A Hierarchy of Banach Spaces with $C(K)$ Calkin Algebras

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ABSTRACT. For every well-founded tree $\mathcal{T}$ having a unique root such that every non-maximal node of it has countable infinitely many immediate successors, we construct a $\mathcal{L}_\infty$-space $X_{\mathcal{T}}$. We prove that, for each such tree $\mathcal{T}$, the Calkin algebra of $X_{\mathcal{T}}$ is homomorphic to $C(\mathcal{T})$, the algebra of continuous functions defined on $\mathcal{T}$, equipped with the usual topology. We use this fact to conclude that, for every countable compact metric space K, there exists a $\mathcal{L}_\infty$-space whose Calkin algebra is isomorphic, as a Banach algebra, to $C(K)$.

INTRODUCTION

The Calkin algebra of a Banach space X is defined to be the quotient algebra $Cal(X) = \mathcal{L}(X)/\mathcal{K}(X)$, where $\mathcal{L}(X)$ denotes the algebra of all bounded linear operators defined on X, and $\mathcal{K}(X)$ denotes the ideal of all the compact ones. It is named after J. W. Calkin, who proved in [3] that the only non-trivial closed ideal of the bounded linear operators on ℓ_2 is the one of the compact operators. It is an important example of a unital algebra: for example, an operator $U \in \mathcal{L}(X)$ is Fredholm if and only if the class $[U]$ of U in the Calkin algebra is invertible. A question that arises is the following: given A a unital Banach algebra, does there then exist a Banach space X so that the Calkin algebra of X is isomorphic, as a Banach algebra, to A?

The very first Banach space for which the Calkin algebra was explicitly described is the Argyros-Haydon space X_{AH} [1] whose main feature is that it has the "scalar plus compact" property; that is, the ideal of the compact operators is of co-dimension one in the algebra of all bounded operators, and hence its Calkin algebra is one-dimensional. The aforementioned construction is based on a combination of the methods used by W. T. Gowers and B. Maurey in [6] and by

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J. Bourgain and F. Delbaen in [2]; hence, the resulting space X_{AH} is hereditarily indecomposable (HI) as well as a $\mathcal{L}_\infty$-space. It is worth mentioning that for every natural number k, by carefully taking $X_1, \dots, X_k$ versions of the Argyros-Haydon space, the Calkin algebra of the direct sum of these spaces is k-dimensional. In 2013, M. Tarbard [11], combining the techniques from [1] and [5], provided an example of a Banach space $\mathfrak{X}_\infty$, such that its Calkin algebra is isometric, as a Banach algebra, to the convolution algebra $\ell_1(\mathbb{N}_0)$.

Since the space ℓ_1 occurs as a Calkin algebra, one may ask whether the same is true for c. To make the question more precise, does there exist a Banach space such that its Calkin algebra is isomorphic, as a Banach algebra, to $C(\omega)$? Or more generally, one may ask for what topological spaces K the algebra $C(K)$ is isomorphic to the Calkin algebra of some Banach space.

The above question is the one under consideration in the present paper. In particular we prove the following result.

Theorem A. *Let $\mathcal{T}$ be a well-founded tree with a unique root, such that every non-maximal node of $\mathcal{T}$ has countable infinitely many immediate successors. Then, there exists a $\mathcal{L}_\infty$-space $X_{\mathcal{T}}$ with the following properties:*

(i) *The dual of $X_{\mathcal{T}}$ is isomorphic to ℓ_1.*
(ii) *There exists a family of norm-one projections $(I_s)_{s\in\mathcal{T}}$ such that every operator defined on the space is approximated by a sequence of operators, each one of which is a linear combination of these projections plus a compact operator.*
(iii) *There also exists a bounded, one-to-one, and onto algebra homomorphism $\Phi : Cal(X_{\mathcal{T}}) \to C(\mathcal{T})$, where $C(\mathcal{T})$ denotes the algebra of all continuous functions defined on the compact topological space $\mathcal{T}$. In other words, the Calkin Algebra of $X_{\mathcal{T}}$ is isomorphic, as a Banach algebra, to $C(\mathcal{T})$.*

As an application, we obtain the following theorem.

Theorem B. *For every countable compact metric space K, there exists a $\mathcal{L}_\infty$-space X, with X^* isomorphic to ℓ_1, so that its Calkin algebra is isomorphic, as a Banach algebra, to $C(K)$.*

The construction of the spaces is done recursively on the order of the tree $\mathcal{T}$. On the basic recursive step (i.e., in the case where $\mathcal{T}$ is a singleton), the space $X_{\mathcal{T}}$ is in fact the Argyros-Haydon space. In the general case, the space $X_{\mathcal{T}}$ is the direct sum $(\sum \oplus X_{\mathcal{T}_n})_{\mathrm{AH}}$, where the trees $\mathcal{T}_n$ have order smaller than the one of $\mathcal{T}$, and the outside norm is the Argyros-Haydon one, as it was defined by the third named author in [12]. Given the space X_{AH} from [1] and the definition of the direct sum $(\sum \oplus X_n)_{\mathrm{AH}}$ of a sequence of Banach spaces from [12], the definition of the spaces $X_{\mathcal{T}}$ can be formulated quite easily; however, the proofs involve many details from these constructions, and are in some cases quite technical.

The paper is divided into five sections. The preliminary section is the most lengthy, and mainly discusses the tools from [1] and [12] that are essential to obtain the result in this paper. The basic properties of constructions from these papers are presented, while we also make some remarks not mentioned there. Moreover, some basic facts about trees are included. Section 2 is devoted to the definition of the spaces $X_{\mathcal{T}}$ and the proof of some of their essential properties. Section 3 focuses on the study of the operators defined on these spaces, and concludes with the fact that, for every such space $X_{\mathcal{T}}$, there exists a set of bounded norm-one projections $\{I_s \mid s \in \mathcal{T}\}$ so that every operator on $X_{\mathcal{T}}$ is approximated by operators, each one of which is a linear combination of these projections plus a compact operator. In the fourth section, using the aforementioned result, we prove that the Calkin algebra of $X_{\mathcal{T}}$ is homomorphic to $C(\mathcal{T})$, the algebra of continuous functions on $\mathcal{T}$. In the fifth, and final, section we use the properties of our construction to deduce the main result: that is, that for every countable compact metric space K, there exists a Banach space whose Calkin algebra is isomorphic, as a Banach algebra, to $C(K)$.

1. Preliminaries

We begin with a preliminaries section that includes many facts and estimations concerning Bourgain-Delbaen spaces, the Argyros-Haydon space, as well as the Argyros-Haydon sum of a sequence of Banach spaces as it was defined in [12]. The section is unavoidably technical and extensive, as our methods rely on details concerning the just-mentioned constructions. We also mention some basics concerning trees.

