\mathrm{supp}(x_{\bigcup_{i=1}^j B_j}), \max\mathrm{supp}(x_A)]} x_A; \end{aligned} \tag{120}$$

that is, $\|u_A\| \le 1$ and by Definition 9.5(b) we obtain

$$\max(\bar{B}_j) \le \max\Big(\bigcup_{i=1}^j B_i\Big) \le \max\mathrm{supp}(x_{\bigcup_{i=1}^j B_i}) < \min\mathrm{supp}(u_A) \qquad \text{and}$$

$$\max\mathrm{supp}(u_A) \le \max\mathrm{supp}(x_A) < \min\big\{m \in M : m > \max(A)\big\} \le \max(\bar{B}_{j+2}),$$

which yields

$$\mathrm{ran}(u_A) \subset \big(\max(\bar{B}_j), \max(\bar{B}_{j+2})\big). \tag{121}$$

Furthermore, we have $\zeta(\beta(\gamma+1), A) = \zeta(\beta, \bar{B}_{j+1})\zeta(\beta\gamma, A \setminus (\bigcup_{i=1}^j B_i))$, and since $\zeta(\beta, \bar{B}_{j+1}) = \zeta(\beta, \bar{B}_j)$ (by $l_1(\bar{B}_j) > 0$), we obtain

$$\frac{\zeta(\beta(\gamma+1), A)}{\zeta(\beta, \bar{B}_j)} = \zeta\Big(\beta\gamma, A \setminus \Big(\bigcup_{i=1}^j B_i\Big)\Big). \tag{122}$$

We deduce

$$y^{(j)}_{\Phi,E} = \sum_{C\in\mathcal{E}_{\beta,\gamma}(B_{j+1})} \zeta(\beta\gamma, C)u_{(\bigcup_{i=1}^{j} B_i)\cup C}.$$

The above in combination with (121), (122), and the fact that $\|u_A\| \le 1$ yield that $\|y^{(j)}_{\Phi,E}\| \le 1$ and $\operatorname{ran}(y^{(j)}_{\Phi,E}) \subset (\max(\bar{B}_j), \max(\bar{B}_{j+2}))$. □

LEMMA 11.4

Let γ, $M \in [N]^\omega$, A_0, Φ, and c be as in the statement of Lemma 11.2. Let $E \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [M]^{<\omega})$, $(B_j)_{j=1}^{d}$, $\bar{B}$, and $(\bar{B}_j)_{j=1}^{d}$ be as in the statement of Lemma 11.3, and let $\Phi^{(j)}_E$, $j = 1,\ldots,d-1$, and $M^{(j)} \in [M]^\omega$ be defined as in (113). For $j = 1,\ldots,d-1$ with $l_1(\bar{B}_j) > \emptyset$ denote by

$$\big((z^{(k)}_{\Phi^{(j)}_E,C})_{k=1}^{s(\beta,\gamma',C)}\big)_{C\in\mathcal{E}_{\beta,\gamma'}(B_{j+1})}$$

the block step decomposition of B_{j+1} with respect to $\Phi^{(j)}_E$. Then for $C \in \mathcal{E}_{\beta,\gamma}(B_{j+1})$ we have $s(\beta,\gamma+1,(\bigcup_{i=1}^{j} B_i)\cup C) = s(\beta,\gamma,C)+1$ and

$$z^{(k)}_{\Phi^{(j)}_E,C} = x^{(k+1)}_{\Phi,(\bigcup_{i=1}^{j} B_i)\cup C}, \quad \text{for } k = 1,\ldots,s(\beta,\gamma,C).$$

Proof

Fix $C \in \mathcal{E}_{\beta,\gamma}(B_{j+1})$. By (103b), if we set $A = (\bigcup_{i=1}^{j} B_i)\cup C$, then $s(\beta,\gamma+1,A) = s(\beta,\gamma,C)+1$, $\mathrm{Cp}_{\beta,\gamma+1}(A,i+1) = \mathrm{Cp}_{\beta,\gamma}(A,i)$ for $i = 1,\ldots,s(\beta,\gamma,C)$, and $\mathrm{Cp}_{\beta,\gamma+1}(A,1) = \bigcup_{i=1}^{j} B_i$. Fix $1 \le k \le s(\beta,\gamma,C)$. Let $\{z_C : C \in \mathcal{S}_{\beta\gamma} \cap [M^{(j)}]^{<\omega}\}$ be the family generating $\Phi^{(j)}_E$, and let $n_0 = \max\operatorname{supp}(x_{\bigcup_{i=1}^{j} B_i})$. Then by Definition 11.1 and Lemma 9.7 we have

$$x^{(k+1)}_{\Phi,A} = P_{I(A,k+1)}(x_A)$$

and

$$z^{(k)}_{\Phi^{(j)}_E,C} = P_{I(C,k)}(z_C) = P_{I(C,k)}\big(P_{(n_0,\infty)}(x_{\bigcup_{i\le j} B_i\cup C})\big),$$

with (if $k=1$, replace $\max\operatorname{supp}(z_{\bigcup_{i=1}^{k-1}\mathrm{Cp}_{\beta,\gamma}(C,i)})$ by n_0)

$$\begin{aligned} I(C,k) &= \big(\max\operatorname{supp}(z_{\bigcup_{i=1}^{k-1}\mathrm{Cp}_{\beta,\gamma}(C,i)}), \max\operatorname{supp}(z_{\bigcup_{i=1}^{k}\mathrm{Cp}_{\beta,\gamma}(C,i)})\big] \\ &= \big(\max\operatorname{supp}(P_{(n_0,\infty)}x_{(\bigcup_{i=1}^{j} B_i)\cup(\bigcup_{i=1}^{k-1}\mathrm{Cp}_{\beta,\gamma}(C,i))}), \\ &\qquad \max\operatorname{supp}(P_{(n_0,\infty)}x_{(\bigcup_{i=1}^{j} B_i)\cup(\bigcup_{i=1}^{k}\mathrm{Cp}_{\beta,\gamma}(C,i))})\big] \end{aligned}$$