We start with the definition of a $\mathcal{L}_\infty$-space, which first appeared in [8]. We recall that, for Banach spaces Z, W and a constant $C > 0$, we say that Z is C-isomorphic to W, denoted as $Z \simeq^C W$, if there exists an onto linear isomorphism $T : Z \to W$ such that $\|T\|\,\|T^{-1}\| \leq C$.

Definition 1.1. We say that a separable Banach space X is a $\mathcal{L}_{\infty,C}$-space where $C > 0$ is a constant, if there exists a strictly increasing sequence $(Y_n)_{n\in\mathbb{N}}$ of subspaces of X such that $Y_n \simeq^C \ell_\infty(\dim Y_n)$ for every $n \in \mathbb{N}$ and $X = \overline{\bigcup_{n\in\mathbb{N}} Y_n}$.

Using the Bourgain-Delbaen method of constructing $\mathcal{L}_\infty$-spaces X [2], we can specify the constant $C > 0$ as well as the sequence $(Y_n)_{n\in\mathbb{N}}$ that correspond to X in Definition 1.1. We say that a space X is a BD-$\mathcal{L}_\infty$-space if it is constructed via the BD-method. Two fixed parameters $0 < a < 1$, $0 < b < \frac{1}{2}$ with $a + b > 1$ are used, and there exist the following:

(i) A sequence $(\Delta_n)_{n\in\mathbb{N}}$ of pairwise disjoint subsets of $\mathbb{N}$, where we denote their union by Γ, and set $\Gamma_n = \bigcup_{i=1}^n \Delta_i$ for each $n \in \mathbb{N}$;

(ii) Linear extension operators $i_n : \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma)$ with $\|i_n\|\,\|i_n^{-1}\| \leq 1/(1-2b)$ for every $n \in \mathbb{N}$.

The above assignations are such that $X = \overline{\bigcup_{n\in\mathbb{N}} Y_n}$, where $Y_n = i_n(\ell_\infty(\Gamma_n))$ for every $n \in \mathbb{N}$. In particular, the BD-space X is a $\mathcal{L}_{\infty,C}$-space where $C = 1/(1-2b)$.

1.1. The space $X_{\mathbf{AH}}$.

We denote by $\mathfrak{X}_{\mathrm{AH}}$ the Argyros Haydon space constructed in [1]. The space X_{AH} is a separable HI $\mathcal{L}_{\infty,2}$-space such that $\mathfrak{X}_{\mathrm{AH}}^* \simeq^2 \ell_1$. The construction is based on two fixed strictly increasing sequences of natural numbers $(m_j, n_j)_{j\in\mathbb{N}}$ (with $m_1 \geq 4$), and it is a generalization of the BD method for parameters $a = 1$, using instead of b the sequence $(1/m_j)_{j\in\mathbb{N}}$. In particular, there exists a sequence $(\Delta_n)_{n\in\mathbb{N}}$ and linear operators i_n as above having the following properties:

(i) To each element $\gamma \in \Gamma$ are assigned the rank of γ, $\mathrm{rank}(\gamma)$, the weight of γ, $w(\gamma)$, and the age of γ, $a(\gamma)$, so that
   (a) $\mathrm{rank}(\gamma) = n$ whenever $\gamma \in \Delta_n$;
   (b) $w(\gamma) = m_j$ for some $j \le n$ whenever $\mathrm{rank}(\gamma) = n$;
   (c) $a(\gamma) = a \le n_j$.

(ii) $\|i_n\|\,\|i_n^{-1}\| \le 2$.

Moreover, the space $\mathfrak{X}_{\mathrm{AH}}$ admits an FDD $(M_n)_{n\in\mathbb{N}}$ where $M_n = i_n(\ell_\infty(\Delta_n))$ is isometric to $\ell_\infty(\Delta_n)$. Using the FDD, we define the range of an element x in $\mathfrak{X}_K$ as the minimum interval I of $\mathbb{N}$ such that $x \in \sum_{n\in I} \oplus M_n$. A bounded sequence $(x_k)_k$ in $\mathfrak{X}_K$ is called a block sequence if $\max \operatorname{ran} x_k < \min \operatorname{ran} x_{k+1}$ for every k.

1.2. AH-$\mathcal{L}_\infty$ sums of separable Banach spaces. In this subsection, we recall the basic components from [12], namely, constructing sums $(\sum_n \oplus X_n)_{\mathrm{AH}}$ where $(X_n)_n$ is a sequence of separable Banach spaces. We start with the following notation.

Notation 1.2. Let $(E_n, \|\cdot\|_{E_n})_{n=1}^\infty$ be sequences of separable Banach spaces. By $(\sum_n \oplus E_n)_\infty$, we denote the space of vector elements $\vec{x} = (x_n)_{n=1}^\infty$ such that the n-th coordinate of $\vec{x}$ is in E_n for all $n \in \mathbb{N}$, and $\|\vec{x}\|_\infty = \sup \|x_n\|_{E_n}$ is finite. By $c_{00}(\sum_n \oplus E_n)$, we denote the subspace of $(\sum_n \oplus E_n)_\infty$ consisting of all $\vec{x} = (x_n)_{n=1}^\infty$ for which there exists $n_0 \in \mathbb{N}$ with the property that $x_n = 0$ for every $n \ge n_0$. For a vector $\vec{x} = (x_n)_{n=1}^\infty \in c_{00}(\sum_n \oplus E_n)$, we define the support of x as

$$\operatorname{supp} x = \{n \in \mathbb{N} \mid x_n \neq 0\}.$$

For every finite interval $J \subset \mathbb{N}$, we denote by R_J the natural restriction map

$$R_J : \Big(\sum_n \oplus E_n\Big)_\infty \to \Big(\sum_{n\in J} \oplus E_n\Big)_\infty$$

defined as $R_J(\vec{x}) = (x_n)_{n\in J}$ for every $\vec{x} = (x_n)_{n=1}^\infty \in c_{00}(\sum_n \oplus E_n)$. For I, J subsets of $\mathbb{N}$ we say that I, J are successive (denoted $I < J$), if $\max I < \min J$. For $\vec{x}, \vec{y}, \vec{z}$ vector elements of $c_{00}(\sum_n \oplus E_n)$ such that $\operatorname{supp} \vec{x} < \operatorname{supp} \vec{y} < \operatorname{supp} \vec{z}$, we denote by $(\vec{x}, \vec{y}, \vec{z})$ the vector $\vec{x} + \vec{y} + \vec{z} \in c_{00}(\sum_n \oplus E_n)$.

By $(\sum_{n=1}^\infty \oplus E_n)_1$, we denote the space of vector elements $\vec{x} = (x_n)_{n=1}^\infty$ such that the n-th coordinate of $\vec{x}$ is in E_n and $\|\vec{x}\|_1 = \sum_n \|x_n\|_{E_n}$ is finite. If the elements of $c_{00}(\sum_n \oplus E_n)$ are considered to be functionals (i.e., E_n are dual spaces), we use letters as $\vec{f}, \vec{g}, \vec{h}$, and so on for their representation. For $\vec{x} \in (\sum_n \oplus E_n)_\infty$ and $\vec{f} \in c_{00}(\sum_n \oplus E_n^*)$, we denote by $\vec{f}(\vec{x})$ the inner product $\sum_n f_n(x_n)$.

We continue with the definition of spaces $(\sum_{n=1}^\infty \oplus X_n)_{\mathrm{BD}}$ for a sequence $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ of separable Banach spaces.

Definition 1.3. Let $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of separable Banach spaces. A Banach space $\mathcal{Z}$ is called a Bourgain Delbaen(BD)-$\mathcal{L}_{\infty,C}$-sum of the sequence $(X_n, \|\cdot\|_n)_n$, denoted as $\mathcal{Z} = (\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{BD}}$, if there exists a sequence $(\Delta_n)_{n\in\mathbb{N}}$ of finite, pairwise disjoint subsets of $\mathbb{N}$, and the following hold:

(i) The space $\mathcal{Z}$ is a subspace of $(\sum_{n=1}^{\infty} \oplus (X_n \oplus \ell_\infty(\Delta_n))_\infty)_\infty = \mathcal{Z}_\infty$.

(ii) For every n, there exists a linear extension operator

$$i_n : \Big(\sum_{k=1}^{n} \oplus (X_k \oplus \ell^\infty(\Delta_k))_\infty \Big)_\infty \to \Big(\sum_{n=1}^{\infty} \oplus (X_n \oplus \ell_\infty(\Delta_n))_\infty \Big)_\infty$$

such that the following hold:

(a) $\|i_n\| \leq C$ for every $n \in \mathbb{N}$;

(b) Each $\vec{x} \in (\sum_{k=1}^{n} \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty)_\infty$ satisfies the following:

(ι) $R_{[1,n]}(i_n(\vec{x})) = \vec{x}$ while $R_{(n,\infty)}(i_n(\vec{x}))$ is an element of

$$\Big(\sum_{k=n+1}^{\infty} \oplus (\{0\} \oplus \ell_\infty(\Delta_k))_\infty \Big)_\infty;$$

($\iota\iota$) $i_l(R_{[1,l]} i_n(\vec{x})) = i_n(\vec{x})$ for every $l \geq n+1$.

(iii) Setting $Y_n = i_n[(\sum_{k=1}^{n} \oplus (X_k \oplus \ell^\infty(\Delta_k))_\infty)_\infty]$, the union $\bigcup_n Y_n$ is dense in $\mathcal{Z}$.

Note that a space $\mathcal{Z} = (\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{BD}}$ can be obtained by modifying the original Bourgain-Delbaen $\mathcal{L}_\infty$ method of construction in [2] as was described in [12]. In particular, applying this modification to the BD-method of Argyros-Haydon, we can construct BD-$\mathcal{L}_{\infty,2}$ sums $(\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{BD}}$ of separable Banach spaces X_n, denoted as $(\sum \oplus X_n)_{\mathrm{AH}}$. The next result is proved in ([12]).

Proposition 1.4. *Let $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of separable Banach spaces. The space $\mathcal{Z} = (\sum \oplus X_n)_{\mathrm{AH}}$ has the following properties:*

(i) *$\mathcal{Z} = (\sum \oplus X_n)_{\mathrm{AH}}$ admits a shrinking Schauder Decomposition $(Z_n)_{n\in\mathbb{N}}$ such that each $Z_n = i_n[(X_n \oplus \ell_\infty(\Delta_n))_\infty]$.*

(ii) *Every horizontally block subspace of $\mathcal{Z}$ is* HI.

(iii) *The dual $\mathcal{Z}^*$ is 2-isomorphic to $(\sum_{n=1}^{\infty} \oplus (X_n^* \oplus \ell_1(\Delta_n))_1)_1$.*

Definition 1.5. For an element z of $\mathcal{Z} = (\sum \oplus X_n)_{\mathrm{AH}}$, we define the range of z, denoted by $\operatorname{ran} z$, as the minimum interval $I \subset \mathbb{N}$ such that $z \in \sum_{n\in I} \oplus Z_n$.

Because $\mathcal{Z}$ is a subspace of $\mathcal{Z}_\infty$, we can consider the restriction mappings $R_{[1,n]} : \mathcal{Z} \to (\sum_{k=1}^{n} \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty)_\infty$. We denote by $P_{[1,n]} : \mathcal{Z} \to \mathcal{Z}$ the projections associated with the Schauder Decomposition $(Z_n)_{n\in\mathbb{N}}$ that are defined as $P_{[1,n]} = i_n \circ R_{[1,n]}$. We write P_n instead of $P_{\{n\}}$, and for every $k \leq m$, we define $P_{(k,m]} = \sum_{i=k+1}^{m} P_i = i_m \circ R_{[1,m]} - i_k \circ R_{[1,k]}$.

We also identify $(\sum_{n=k}^{m} \oplus (X_n \oplus \{0\})_\infty)_\infty$ with $(\sum_{n=k}^{m} \oplus X_n)_\infty$ and similarly $(\sum_{n=k}^{m} \oplus (\{0\} \oplus \ell_\infty(\Delta_n))_\infty)_\infty$ with $(\sum_{n=k}^{m} \oplus \ell_\infty(\Delta_n))_\infty$. For $\gamma \in \Delta_n$ we denote by e_γ^* the usual vector element of $\ell_1(\Delta_n)$. We shall also extend each element

$b^* = \sum_\gamma a_\gamma e^*_\gamma \in (\sum_{n=k}^m \oplus \ell_1(\Delta_n))_1$ to a functional $\vec{b}^* : \mathcal{Z} \to \mathbb{R}$ defined as $\vec{b}^* = b^* \circ R_{[1,n]}$. Similarly, we extend every $\vec{f} \in (\sum_{n=k}^m \oplus X^*_n)_1$ to a functional $\vec{f} : \mathcal{Z} \to \mathbb{R}$.