$$= \big(\max \operatorname{supp}(x_{(\bigcup_{i=1}^{j} B_i)\cup(\bigcup_{i=1}^{k-1} \mathrm{Cp}_{\beta,\gamma}(C,i))}),$$
$$\max \operatorname{supp}(x_{(\bigcup_{i=1}^{j} B_i)\cup(\bigcup_{i=1}^{k} \mathrm{Cp}_{\beta,\gamma}(C,i))})\big]$$
$$= I(A, k+1),$$

where we used $n_0 \le \max \operatorname{supp}(x_{\bigcup_{i=1}^{j} B_i})$, which follows from Definition 9.5(b). Hence,

$$z^{(k)}_{\Phi_E^{(j)},C} = P_{I(A,k+1)} P_{(n_0,\infty)}(x_A) = P_{I(A,k+1)}(x_A) = x^{(k+1)}_{\Phi,A}. \qquad \square$$

PROPOSITION 11.5

Assume that $\mathrm{Sz}(X) \le \omega^{\beta}$. *Then, for every* $1 \ge c > 0$ *and* $M \in [\mathbb{N}]^{\omega}$, *there exists* $N \in [M]^{\omega}$ *with the following property: for every* $\gamma \le \beta$, $L \in [N]^{\omega}$, $A_0 \subset N$ *that is either empty or a singleton, every semiembedding* $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega} \to X$ *from* $\mathcal{S}_{\beta\gamma} \cap [L]^{<\omega}$ *into* X, *starting at* A_0, *that is* c*-refined, every* $E \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega})$, *and every* (β,γ)*-special family of convex combinations* $\{r(A,k) : A \in \mathcal{E}_{\beta,\gamma}(A_0 \cup E), 1 \le k \le s(\beta,\gamma,A)\}$, *we have*

$$\sum_{A \in \mathcal{E}_{\beta,\gamma}(A_0 \cup E)} \zeta(\beta\gamma, A) \sum_{k=1}^{s(\beta,\gamma,A)} r(A,k) \|x^{(k)}_{\Phi,A}\| \ge \frac{c}{3}, \tag{123}$$

where $(x^{(k)}_{\Phi,A})_{A \in \mathcal{E}_{\beta,\gamma(A_0,E)}}$ *is the block step decomposition of* E *for* Φ *(Definition 11.1).*

Proof

Fix $M \in [\mathbb{N}]^{\omega}$ and $1 \ge c > 0$. Choose $\varepsilon > 0$ such that $c/2 - \varepsilon > c/3$, and then apply Theorem 6.1 to find $N \in [M]^{\infty}$ such that (67) is satisfied for that ε and β and, moreover,

$$\Big(\frac{c}{2} - \varepsilon - \frac{4}{\min(N)}\Big) \prod_{m \in N} \Big(1 - \frac{1}{m}\Big) > \frac{c}{3}. \tag{124}$$

We claim that this is the desired set. We shall prove by transfinite induction on γ the following statement: if $\gamma \le \beta$, $L \in [N]^{\omega}$, $A_0 \subset L$ that is either empty or a singleton, and $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega} \to X$ is a semiembedding of $\mathcal{S}_{\beta\gamma} \cap [L]^{<\omega}$ into X, starting at A_0, that is c-refined, then for any $E \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega})$ and any (β,γ)-special family of convex combinations $\{r(A,k) : A \in \mathcal{E}_{\beta,\gamma}(A_0 \cup E), 1 \le k \le s(\beta,\gamma,A)\}$ we have

$$\sum_{A\in\mathcal{E}_{\beta,\gamma}(A_0\cup E)} \zeta(\beta\gamma, A) \sum_{k=1}^{s(\beta,\gamma,A)} r(A,k)\|x^{(k)}_{\Phi,A}\| \geq \Big(\frac{c}{2} - \varepsilon - \frac{4}{\min(L)}\Big) \prod_{m\in L}^{\infty}\Big(1-\frac{1}{m}\Big). \tag{125}$$

In conjunction with (124), this will yield the desired result. Let $\gamma = 1$, let $L \subset N$, let A_0 be a subset of L that is either empty or a singleton, and let $\Phi : \mathcal{S}_\beta(A_0) \cap [L]^{<\omega} \to X$ be a semiembedding of $\mathcal{S}_\beta \cap [L]^{<\omega}$ into X, starting at A_0, that is c-refined. Let $E \in \mathrm{MAX}(\mathcal{S}_\beta(A_0) \cap [N]^{<\omega})$.

By (103a), for each $A \in \mathcal{E}_{\beta,1}(A_0 \cup E)$ we obtain that the block step decomposition of x_A is just $(x^{(1)}_{\Phi,A}) = (x_A)$, and hence, if $\{r(A,k) : A \in \mathcal{E}_{\beta,1}(A_0 \cup E), 1 \leq k \leq s(\beta,1,A)\}$ is a $(\beta,1)$-special family of convex combinations, then $r(A,1) = 1$. Hence,

$$\begin{aligned}
&\sum_{A\in\mathcal{E}_{\beta,1}(A_0\cup E)} \zeta(\beta, A) \sum_{k=1}^{s(\beta,1,A)} r(A,k)\|x^{(k)}_{\Phi,A}\| \\
&= \sum_{A\in\mathcal{E}_{\beta,1}(A_0\cup E)} \zeta(\beta, A)\|x_A\| \\
&\geq \Big\|\sum_{A\preceq A_0\cup E} \zeta(\beta, A)x_A\Big\| - \sum_{\substack{A\preceq A_0\cup E\\ A\notin\mathcal{E}_{\beta,1}(A_0\cup E)}} \zeta(\beta, A) \\
&\geq \big\|\Phi(E)\big\| - \frac{2}{\min(A_0 \cup E)} \quad \text{(by (108))} \\
&\geq \frac{c}{2} - \frac{c}{2}\zeta(\beta, A_0) - \frac{2}{\min(A_0 \cup B)} \quad \text{(by Definition 9.5(d))} \\
&> \frac{c}{2} - \frac{3}{\min(L)} \quad \text{(by (33))}.
\end{aligned}$$

To verify the induction step, first let $\gamma < \beta$ be an ordinal number for which the conclusion holds. Let $L \subset N$, let A_0 be a subset of L that is either empty or a singleton, and let $\Phi : \mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [L]^{<\omega} \to X$ be a semiembedding of $\mathcal{S}_{\beta(\gamma+1)} \cap [L]^{<\omega}$ into X, starting at A_0, that is c-refined. Let $E \in \mathrm{MAX}(\mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [L]^{<\omega})$, and let $\{r(A,k) : A \in \mathcal{E}_{\beta,\gamma+1}(A_0 \cup E), 1 \leq k \leq s(\beta,\gamma+1,A)\}$ be a $(\beta,\gamma+1)$-special family of convex combinations. Let $A_0 \cup E = \bigcup_{j=1}^d B_j$, where $B_1 < \cdots < B_d$ are in $\mathrm{MAX}(\mathcal{S}_{\beta\gamma} \cap [L]^{<\omega})$ and $\bar{B} = \min\{B_j : 1 \leq j \leq d\}$ is in $\mathrm{MAX}(\mathcal{S}_\beta)$. By Lemma 11.3 and the choice of the set N, we obtain

$$\Big\| \sum_{\substack{1\le j<d\\ l_1(\bar{B}_j)>0}} \zeta(\beta,\bar{B}_{j+1}) y^{(j)}_{\Phi,E} \Big\| < \varepsilon. \tag{126}$$

Combining first Lemmas 11.2 and 11.3, then applying Definition 9.5(d) and (33), and finally by using (126), we deduce

$$\begin{aligned}
&\sum_{A\in\tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0\cup E)} \zeta\big(\beta(\gamma+1),A\big)\|x^{(1)}_{\Phi,A}\| \\
&\ge \Big\| \sum_{A\in\tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0\cup E)} \zeta\big(\beta(\gamma+1),A\big)x^{(1)}_{\Phi,A} \Big\| \\
&\ge \big\|\Phi(E)\big\| - \Big\| \sum_{\substack{1\le j<d\\ l_1(\bar{B}_j)>0}} \zeta(\beta,\bar{B}_{j+1}) y^{(j)}_{\Phi,E} \Big\| - \frac{3}{\min(A_0\cup E)} \\
&\ge \frac{c}{2} - \frac{c}{2}\zeta\big(\beta(\gamma+1),A_0\big) - \varepsilon - \frac{3}{\min(A_0\cup E)} \ge \frac{c}{2} - \varepsilon - \frac{4}{\min(L)}.
\end{aligned} \tag{127}$$

We distinguish two cases. If $r(A,1)=1$ for all $A\in\mathcal{E}_{\beta,\gamma+1}(A_0\cup E)$, then

$$\begin{aligned}
&\sum_{A\in\mathcal{E}_{\beta,\gamma+1}(A_0\cup E)} \zeta\big(\beta(\gamma+1),A\big) \sum_{k=1}^{s(\beta,\gamma+1,A)} r(A,k)\|x^{(k)}_{\Phi,A}\| \\
&= \sum_{A\in\mathcal{E}_{\beta,\gamma+1}(A_0\cup E)} \zeta\big(\beta(\gamma+1),A\big)\|x^{(1)}_{\Phi,A}\| \\
&\ge \sum_{A\in\tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0\cup E)} \zeta\big(\beta(\gamma+1),A\big)\|x^{(1)}_{\Phi,A}\|.
\end{aligned}$$

By (127), the result follows in that case.

Otherwise we have $r_1 = r(A,1) < 1$ for all $A\in\mathcal{E}_{\beta,\gamma+1}(A_0\cup E)$. For $1\le j<d$, with $l_1(\bar{B}_j)>0$, define $L^{(j)} = L\cap(\max(A_0\cup(\bigcup_{1\le i\le j} B_i)),\infty)$ and $\Phi^{(j)}_E : \mathcal{S}_{\beta\gamma}\cap[L^{(j)}]^{<\omega}\to X$ as in (113). By Lemma 9.7, each $\Phi^{(j)}_E$ is a semiembedding of $\mathcal{S}_{\beta\gamma}\cap[L^{(j)}]^{<\omega}$ into X, starting at $\emptyset$, that is c-refined.

By Lemma 10.9, the family $\{r^{(j)}(C,k) : C\in\mathcal{E}_{\beta,\gamma}(B_{j+1})\}$, with

$$r^{(j)}(C,k) = \frac{1}{1-r(A^{(1)},1)} r\Big(\Big(\bigcup_{i=1}^{j} B_i\Big)\cup C, k+1\Big)$$

for $k=1,\dots,s(\beta,\gamma,C) = s(\beta,\gamma,(\bigcup_{i=1}^{j} B_i)\cup C)-1$ and some $A^{(1)}\in\mathcal{E}_{\beta,\gamma+1}(A_0\cup E)$, is a (β,γ)-special family of convex combinations. Hence, by the inductive assumption applied to the map $\Phi^{(j)}_E$ and Lemma 11.4, we deduce that

$$\sum_{A\in\mathcal{E}^{(j)}_{\beta,\gamma+1}(A_0\cup E)} \zeta\big(\beta(\gamma+1),A\big)\sum_{k=2}^{s(\beta,\gamma+1,A)} r(A,k)\|x^{(k)}_{\Phi,A}\|$$

$$= \zeta(\beta,\bar B_j)\big(1-r(A^{(1)},1)\big)\sum_{C\in\mathcal{E}_{\beta,\gamma}(B_{j+1})} \zeta(\beta\gamma,C)\sum_{k=1}^{s(\beta,\gamma,C)} r^{(j)}(C,k)\|z^{(k)}_{\Phi^{(j)}_E,C}\|$$

$$\geq \zeta(\beta,\bar B_j)\big(1-r(A^{(1)},1)\big)\Big(\frac{c}{2}-\varepsilon-\frac{4}{\min(L^{(j)})}\Big)\prod_{m\in L^{(j)}}\Big(1-\frac{3}{m}\Big). \tag{128}$$