The construction of $\mathcal{Z} = (\sum_{n=1}^\infty \oplus X_n)_{\mathrm{AH}}$ is based on the same parameters $(m_j, n_j)_{j=1}^\infty$ of X_{AH}. More precisely, for every $n \in \mathbb{N}$, each set Δ_{n+1} is the union of pairwise disjoint sets $\Delta_{n+1} = \Delta^0_{n+1} \cup \Delta^1_{n+1}$ satisfying the following:

(i) For every $\gamma \in \Delta^0_{n+1}$, there is $\vec{f} \in (\sum_{k=1}^n \oplus X^*_k)_1$ with $\|\vec{f}\| \leq 1$ so that

$$\vec{e}^*_\gamma = \vec{e}^*_\gamma \circ P_{n+1} + \frac{1}{m_1}\vec{f}. \tag{1.1}$$

(ii) The set Δ^1_{n+1} is defined similarly as the set Δ_n of X_{AH}, and consists of elements with rank, weight, and age depending on $(m_j, n_j)_{j=1}^\infty$. Moreover, as was stated in [12], for every $\gamma \in \Delta^1_{n+1}$ with weight $w(\gamma) = m_j$, there exists a family $\{p_i, q_i, \xi_i, \vec{b}^*_i\}_{i=1}^a$, called the analysis of γ, such that the following hold:
   (a) $0 \leq p_1 < q_1 < p_2 < q_2 < \cdots < p_a < q_a = n$.
   (b) $\xi_i \in \Delta^1_{q_i+1}$ with $w(\xi_i) = m_j$ and $b^*_i \in (\sum_{k=p_i}^{q_i} \oplus \ell_1(\Delta_k))_1$.
   (c) $\vec{e}^*_\gamma = \sum_{i=1}^a \vec{e}^*_{\xi_i} \circ P_{\{q_i+1\}} + (1/m_j) \sum_{i=1}^a \vec{b}^*_i \circ P_{(p_i,q_i]}$.

Remark 1.6. In [12], (1.1) is actually slightly different. More precisely, in that case, for every $n \in \mathbb{N}$ and $\gamma \in \Delta^0_{n+1}$, we have that $\vec{e}^*_\gamma = \vec{e}^*_\gamma \circ P_{n+1} + \vec{f}$; that is, the constant $1/m_1$ is missing. All results from [12] also hold with this modification, the only difference being that in various cases some constants have to be adjusted. This modification is used only to prove Proposition 1.17, which is essential to obtaining the main result of this paper.

For the rest of this subsection, we fix a sequence of separable Banach spaces $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$, and we let $\mathcal{Z} = (\sum_{n=1}^\infty \oplus X_n)_{\mathrm{AH}}$ with Schauder Decomposition $(Z_n)_{n\in\mathbb{N}}$ be as stated in Proposition 1.4.

Lemma 1.7. *Let $n, q \in \mathbb{N}$ with $q > n$ and $\gamma \in \Delta^1_q$ with $w(\gamma) = m_j$ for some $j \in \mathbb{N}$. Consider the functional $g : \mathcal{Z} \to \mathbb{R}$ with $g = \vec{e}^*_\gamma \circ P_{[1,n]}$. Then, one of the following holds:*

(i) $g = 0$.
(ii) *There are $p_1 \leq n$ and $\vec{b}^* \in (\sum_{k=1}^n \oplus \ell_1(\Delta_k))_1$ with $\|\vec{b}^*\| \leq 1$ so that* $g = (1/m_j)\vec{b}^* \circ P_{(p_1,n]}$.
(iii) *There are $p_0 < p_1 \leq n$ and $\gamma' \in \Delta_{p_0}$ and $\vec{b}^*$ as before so that we have* $g = \vec{e}^*_{\gamma'} + (1/m_j)\vec{b}^* \circ P_{[p,n]}$.

Proof. Let $\vec{e}^*_\gamma = \sum_{i=1}^a \vec{e}^*_{\xi_i} \circ P_{\{q_i+1\}} + (1/m_j) \sum_{i=1}^a \vec{b}^*_i \circ P_{(p_i,q_i]}$ as in (c) above. Note that, as in [1, Proposition 4.5], we have that for every $1 \leq i_0 < a$ the following holds:

$$\vec{e}^*_\gamma = \vec{e}^*_{\xi_{i_0}} + \sum_{i=i_0+1}^{a} \vec{e}^*_{\xi_i} \circ P_{\{q_i+1\}} + \frac{1}{m_j} \sum_{i=i_0+1}^{a} \vec{b}^*_i \circ P_{(p_i,q_i]}. \tag{1.2}$$

Let us first assume that $n < q_1 + 1$. If $n \le p_1$, then we easily conclude that $\vec{e}^*_\gamma \circ P_{[1,n]} = 0$, and the first assertion holds. Otherwise, we have $p_1 < n$ and $\vec{e}^*_\gamma \circ P_{[1,n]} = (1/m_j)\vec{b}^*_i \circ P_{(p_1,n]}$; that is, the second assertion holds.

Let us now assume that $q_1 + 1 \le n$, and set $i_0 = \max\{i \mid q_i + 1 \le n\}$. Since $q > a$, by property (a) it follows that $i_0 < a$, and so, using (1.2), we obtain

$$\vec{e}^*_\gamma \circ P_{[1,n]} = \vec{e}^*_{\xi_{i_0}} \circ P_{[0,n]} + \frac{1}{m_j}\vec{b}^*_{i_0+1} \circ P_{(p_{i_0+1},n]}$$

(where if $p_{i_0+1} > n$, then the last part of the right-hand side in the above inequality is zero). Since $\xi_{i_0} \in \Delta^1_{q_{i_0}+1}$ and $i_0 + 1 \le n$, we conclude that $\vec{e}^*_{\xi_{i_0}} \circ P_{[0,n]} = \vec{e}^*_{\xi_{i_0}}$, and hence $g = \vec{e}^*_{\xi_{i_0}} + (1/m_j)\vec{b}^*_{i_0+1} \circ P_{(p_{i_0+1},n]}$. ◻

Lemma 1.8. *The space Z_n is isometric to $(X_n \oplus \ell_\infty(\Delta_n))_\infty$ for every $n \in \mathbb{N}$. More precisely, the operator i_n restricted onto $(X_n \oplus \ell_\infty(\Delta_n))_\infty$, viewed as a subspace of $(\sum_{k=1}^n \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty)_\infty$, is an isometry.*

Proof. Let $z = i_n(\vec{x}) \in Z_n$ where $\vec{x} \in (X_n \oplus \ell_\infty(\Delta_n))_\infty$. We will prove that $\|z\| = \|x\|_\infty$. Observe that, by Definition 1.3, $R_{(n,\infty)}(z) \in (\sum_{k>n} \oplus \ell_\infty(\Delta_n))_\infty$. Let $k > n$ and $\gamma \in \Delta^1_k$, with $w(\gamma) = m_j$, and analyze $\{p_i, q_i, \xi_i, \vec{b}^*_i\}_{i=1}^a$. Note that there exists at most one $1 \le i \le a$ such that either $n = q_i + 1$ or $n \in (p_i, q_i]$. In the first case, we have that $\vec{e}^*_\gamma(z) = \vec{e}^*_{\xi_i}(P_{\{q_i+1\}}z) = e^*_{\xi_i}(x) \le \|x\|_\infty$, and similarly in the second case $\vec{e}^*_\gamma(z) = (1/m_j)\vec{b}^*_i(P_{(p_i,q_i]}z) = (1/m_j)b^*_i(x) \le \|x\|_\infty$. Hence, $\|R_{(n,\infty)}z\| \le \|x\|_\infty$, and since $\|x\|_\infty = \|R_{[1,n]}z\|$, it follows that $\|z\| = \|x\|_\infty$ as promised. ◻

If by $j_n : X_n \to (\sum_{k=1}^n \oplus (X_n \oplus \ell_\infty(\Delta_k))_\infty)_\infty$, we denote the natural embedding, we immediately obtain the following result.