We combine (127) with (128) to obtain

$$\sum_{A\in\mathcal{E}_{\beta,\gamma+1}(A_0\cup E)} \zeta\big(\beta(\gamma+1),A\big)\sum_{k=1}^{s(\beta,\gamma+1,A)} r(A,k)\|x^{(k)}_{\Phi,A}\|$$

$$\geq \sum_{A\in\tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0\cup E)} \zeta\big(\beta(\gamma+1),A\big)\sum_{k=1}^{s(\beta,\gamma+1,A)} r(A,k)\|x^{(k)}_{\Phi,A}\|$$

$$= \sum_{A\in\tilde{\mathcal{E}}_{\beta,\gamma+1}(A_0\cup E)} \zeta\big(\beta(\gamma+1),A\big) r(A,1)\|x^{(1)}_{\Phi,A}\|$$

$$+ \sum_{\substack{1\leq j<d\\ l_1(\bar B_j)>0}}\ \sum_{A\in\mathcal{E}^{(j)}_{\beta,\gamma+1}(A_0\cup E)} \zeta\big(\beta(\gamma+1),A\big)\sum_{k=2}^{s(\beta,\gamma+1,A)} r(A,k)\|x^{(k)}_{\Phi,A}\|$$

$$\geq r(A^{(1)},1)\Big(\frac{c}{2}-\varepsilon-\frac{4}{\min(L)}\Big)$$

$$+ \sum_{\substack{1\leq j<d\\ l_1(\bar B_j)>0}} \zeta(\beta,\bar B_j)\big(1-r(A^{(1)},1)\big)\Big(\frac{c}{2}-\varepsilon-\frac{4}{\min(L)}\Big)\prod_{m\in L^{(j)}}\Big(1-\frac{1}{m}\Big)$$

$$\geq \Big(\sum_{\substack{1\leq j<d\\ l_1(\bar B_j)>0}} \zeta(\beta,\bar B_j)\Big)\Big(\frac{c}{2}-\varepsilon-\frac{4}{\min(L)}\Big)\prod_{m\in L^{(1)}}\Big(1-\frac{1}{m}\Big)$$

$$\geq \Big(1-\frac{1}{\min(\bar B)}\Big)\Big(\frac{c}{2}-\varepsilon-\frac{4}{\min(L)}\Big)\prod_{m\in L^{(1)}}\Big(1-\frac{1}{m}\Big) \quad \text{(by Lemma 3.7(e))}$$

$$\geq \Big(\frac{c}{2}-\varepsilon-\frac{4}{\min(L)}\Big)\prod_{m\in L}\Big(1-\frac{1}{m}\Big).$$

Assume now that $\gamma \leq \beta$ is a limit ordinal number and that the claim holds for all $\gamma' < \gamma$. Let $L \in [N]^\omega$, let A_0 be a subset of L that is either empty or a singleton, and let $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega} \to X$ be a semiembedding of $\mathcal{S}_{\beta\gamma} \cap [L]^{<\omega}$ into X, starting at A_0, that is c-refined. We distinguish between two cases, namely, whether A_0 is a singleton or whether it is empty. In the first case, $A_0 = \{a_0\}$ for some $a_0 \in L$. By Remark 9.6, the map Ψ with $\Psi = \Phi$ can be seen as a semiembedding of $\mathcal{S}_{\beta\eta(\gamma,a_0)} \cap [L]^{<\omega}$ into X, starting at A_0, that is c-refined. If $E \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega})$, then $E \in \mathrm{MAX}(\mathcal{S}_{\beta\eta(\gamma,a_0)}(A_0) \cap [L]^{<\omega})$ and by (110) we have

$$\big((x_{\Psi,A}^{(k)})_{k=1}^{s(\beta,\eta(\gamma,a_0),A)}\big)_{A\in\mathcal{E}_{\beta,\eta(\gamma,a_0)}(A_0\cup E)} = \big((x_{\Phi,A}^{(k)})_{k=1}^{s(\beta,\gamma,A)}\big)_{A\in\mathcal{E}_{\beta,\gamma}(A_0\cup E)},$$

whereas if $\{r(A,k) : A \in \mathcal{E}_{\beta,\gamma}(A_0 \cup E), 1 \leq k \leq s(\beta,\gamma,A)\}$ is a (β,γ)-special family of convex combinations, then by the remark following Definition 10.7, it is a $(\beta,\eta(\gamma,a_0))$-special family of convex combinations as well. Applying the inductive assumption for $\eta(\gamma,a_0) < \gamma$ yields

$$\begin{aligned}
&\sum_{A\in\mathcal{E}_{\beta,\gamma}(A_0\cup E)} \zeta(\beta\gamma,A) \sum_{k=1}^{s(\beta,\gamma,A)} r(A,k)\|x_{\Phi,A}^{(k)}\| \\
&= \sum_{A\in\mathcal{E}_{\beta,\eta(\gamma,a_0)}(A_0\cup E)} \zeta\big(\beta\eta(\gamma,a_0),A\big) \sum_{k=1}^{s(\beta,\eta(\gamma,a_0),A)} r(A,k)\|x_{\Psi,A}^{(k)}\| \\
&\geq \Big(\frac{c}{2} - \varepsilon - \frac{4}{\min(L)}\Big) \prod_{m\in L}^{\infty}\Big(1 - \frac{1}{m}\Big). \qquad (129)
\end{aligned}$$

In the second case, A_0 is empty. Let $B \in \mathrm{MAX}(\mathcal{S}_{\beta\gamma} \cap [L]^{<\omega})$, and set $a_0 = \min(B)$. By Remark 9.6, if $L' = L \cap [a_0,\infty)$, then the map $\Psi : \mathcal{S}_{\beta\eta(\gamma,a_0)}(A_0) \cap [L'] \to X$ with $\Psi(A) = \Phi(A_0 \cup A)$ is a semiembedding of $\mathcal{S}_{\beta\eta(\gamma,a_0)} \cap [L']^{<\omega}$ into X, starting at A_0, that is c-refined. By (111) we obtain

$$\big((x_{\Psi,A}^{(k)})_{k=1}^{s(\beta,\eta(\gamma,a_0),A)}\big)_{A\in\mathcal{E}_{\beta,\eta(\gamma,a_0)}(B)} = \big((x_{\Phi,A}^{(k)})_{k=1}^{s(\beta,\gamma,A)}\big)_{A\in\mathcal{E}_{\beta,\gamma}(A_0\cup E)},$$

whereas if $\{r(A,k) : A \in \mathcal{E}_{\beta,\gamma}(B), 1 \leq k \leq s(\beta,\gamma,A)\}$ is a (β,γ)-special family of convex combinations, then by the remark following Definition 10.8, it is a $(\beta,\eta(\gamma, a_0))$-special family of convex combinations as well. The result follows in the same manner as in (129). ☐

THEOREM 11.6

Assume that X is a reflexive and separable Banach space, with the property that $\mathrm{Sz}(X) \leq \omega^\beta$ and $\mathrm{Sz}(X^*) < \beta$. Then for no $L \in [\mathbb{N}]^\omega$ does there exist a semiembedding of $\mathcal{S}_{\beta 2} \cap [L]^{<\omega}$ into X, starting at $\emptyset$.

Proof
By [32, Main Theorem] we can embed X into a reflexive space Z with basis such that $\mathrm{Sz}(Z) = \mathrm{Sz}(X)$ and $\mathrm{Sz}(Z^*) = \mathrm{Sz}(X^*)$. Thus, we may assume that X has a basis, which must be shrinking and boundedly complete, since X is reflexive. By renorming X, we may assume that the bases of X and X^* are bimonotone. Choose α_0 with $\mathrm{Sz}(X^*) \leq \omega^{\alpha_0} < \beta$. (This is possible due to the form of β.) Note that

$$\mathrm{CB}(\mathcal{S}_{\alpha_0}) = \omega^{\alpha_0} + 1 < \beta + 1 = \mathrm{CB}(\mathcal{F}_{\beta,\beta}). \tag{130}$$

Toward a contradiction, assume that there exists $L \in [\mathbb{N}]^\omega$ and a semiembedding Ψ of $\mathcal{S}_{\beta^2} \cap [L]^{<\omega}$ into X, starting at $\emptyset$. By Lemma 9.8 there exist $1 \geq c > 0$, $M \in [L]^\omega$, and a semiembedding Φ of $\mathcal{S}_{\beta^2} \cap [M]^{<\omega}$ into X, starting at $\emptyset$, that is c-refined. Applying Proposition 11.5, we may pass to a further subset of M, again denoted by M, so that (123) holds. Fix $0 < \varepsilon < c/3$, and apply Theorem 6.1 to the space X^* and the ordinal number a_0 to find a further subset of M, which we again denote by M, so that (67) is satisfied.

By Propositions 2.2 and 2.13 we may pass to a subset of M, again denoted by M, so that $\mathcal{S}_{\alpha_0} \cap [M]^{<\omega} \subset \mathcal{F}_{\beta,\beta}$. By Lemma 10.2 we obtain that, for any $B \in \mathrm{MAX}(\mathcal{S}_{\beta^2} \cap [M]^{<\omega})$ and $A \in \mathcal{E}_{\beta,\beta}(B)$, there exists $\tilde{A} \in \mathrm{MAX}(\mathcal{S}_{\alpha_0})$ with

$$\tilde{A} \preceq \big\{\min\big(\mathrm{Cp}_{\beta,\beta}(A,k)\big) : 1 \leq k \leq s(\beta,\beta,A)\big\}. \tag{131}$$

Choose $B \in \mathrm{MAX}(\mathcal{S}_{\beta^2} \cap [M]^{<\omega})$. We will define a (β,β)-special family of convex combinations $\{r(A,k) : A \in \mathcal{E}_{\beta,\beta}(B), 1 \leq k \leq s(\beta,\beta,A)\}$. For $A \in \mathcal{E}_{\beta,\beta}(B)$ let $\tilde{A} = \{a_1^A, \ldots, a_{d_A}^A\} \in \mathrm{MAX}(\mathcal{S}_{\alpha_0})$ be as in (131). For $1 \leq k \leq s(\beta,\beta,A)$ set

$$r(A,k) = \begin{cases} \zeta(\alpha_0, \tilde{A}_k) & \text{if } k \leq \#\tilde{A}, \\ 0 & \text{otherwise,} \end{cases} \tag{132}$$

where $\tilde{A}_k = \{a_1^A, \ldots, a_k^A\}$ for $1 \leq k \leq \#\tilde{A}$. We will show that this family satisfies Definitions 10.8(a) and 10.8(b). The first assertion is straightforward; to see the second one, let $A^{(1)}, A^{(2)} \in \mathcal{E}_{\beta,\beta}(B)$ such that if $(D_k(A^{(1)}))_{k=1}^{s(\beta,\beta,A)}$ and $(D_k(A^{(2)}))_{i=1}^{s(\beta,\beta,A')}$ are the maximal chains of $\mathcal{A}_{\beta,\beta}(B)$ provided by Lemma 10.2, then for some k we have $D_k(A^{(1)}) = D_k(A^{(2)})$. By Lemmas 10.6 and 10.7 we obtain $\min(\mathrm{Cp}_{\beta,\beta}(A^{(1)}, m)) = \min(\mathrm{Cp}_{\beta,\beta}(A^{(2)}, m))$ for $m = 1, \ldots, k$, which implies $\tilde{A}_m^{(1)} = \tilde{A}_m^2$ for $m = 1, \ldots, \min\{k, \#\tilde{A}^{(1)}\}$. By (132) it easily follows that $r(A^{(1)}, k) = r(A^{(2)}, k)$. Since (123) is satisfied, we obtain

$$\sum_{A \in \mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) \|x_{\Phi,A}^{(k)}\| \geq \frac{c}{3}. \tag{133}$$