Corollary 1.9. *The map $i_n \circ j_n : X_n \to \mathcal{Z}$ is an isometric embedding, and hence the space $i_n \circ j_n[X_n]$ is isometric to X_n for every $n \in \mathbb{N}$.*

Let $\pi_n : (\sum_{k=1}^n \oplus (X_n \oplus \ell_\infty(\Delta_n))_\infty)_\infty \to X_n$ denote the natural restriction onto the coordinate X_n.

Lemma 1.10. *For every $n \in \mathbb{N}$, the map $I_n : \mathcal{Z} \to \mathcal{Z}$ with $I_n = i_n \circ j_n \circ \pi_n \circ R_{[1,n]}$ is a norm-one projection onto $i_n \circ j_n[X_n]$ with*

$$\ker I_n = i_n[\ell_\infty(\Delta_n)] \oplus \sum_{k \ne n} \oplus Z_k,$$

where $\ell_\infty(\Delta_n)$ is viewed as a subspace of $(\sum_{k=1}^n \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty)_\infty$ in the canonical way.

Proof. The fact that I_n is a projection, with the image and kernel mentioned above, follows from its definition. To see that $\|I_n\| = 1$, let $z \in \mathcal{Z}$, and let $\vec{x} = R_{[1,n]}(z)$. Note that $I_n(z) = i_n \circ j_n \circ \pi_n(\vec{x})$, and by Corollary 1.9, we have that $\|I_n(z)\| \leq \|j_n \circ \pi_n(\vec{x})\| \leq \|\vec{x}\| \leq \|z\|$, and hence $\|I_n\| = 1$. □

Remark 1.11. Note that, for every $n \in \mathbb{N}$, the difference $P_n - I_n$ is a finite rank operator. More precisely, $(P_n - I_n)(\mathcal{Z}) = i_n[\ell_\infty(\Delta_n)]$ (where $\ell_\infty(\Delta_n)$ is viewed as a subspace of $(\sum_{k=1}^n \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty)_\infty$ in the canonical way).

Remark 1.12. For $n \in \mathbb{N}$ and a bounded linear $T_n : X_n \to X_n$, we may identify T_n with the map $\tilde{T}_n = i_n \circ j_n \circ T_n \circ \pi_n \circ R_{[1,n]}$ that is defined on $\mathcal{Z}$. It follows that $\|T_n\| = \|\tilde{T}_n\|$. Also, for $n \in \mathbb{N}$ and bounded linear operators $T_k : X_k \to X_k$, $k = 1, \dots, n$, we may define the operator $\sum_{k=1}^n \oplus T_k = \sum_{k=1}^n \tilde{T}_k$. Proposition 1.17 below provides a relation of the norm of $\sum_{k=1}^n \oplus T_k$, in the Calkin algebra of $\mathcal{Z}$, to the norms of the T_k, $k = 1, \dots, n$.

1.3. AH(L)-$\mathcal{L}_\infty$ sums of separable Banach spaces for L an infinite subset of the natural numbers. The construction of $\mathfrak{X}_{\mathrm{AH}}$ is based on a sequence of parameters $(m_j, n_j)_{j\in\mathbb{N}}$ satisfying certain lacunarity conditions that are preserved under taking a subsequence $(m_j, n_j)_{j\in L}$, where L is an infinite subset of $\mathbb{N}$. Hence, for every such L, the space $\mathfrak{X}_{\mathrm{AH}(L)}$ can been defined by using as parameters the sequence $(m_j, n_j)_{j\in L}$.

In a similar manner, such a sequence $(m_j, n_j)_j$ is used for constructing the Argyros-Haydon sum of a sequence of separable Banach spaces $(X_n)_n$. By using an infinite subset of the natural numbers L and as parameters the sequence $(m_j, n_j)_{j\in L}$, we define the space $(\sum_{n=1}^\infty \oplus X_n)_{\mathrm{AH}(L)}$. Note also that, in (1.1), the constant $1/m_1$ is now replaced with $1/m_{\min L}$.

The statements from the following remark follow from the corresponding proofs in [1, Proposition 3.2, Theorems 3.4 and 3.5, Proposition 5.11].

Remark 1.13. Let L be an infinite subset of the natural numbers. If $\varepsilon = 2/(m_{\min L} - 2)$, then the following more precise estimations are satisfied for the space $\mathfrak{X}_{\mathrm{AH}(L)}$:

(i) The extension operators i_n have norm at most $1 + \varepsilon$.
(ii) The space $\mathfrak{X}_{\mathrm{AH}(L)}$ is a $\mathcal{L}_{\infty,1+\varepsilon}$-space.
(iii) The dual of $\mathfrak{X}_{\mathrm{AH}(L)}$ is $(1 + \varepsilon)$-isomorphic to ℓ_1.

We recall the following result from [1] that is needed for the sequel.

Proposition 1.14. *Let L be an infinite subset of $\mathbb{N}$ and $(z_k)_k$ be a bounded block sequence in $X_{\mathrm{AH}(L)}$. Then, there exists an infinite subset $\tilde{L}$ of L such that, for every $j \in \tilde{L}$, there exists a subsequence $(z_{k_i})_i$ of $(z_k)_k$ satisfying*

$$\Big\| \sum_{i=1}^{n_j} z_{k_i} \Big\| \geq \frac{1}{2m_j} \sum_{i=1}^{n_j} \|z_{k_i}\|.$$

Just as above, the statements from the remark below also follow from the corresponding proofs in [12, Lemma 2.6, Propositions 3.1 and 5.1, Corollary 5.15].

Remark 1.15. Let L be an infinite subset of $\mathbb{N}$ and $(X_n)_n$ be a sequence of separable Banach spaces. If $\varepsilon = 2/(m_{\min L} - 2)$, then the following more precise estimations are satisfied for the space $(\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(L)}$:

(i) The extension operators i_n have have norm at most $1 + \varepsilon$.
(ii) The space $(\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(L)}$ is a BD-$\mathcal{L}_{\infty,1+\varepsilon}$ sum of $(X_n)_n$.
(iii) The dual of $(\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(L)}$ is $(1+\varepsilon)$-isomorphic to the space

$$\Big(\sum_{n=1}^{\infty} \oplus (X_n^* \oplus \ell_1(\Delta_n))_1\Big)_1,$$

where Δ_n are finite sets.