For each $A \in \mathcal{E}_{\beta,\beta}(B)$ and $k = 1, \ldots, d_A$ choose $f_A^{(k)}$ in S_{X^*}, with $f_A^{(k)}(x_A^{(k)}) = \|x_A^{(k)}\|$ and

$$\begin{aligned}\operatorname{ran}(f_A^{(k)}) &\subset \operatorname{ran}(x_{\Phi,A}^{(k)})\\ &\subset \big(\max \operatorname{supp}(x_{\bigcup_{i=1}^{k-1} \mathrm{Cp}_{\beta,\beta}(A,i)}), \max \operatorname{supp}(x_{\bigcup_{i=1}^{k} \mathrm{Cp}_{\beta,\beta}(A,i)})\big]\\ &\subset \big(\min\big(\mathrm{Cp}_{\beta,\beta}(A,k-1)\big), \min\big(\mathrm{Cp}_{\beta,\beta}(A,k+1)\big)\big)\\ &= \big(\max(\tilde{A}_{k-1}), \max(\tilde{A}_{k+1})\big),\end{aligned}$$

where the third inclusion follows from Definition 9.5(b). As (67) is satisfied, we obtain that for all $A \in \mathcal{E}_{\beta,\beta}(B)$

$$\Big\|\sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) f_A^{(k)}\Big\| < \varepsilon. \tag{134}$$

We finally calculate

$$\begin{aligned}\frac{c}{3} &\le \sum_{A\in\mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k)\|x_{\Phi,A}^{(k)}\| \quad \text{(by (133))}\\ &= \sum_{A\in\mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) f_A^{(k)}(x_{\Phi,A}^{(k)}) \quad \text{(by the choice of } f_A^{(k)})\\ &= \sum_{A\in\mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) f_A^{(k)}\Big(\sum_{m=1}^{s(\beta,\beta,A)} x_{\Phi,A}^{(m)}\Big)\\ &\quad \text{(since } \operatorname{ran}(f_A^{(k)}) \subset \operatorname{ran}(x_{\Phi,A}^{(k)}))\\ &= \sum_{A\in\mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) f_A^{(k)}(x_A) \quad \text{(by (109))}\\ &\le \sum_{A\in\mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A)\Big\|\sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) f_A^{(k)}\Big\| < \varepsilon \quad \text{(by (134))}.\end{aligned}$$

This contradiction completes the proof. ☐

Before proving Corollary 1.2 we will need the following observation.

PROPOSITION 11.7

Let X be a Banach space, let $\alpha < \omega_1$, and let L be an infinite subset of the natural *numbers so that there exist numbers $0 < c < C$ and a map $\Phi : \mathcal{S}_\alpha \cap [L]^{<\omega} \to X$ that is a c-lower-$d_{\infty,\alpha}$ and C-upper-$d_{1,\alpha}$ embedding. Then for every $\beta < \alpha$ there exist $n \in \mathbb{N}$ and a map $\Phi_\beta : \mathcal{S}_\beta \cap [L \cap (n,\infty)]^{<\omega} \to X$ that is a c-lower-$d_{\infty,\beta}$ and C-upper-$d_{1,\beta}$ embedding.*

Before proving Proposition 11.7 we need some preliminary observations. The first one can be easily shown, and we omit a proof.

LEMMA 11.8
Let α be an ordinal number with $A \subset [0,\alpha]$ satisfying:
(a) *$\alpha \in A$ and*
(b) *if $\beta \in A$ and $\gamma < \beta$, then there is $\gamma \le \eta < \beta$ with $\eta \in A$.*
Then $A = [0,\alpha]$.

LEMMA 11.9
Let $\alpha < \omega_1$ be a limit ordinal number. Then there exists a sequence of successor ordinal numbers $(\mu(\alpha,n))_n$ satisfying the following statements.
(a) *$\mu(\alpha,n) < \alpha$ for all $n \in \mathbb{N}$ and $\lim_n \mu(\alpha,n) = \alpha$.*
(b) *$\mathcal{S}_\alpha = \{A \in [\mathbb{N}]^{<\omega} : A \in \mathcal{S}_{\mu(\alpha,\min(A))}\} \cup \{\emptyset\}$ and $\mathcal{S}_{\mu(\alpha,n)} \cap [[n,\infty)]^{<\omega} \subset \mathcal{S}_\alpha$ for all $n \in \mathbb{N}$.*
(c) *For $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$, $z_{(\alpha,A)} = z_{(\mu(\alpha,\min(A)),A)}$.*

Proof
We define $(\mu(\alpha,n))_n$ by transfinite recursion on the set of countable limit ordinal numbers. For $\alpha = \omega$ we set $(\mu(\omega,n))_n = (\lambda(\omega,n))_n$. If α is a limit ordinal such that for all $\alpha' < \alpha$ the corresponding sequence has been defined, set for each $n \in \mathbb{N}$

$$\mu(\alpha,n) = \begin{cases} \lambda(\alpha,n) & \text{if } \lambda(\alpha,n) \text{ is a successor ordinal number,} \\ \mu(\lambda(\alpha,n),n) & \text{otherwise.} \end{cases}$$