The next statement is proved in [12, Proposition 6.3], where the constant $1/m_{\min L}$ results from the adjustment stated in Remark 1.6.

Proposition 1.16. *Let $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of separable Banach spaces, L be an infinite subset of $\mathbb{N}$, $\mathcal{Z} = (\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(L)}$, and $(z_k)_k$ be a bounded block sequence in $\mathcal{Z}$. Then, there exists an infinite subset $\tilde{L}$ of L such that, for every $j \in \tilde{L}$, there exists a subsequence $(z_{k_i})_{i\in\mathbb{N}}$ of $(z_k)_k$ satisfying*

$$\Big\|\sum_{i=1}^{n_j} z_{k_i}\Big\| \geq \frac{1}{m_{\min L}} \cdot \frac{1}{2m_j} \sum_{k=1}^{n_j} \|z_{k_i}\|.$$

The following proposition essentially states that the norm of diagonal operators defined on $\mathcal{Z} = (\sum_n \oplus X_n)_{\mathrm{AH}(L)}$, in the Calkin algebra of $\mathcal{Z}$, is a sufficient approximation of the supremum of the norms of the operator when restricted onto the coordinates X_n. This is the only part of this paper where the modification of (1.1) stated in Remark 1.6 is needed.

Proposition 1.17. *Let $(X_n\|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of separable Banach spaces, L be an infinite subset of $\mathbb{N}$, $\mathcal{Z} = (\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(L)}$. Let also $n \in \mathbb{N}$, $T_k : X_k \to X_k$ be bounded linear operators for $k = 1, \dots, n$ and $\lambda \in \mathbb{R}$. Define $T : \mathcal{Z} \to \mathcal{Z}$ with $T = \sum_{k=1}^{n} \oplus T_k + \lambda P_{(n,+\infty)}$ (see Remark 1.12). Then, there exists a compact operator $K : \mathcal{Z} \to \mathcal{Z}$ so that*

$$\|T - K\| \leq \Big(1 + \frac{4}{m_{\min L} - 2}\Big) \max\{\max_{1\leq k\leq n} \|T_k\|, |\lambda|\}.$$

Proof. Consider the operators

$$S_n^1, S_n^2 : \Big(\sum_{k=1}^{n} \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty\Big)_\infty \to \Big(\sum_{k=1}^{n} \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty\Big)_\infty$$

such that if $x = (x_k, z_k)_{k=1}^n$, then $S_n^1 x = (x_k, 0)_{k=1}^n$ and $S_n^2 x = (0, z_k)_{k=1}^n$. Define $A_n^1 = i_n \circ S_n^1 \circ R_{[1,n]}$ and $A_n^2 = i_n \circ S_n^2 \circ R_{[1,n]}$.

Observe that A^2_n is a finite-rank operator. Also note that $P_n = A^1_n + A^2_n$, and hence $P_n - A^1_n$ is a finite-rank operator as well.

Define $K = (P_n - A^1_n) \circ \sum_{k=1}^n \oplus T_k - \lambda A^2_n$, which is a finite-rank operator. Then,

$$T - K = A^1_n \circ \sum_{k=1}^{n} \oplus T_k + \lambda(P_{(n,+\infty)} + A^2_n). \tag{1.3}$$

We shall show that

$$\|T - K\| \leq (1+\delta)\max\{\max \|T_k\|, |\lambda|\},$$

where $\delta = 4/(m_{\min L} - 2)$. Let x be an element of $\mathcal{Z}$ with $\|x\| = 1$, and consider $x = (x_k, z_k)_{k=1}^\infty$ as a vector in $\mathcal{Z}_\infty = (\sum_{k=1}^\infty (X_k \oplus \ell_\infty(\Delta_k))_\infty)_\infty$. Observe that

$$\Big(A^1_n \circ \sum_{k=1}^{n} \oplus T_k\Big)x = i_n((T_k x_k, 0)_{k=1}^n) \quad \text{and} \quad \lambda A^2_n x = i_n((0, \lambda z_k)_{k=1}^n). \tag{1.4}$$

Set $(T - K)x = y$; we shall prove that $\|y\| \leq (1+\delta)\max\{\max\|T_k\|, |\lambda|\}$. By (1.3), (1.4), and $P_n = i_n \circ R_n$, we conclude that

$$y = i_n((T_k x_k, \lambda z_k)_{k=1}^n) + \lambda(x - i_n \circ R_n x). \tag{1.5}$$

Write $y = (y_k, w_k)_{k=1}^\infty$ as a vector in $\mathcal{Z}_\infty$. Note that $R_n(x - i_n \circ R_n x) = 0$; therefore, by (1.5) we obtain that $R_n y = R_n \circ i_n((T_k x_k, \lambda z_k)_{k=1}^n) = (T_k x_k, \lambda z_k)_{k=1}^n$, and hence,

$$\|R_n y\| \leq \max\{\max\|T_k\|, |\lambda|\}.$$

Also, by (1.5) and Definition 1.3(ii)(b)(ι) we obtain that, for $k > n$, we have that $y_k = \lambda x_k$. All that remains to be shown is that, for $q > n$, we have that $\|w_q\| \leq (1+\delta)\max\{\max\|T_k\|, |\lambda|\}$. Fix $q > n$; it suffices to show that

$$|\vec{e}_\gamma(y)| \leq (1+\delta)\max\{\max\|T_k\|, |\lambda|\} \quad \text{for all } \gamma \in \Delta_q.$$

To that end, let $\gamma \in \Delta_q$.

Using (1.5), rewrite y as

$$y = i_n((T_k x_k - \lambda x_k, 0)_{k=1}^n) + \lambda x.$$

We conclude that

$$\begin{aligned} |\vec{e}_\gamma(y)| &= \big|\lambda \vec{e}_\gamma(x) + \vec{e}_\gamma(i_n((T_k x_k - \lambda x_k, 0)_{k=1}^n))\big| \\ &\leq |\lambda| + \big|\vec{e}_\gamma(i_n((T_k x_k - \lambda x_k, 0)_{k=1}^n))\big|. \end{aligned} \tag{1.6}$$

We distinguish two cases concerning γ. We first study the case where $\gamma \in \Delta_q^0$. In this case, there exists $\vec{f} \in (\sum_{k=1}^{q-1} \oplus X_k^*)_1$ with $\|\vec{f}\| \leq 1$, such that we have $\vec{e}_\gamma^* = \vec{e}_\gamma^* \circ P_{\{q\}} + (1/m_{\min L})\vec{f}$ (recall that in the construction of $(\sum_k \oplus X_k)_{\mathrm{AH}(L)}$, in (1.1) the constant $1/m_1$ is replaced with $1/m_{\min L}$). It follows that

$$
\begin{aligned}
(1.7) \qquad & |\vec{e}_\gamma(i_n((T_k x_k - \lambda x_k, 0)_{k=1}^n))| \\
&= \frac{1}{m_{\min L}} |\vec{f}(i_n((T_k x_k - \lambda x_k, 0)_{k=1}^n))| \\
&\leq \frac{1}{m_{\min L}} (|\lambda| + \max \|T_k\|) \\
&\leq \frac{2}{m_{\min L}} \max\{\max \|T_k\|, |\lambda|\}.
\end{aligned}
$$

Combining (1.6) with (1.7), we obtain the desired bound.