The fact that (b), (c), and the first part of (a) hold is proved easily by transfinite induction using (23) in Corollary 2.7 and the definition of repeated averages. To show that $\lim_n \mu(\alpha,n) = \alpha$, we will show that for arbitrary $L \in [\mathbb{N}]^\omega$ we have $\sup_{n\in L} \mu(\alpha,n) = \alpha$. Fix $L \in [\mathbb{N}]^\omega$ and $\beta < \alpha$. Then, since $\mathrm{CB}(\mathcal{S}_\alpha \cap [L]^{<\omega}) = \omega^\alpha + 1 > \omega^\beta + 1$, we have $\emptyset \in (\mathcal{S}_\alpha \cap [L]^{<\omega})^{(\omega^\beta+1)}$, and hence, there exists $n \in L$ with $\{n\} \in (\mathcal{S}_\alpha \cap [L]^{<\omega})^{(\omega^\beta)}$. Using (4) we obtain

$$\begin{aligned} \emptyset &\in \big(\mathcal{S}_\alpha \cap [L]^{<\omega}\big)^{(\omega^\beta)}\big(\{n\}\big) \subset \mathcal{S}_\alpha^{(\omega^\beta)}\big(\{n\}\big) = \big(\mathcal{S}_\alpha\big(\{n\}\big)\big)^{(\omega^\beta)} \\ &= \big(\mathcal{S}_{\mu(\alpha,n)}\big(\{n\}\big)\big)^{(\omega^\beta)} = \mathcal{S}_{\mu(\alpha,n)}^{(\omega^\beta)}\big(\{n\}\big), \end{aligned}$$

which implies $\{n\} \in \mathcal{S}_{\mu(\alpha,n)}^{(\omega^\beta)}$, that is, $\mathrm{CB}(\mathcal{S}_{\mu(\alpha,n)}) \geq \omega^\beta + 1$, which yields $\mu(\alpha, n) \geq \beta$. □

Proof of Proposition 11.7
We shall first treat two very specific cases. In the first case, $\alpha = \beta + 1$. Fix $n_0 \geq 2$ with $n_0 \in L$, and fix $B_0 \in \mathrm{MAX}(\mathcal{S}_\beta \cap [L]^{<\omega})$ with $\min(B_0) = n_0$. Define $n = \max(B_0)$ and $\Phi_\beta : \mathcal{S}_\beta \cap [L \cap (n, \infty)]^{<\omega} \to X$ with $\Phi_\beta(A) = n_0 \Phi(B_0 \cup A)$. Then Φ_β is the desired embedding.

In the second case, α is a limit ordinal, and for some $n_0 \geq 2$ with $n_0 \in L$ we have $\beta + 1 = \mu(\alpha, n_0)$. Fix $B_0 \in \mathrm{MAX}(\mathcal{S}_\beta \cap [L]^{<\omega})$ with $\min(B_0) = n_0$, define $n = \max(B_0)$, and define $\Phi_\beta : \mathcal{S}_\beta \cap [L \cap (n, \infty)]^{<\omega} \to X$ with $\Phi_\beta(A) = n_0 \Phi(B_0 \cup A)$. Then, by using the properties of $(\mu(\alpha, k))_k$, it can be seen that Φ_β is well defined and is the desired embedding.

In the general case, define A to be the set of all $\beta \leq \alpha$ for which such an n and Φ_β exist. Since $\alpha \in A$, it remains to show that A satisfies Lemma 11.8(b). Indeed, fix $\beta \in A$ and $\gamma < \beta$. If $\beta = \eta + 1$, then by the first case we can deduce that $\eta \in A$ and $\gamma \leq \eta < \beta$. Otherwise, β is a limit ordinal. Let Φ_β and n_β witness the fact that $\beta \in A$, and by Lemma 11.9(a) we may choose $n \in L$ with $n > n_\beta$ such that $\mu(\beta, n) > \gamma + 1$. If η is the predecessor of $\mu(\beta, n)$, then by the second case we deduce that $\eta \in A$ and $\gamma \leq \eta < \beta$. □

Proof of Corollary 1.2
We first recall a result by Causey [12, Theorem 6.2], which says that for a countable ordinal ξ it follows that $\gamma = \omega^\xi$ is the Szlenk index of some separable Banach space X if and only if ξ is not of the form $\xi = \omega^\eta$, with η being a limit ordinal. Since $\alpha = \omega^{\omega^{\omega^\alpha}}$, α cannot be the Szlenk index of some separable Banach space.

(a)⇒(b). From (a) and Causey's result we have $\mathrm{Sz}(X) < \omega^\alpha$ and $\mathrm{Sz}(X^*) < \omega^\alpha$, and thus, there exists a $\xi < \omega_1$ with $\beta = \omega^{\omega^\xi} \leq \omega^{\omega^{\xi+1}} < \alpha$ such that $\mathrm{Sz}(X) < \omega^\beta$ and $\mathrm{Sz}(X) < \beta$. Thus, it follows from Theorem 11.6 that for no $L \in [\mathbb{N}]^\omega$ are there numbers $0 < c < C$ and a map $\Phi : \mathcal{S}_{\beta^2} \cap [L]^{<\omega} \to X$ that is a c-lower-d_{∞,β^2} and C-upper-d_{1,β^2} embedding. Since $\beta^2 \leq \omega^{\omega^{\xi+1}} < \alpha$ Proposition 11.7 yields our claim.

(b)⇒(a). This follows from Theorems 8.1 and 8.3. □

To prove Corollary 1.3 recall that every separable Banach space is isometrically equivalent to a subspace of $C[0,1]$, the space of continuous functions on $[0,1]$. The set $\mathcal{SB}$ of all closed subspaces of $C[0,1]$ is given the *Effros–Borel structure*, which is the σ-algebra generated by the sets $\{F \in \mathcal{SB} : F \cap U \neq \emptyset\}$, where U ranges over all open subsets of $C[0,1]$.