We now assume that $\gamma \in \Delta_q^1$. Applying Lemma 1.7, we obtain that either $\vec{e}_\gamma(i_n((T_k x_k - \lambda x_k, 0)_{k=1}^n)) = 0$ or that there exist $j \in L$, $1 \leq p \leq n$, and $\vec{b}^* \in (\sum_{k=1}^n \oplus \ell_1(\Delta_k))_1$ with $\|\vec{b}^*\| \leq 1$ so that

$$
\vec{e}_\gamma(i_n((T_k x_k - \lambda x_k, 0)_{k=1}^n)) = \frac{1}{m_j} \vec{b}^* \circ P_{[p,n]} \circ i_n((T_k x_k - \lambda x_k, 0)_{k=1}^n),
$$

and therefore we obtain

$$
\begin{aligned}
(1.8) \qquad & |\vec{e}_\gamma(i_n((T_k x_k - \lambda x_k, 0)_{k=1}^n))| \\
&\leq \frac{1}{m_j} \|P_{[p,n]}\| (|\lambda| + \max \|T_k\|) \\
&\leq \frac{1}{m_{\min L}} \frac{2 m_{\min L}}{m_{\min L} - 2} (|\lambda| + \max \|T_k\|) \\
&= \frac{2}{m_{\min L} - 2} (|\lambda| + \max \|T_k\|) \\
&\leq \frac{4}{m_{\min L} - 2} \max\{\max \|T_k\|, |\lambda|\}.
\end{aligned}
$$

Finally, combining (1.6) and (1.8), we conclude that

$$
|\vec{e}_\gamma(y)| \leq \left(1 + \frac{4}{m_{\min L} - 2}\right) \max\{\max_{1 \leq k \leq n} \|T_k\|, |\lambda|\},
$$

which completes the proof. □

1.4. Well-founded trees. A tree is an ordered space $(\mathcal{T}, \leq)$ such that, for every $t \in \mathcal{T}$, the set $\{s \in \mathcal{T} \mid s \leq t\}$ is well ordered. An element $t \in \mathcal{T}$ will be called a *node of* $\mathcal{T}$, and a minimal node will be called a *root*. A subset S of $\mathcal{T}$ is called a *downwards closed subtree of* $\mathcal{T}$ if, for every $t \in S$, the set $\{s \in \mathcal{T} \mid s \leq t\}$ is a subset of S. For every node t, we denote by $\mathrm{succ}(t)$ the set of the immediate successors of t, while by $\mathcal{T}_t$, we denote the set $\{s \in \mathcal{T} \mid t \leq s\}$. Note that the set $\mathcal{T}_t$, with the induced ordering, is also a tree. If t is a node of $\mathcal{T}$ and $s \in \mathrm{succ}(t)$, then t is called the *immediate predecessor of* s, and is denoted by s^-. For a node t, we define the height of t, denoted by $|t|$, to be the order type of the well-ordered set $\{s \in \mathcal{T} \mid s \leq t\}$. In particular, if t is a root of $\mathcal{T}$, then $|t| = 0$. A tree is called *well founded* if it does not contain any infinite chains. Note that in this case, $|t|$ is finite for every $t \in \mathcal{T}$.

Remark 1.18. From now on, unless stated otherwise, every tree $\mathcal{T}$ will be assumed to be well founded and have a unique root, denoted by $\varnothing_{\mathcal{T}}$; and for every non-maximal node t, the set $\mathrm{succ}(t)$ will be assumed to be infinitely countable. Note that such a tree is either infinitely countable or a singleton.

A tree $\mathcal{T}$ is equipped with the compact Hausdorff topology having the sets $\mathcal{T}_t$, $t \in \mathcal{T}$ as a subbase. Then, a node is an isolated point if and only if the set $\mathrm{succ}(t)$ is finite, which (following the convention from Remark 1.18) is the case if and only if t is a maximal node. Moreover, the set of all maximal nodes of $\mathcal{T}$ is dense in $\mathcal{T}$.

For a tree $\mathcal{T}$ the derivative $\mathcal{T}'$ of $\mathcal{T}$ is defined to be the downwards closed subtree of all non-maximal nodes of $\mathcal{T}$. For an ordinal number α, the derivative of order α of $\mathcal{T}$, which is also a downwards closed subtree of $\mathcal{T}$, is defined recursively. For $\alpha = 0$, set $\mathcal{T}^0 = \mathcal{T}$. Assuming that, for an ordinal number α, the derivatives $\mathcal{T}^\beta$ have been defined for all $\beta < \alpha$, define $\mathcal{T}^\alpha = \bigcap_{\beta<\alpha} \mathcal{T}^\beta$ if α is a limit ordinal number, and $\mathcal{T}^\alpha = (\mathcal{T}^\beta)'$ if α is a successor ordinal number with $\alpha = \beta + 1$. The rank of the tree $\mathcal{T}$, denoted by $\rho(T)$, is defined to be the smallest ordinal number α satisfying $\mathcal{T}^\alpha = \mathcal{T}^{\alpha+1} = \emptyset$. Note that $\rho(\mathcal{T})$ is a countable ordinal number.

For a node t, the rank of t, with respect to the tree $\mathcal{T}$, denoted by $\rho_{\mathcal{T}}(t)$, is also defined to be the supremum of all ordinal numbers α such that $t \in \mathcal{T}^\alpha$. Observe that the supremum is actually obtained: that is, if $\rho_{\mathcal{T}}(t) = \alpha$, then $t \in \mathcal{T}^\alpha$. The function $\rho_{\mathcal{T}}$ satisfies the following properties: $\rho_{\mathcal{T}}(t) = 0$ if and only if t is a maximal node, and otherwise $\rho_{\mathcal{T}}(t) = \sup\{\rho_{\mathcal{T}}(s) + 1 \mid t < s\} = \sup\{\rho_{\mathcal{T}}(s) + 1 \mid s \in \mathrm{succ}(t)\}$. Moreover, $\rho(\mathcal{T}) = \sup\{\rho_{\mathcal{T}}(t) + 1 \mid t \in \mathcal{T}\} = \rho_{\mathcal{T}}(\varnothing_{\mathcal{T}}) + 1$. Furthermore, if s is a node of $\mathcal{T}$ and we consider the tree $\mathcal{T}_s$ with the induced ordering, then $\rho_{\mathcal{T}}(t) = \rho_{\mathcal{T}_s}(t)$ for every $t \in \mathcal{T}_s$.