Proof of Corollary 1.3
By [26, Theorem D] the set

$$\mathcal{C}_\alpha = \big\{X \in \mathcal{SB} : X \text{ reflexive and } \max\big(\mathrm{Sz}(X), \mathrm{Sz}(X^*)\big) \le \alpha\big\}$$

is analytic. So, by Souslin's separation theorem (cf. [18, Theorem 14.11]) it is left to show that its complement is also analytic. Since by Corollary 1.2

$$\begin{aligned}\mathcal{SB} \setminus \mathcal{C}_\alpha = \{X \in \mathcal{SB} : X \text{ not reflexive}\} \\ \cup \big\{X \in \mathcal{SB} : X \text{reflexive and} \max\big(\mathrm{Sz}(X), \mathrm{Sz}(X^*)\big) > \alpha\big\} \\ = \{X \in \mathcal{SB} : X \text{ not reflexive}\} \\ \cup \big\{X \in \mathcal{SB} : (\mathcal{S}_\alpha, d_{1,\alpha}) \text{ bi-Lipschitzly embeds into } X\big\}\end{aligned}$$

and since by [9, Corollary 3.3] the set of reflexive spaces in $\mathcal{SB}$ is coanalytic, we deduce our claim from the well-known and easy-to-show observation that the set of $X \in \mathcal{SB}$ in which a fixed (M, d) separable metric space embeds is analytic. ☐

12. Final comments and open questions

The proof of Theorem A yields the following equivalences. The statement that (a)⇒(b) follows from Proposition 7.2, (d)⇒(a) follows from Theorem 11.6, and (b)⇒(c)⇒(d) is trivial.

COROLLARY 12.1
For a separable Banach space X the following statements are equivalent.
(a) *X is not reflexive.*
(b) *For all $\alpha < \omega_1$ there exists for some numbers $0 < c < C$ a c-lower $d_{\infty,\alpha}$, C-upper $d_{1,\alpha}$ embedding of $\mathcal{S}_\alpha$ into X.*
(c) *For all $\alpha < \omega_1$ there exist a map $\Psi_\alpha : \mathcal{S}_\alpha \to X$ and some $0 < c \le 1$ such that, for all $A, B, C \in \mathcal{S}_\alpha$ with the property that $A \succeq C$, $B \succeq C$, and $A \setminus C < B \setminus C$,*

$$c d_{1,\alpha}(A, B) \le \big\|\Psi(A) - \Psi(B)\big\| \le d_{1,\alpha}(A, B).$$

(d) *For all $\alpha < \omega_1$, there exist an $L \in [\mathbb{N}]^{<\omega}$ and a semiembedding $\Psi_\alpha : \mathcal{S}_\alpha \cap [L]^{<\omega} \to X$.*

As mentioned before we can consider for $\alpha < \omega_1$ and $A \in \mathcal{S}_\alpha$ the vector z_A to be an element in ℓ_1^+, with $\|x\|_{\ell_1} \le 1$. We define

$$T_\alpha = \big\{(A, B) \in \mathcal{S}_\alpha \times \mathcal{S}_\alpha : \exists C \preceq A \text{ and } C \preceq A, \text{ with } A \setminus C < B \setminus C\big\}.$$

We note that $d_{1,\alpha}(A, B) = \|z_A - z_B\|_1$ for $(A, B) \in T_\alpha$. Using this notation we deduce the following sharpening of [29, Theorem 3.1] from Corollary 12.1.

COROLLARY 12.2
Let X be a separable Banach space. Then the following are equivalent.
(a) *X is not reflexive.*
(b) *For all $\alpha < \omega_1$ there exist a map $\Psi_\alpha : \mathcal{S}_\alpha \to X$ and some $0 < c \le 1$ such that*

$$c d_{1,\alpha}(A, B) \le \left\| \Psi_\alpha(A) - \Psi_\alpha(B) \right\| \le d_{1,\alpha}(A, B) \quad \text{whenever } (A, B) \in \mathcal{T}_\alpha.$$

We note that James's space J is a nonreflexive space for which it is not hard to see that $\mathrm{Sz}(J) = \mathrm{Sz}(J^*) = \omega$. Theorem C and Corollary 1.2 are therefore not true if we omit the requirement that X is reflexive. Nevertheless, the following variation of Corollary 1.2 holds.

COROLLARY 12.3
Assume that $\alpha < \omega_1$ is an ordinal for which $\alpha = \omega^\alpha$. Then the following statements are equivalent for a separable Banach space X.
(a) *X is reflexive and* $\max(\mathrm{Sz}(X), \mathrm{Sz}(X^*)) \le \alpha$.
(b) *There is no map $\Psi : \mathcal{S}_\alpha \to X$ with $0 < c \le 1$ such that, for all $A, B, C \in \mathcal{S}_\alpha$ with the property that $C \preceq A$, $C \preceq B$, and $A \setminus C < B \setminus C$, we have*

$$c d_{1,\alpha}(A, B) \le \left\| \Psi(A) - \Psi(B) \right\| \le d_{1,\alpha}(A, B).$$

Proof
Let $\Psi : \mathcal{S}_\alpha \to X$ satisfy the condition stated in (b) for some $c > 0$. Then $\tilde{\Psi} = (\Psi - \Psi(\emptyset))/2$ also has this property for $c/2$ and maps $\mathcal{S}_\alpha$ into B_X with $\tilde{\Psi}(\emptyset) = 0$.

(a)⇒(b). This follows from Theorem 11.6, [12, Theorem 6.2], and the same argument involving Proposition 11.7 in the proof of Corollary 1.2.

(b)⇒ (a). This follows from Proposition 7.2 and Theorems 8.1 and 8.3. □

Remark
The statement of Corollary 12.3 also holds for $\alpha = \omega$. This can be seen from the proof of [7, Main Result].

We finish by stating three open problems.

PROBLEMS 12.4
(a) *Does there exists a family of metric spaces (M_i, d_i) which is a family of test spaces for reflexivity in the sense of [27], that is, for which it is true that a separable Banach space X is reflexive if and only if not all of the M_i's uniformly bi-Lipschitzly embed into X?*
(b) *Does there exist a countable family of metric spaces (M_i, d_i) which is a family of test spaces for reflexivity?*

(c) *It follows from Theorem B that if X is a separable Banach space with non-separable bidual, then $(\mathcal{S}_\alpha, d_{1,\alpha})$ bi-Lipschitzly embeds into X for all $\alpha < \omega_1$. Is the converse true, or in Ostrovskii's language, are the spaces $(\mathcal{S}_\alpha, d_{1,\alpha})$, $\alpha < \omega_1$, test spaces for spaces with separable biduals?*

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