Remark 1.19. It is more convenient to use the rank $\rho_{\mathcal{T}}(\varnothing_{\mathcal{T}})$ of the root of a tree $\mathcal{T}$ instead of the rank $\rho(\mathcal{T})$ of the tree itself. Therefore, unless stated otherwise, by rank of a tree $\mathcal{T}$ we shall mean the ordinal number $o(\mathcal{T}) = \rho_{\mathcal{T}}(\varnothing_{\mathcal{T}})$. Under this notation, we have that $o(\mathcal{T}) = 0$ if and only if $\mathcal{T}$ is a singleton, and otherwise $o(\mathcal{T}) = \sup\{o(\mathcal{T}_s) + 1 \mid s \in \mathcal{T}, \text{ with } s \neq \varnothing_{\mathcal{T}}\}$.

2. The Spaces $X_{(\mathcal{T},L,\varepsilon)}$

For every tree $\mathcal{T}$, infinite subset of the natural numbers L, and positive real number ε, we define a Banach space $X_{(\mathcal{T},L,\varepsilon)}$. Let us for now forget the parameter ε, which represents the declination of the space's dual from ℓ_1, and concentrate on the set L, which plays a significant role in proving the properties of the space; thus, let us for the moment denote the spaces in the form $X_{(\mathcal{T},L)}$. The purpose of this set L lies in the fact that, as already mentioned, the Argyros-Haydon space is constructed using a sequence of pairs of natural numbers $(m_j, n_j)_j$, called *weights*; and using a subsequence $(m_j, n_j)_{j\in L}$ of these weights, one can define the space $X_{\mathrm{AH}(L)}$. As is proved in [1], if the intersection of two sets L, M is finite, then every operator from $X_{\mathrm{AH}(L)}$ to $X_{\mathrm{AH}(M)}$ is compact. Returning now to the recursive definition of the spaces (which we for now denote by $X_{(\mathcal{T},L)}$), in the basic recursive step the space $X_{(\mathcal{T},L)}$ is the space $X_{\mathrm{AH}(L)}$, while in the general case the space $X_{(\mathcal{T},L)}$ is the direct sum $(\sum \oplus X_{(\mathcal{T}_n,L_n)})_{\mathrm{AH}(L_0)}$, where $(L_n)_n$ defines a partition of L. This method of construction endows these spaces with the following properties: every operator defined on $X_{(\mathcal{T},L)}$ is a multiple of the identity plus a horizontally compact operator (see Definition 3.4), and if $\mathcal{S}$ is a tree while M is such that its intersection with L is finite, then every operator $T : X_{(\mathcal{T},L)} \to X_{(\mathcal{S},M)}$ is compact. These facts are the main tools used to prove Theorem A in the Introduction.

Definition 2.1. By transfinite recursion on the order $o(\mathcal{T})$ of a tree $\mathcal{T}$, we define the spaces $X_{(\mathcal{T},L,\varepsilon)}$ for every L infinite subset of the natural numbers and ε positive real number. We distinguish two cases, namely, the basic step and the general inductive step:

(i) Let $\mathcal{T}$ be a tree with $o(\mathcal{T}) = 0$ (i.e., $\mathcal{T} = \{\varnothing_{\mathcal{T}}\}$), L be an infinite subset of the natural numbers, and ε be a positive real number. Let $\delta > 0$ with $(1+\delta)^2 < 1+\varepsilon$. Choose L' an infinite subset of L such that $4/(\min L' - 2) < \delta$, and define $X_{(\mathcal{T},L,\varepsilon)} = X_{\mathrm{AH}(L')}$.

(ii) Let $\mathcal{T}$ be a tree with $0 < \alpha = o(\mathcal{T})$, L be an infinite subset of the natural numbers, and ε be a positive real number. Assume that, for every tree $\mathcal{S}$ with $o(\mathcal{S}) < \alpha$, for every infinite subset of the natural numbers M and positive real number ε', the space $X_{(\mathcal{S},M,\varepsilon')}$ has been defined. Choose $\{s_n \mid n \in \mathbb{N}\}$, an enumeration of the set $\mathrm{succ}(\varnothing_{\mathcal{T}})$, $\delta > 0$ with $(1+\delta)^2 < 1+\varepsilon$, L' an infinite subset of L with $4/(\min L' - 2) < \delta$, and a partition of L' into infinite sets $(L_n)_{n=0}^{\infty}$. Define $X_{s_n} = X_{(\mathcal{T}_{s_n},L_n,\delta)}$ and

$$X_{(\mathcal{T},L,\varepsilon)} = \Big(\sum_{n=1}^{\infty} \oplus X_{s_n}\Big)_{\mathrm{AH}(L_0)}.$$

Remark 2.2. Proposition 1.4(ii) and a transfinite induction on $o(\mathcal{T})$ yield that the space $X_{(\mathcal{T},L,\varepsilon)}$ is HI-saturated, for all $\mathcal{T}, L, \varepsilon$ as in Definition 2.1.

Proposition 2.3. *Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1. Then, $X^*_{(\mathcal{T},L,\varepsilon)}$ is $(1+\varepsilon)$-isomorphic to ℓ_1.*

Proof. We use transfinite induction on the order $o(\mathcal{T})$ of a tree $\mathcal{T}$. In the case that $o(\mathcal{T}) = 0$, the result follows by Remark 1.13. We now assume that $o(\mathcal{T}) = \alpha > 0$, and that for every tree S with $o(S) < \alpha$, every infinite subset M of $\mathbb{N}$, and every $\varepsilon' > 0$, the dual space $X^*_{(S,M,\varepsilon')}$ is $(1+\varepsilon')$-isomorphic to ℓ_1. Let $\{s_n \mid n \in \mathbb{N}\}$, L', $(L_n)_{n=0}^\infty$, $\delta > 0$, and X_{s_n} as in Definition 2.1(ii) such that

$$X_{(\mathcal{T},L,\varepsilon)} = \Big(\sum_{n=1}^{\infty} \oplus X_{s_n}\Big)_{\mathrm{AH}(L_0)}.$$

Observe that since $L_0 \subset L'$, it follows that $2/(\min L_0 - 2) < \delta$, and hence by Proposition 1.4 and Remark 1.15, there exist $(\Delta_n)_{n\in\mathbb{N}}$ pairwise disjoint subsets of $\mathbb{N}$ such that

$$X^*_{(\mathcal{T},L,\varepsilon)} \simeq^{1+\delta} \Big(\sum_{n=1}^{\infty} \oplus (X^*_{s_n} \oplus \ell_1(\Delta_n))_1\Big)_1.$$

Since $X_{s_n} = X_{(\mathcal{T}_{s_n},L_n,\delta)}$ and $o(\mathcal{T}_{s_n}) < \alpha$ for every $n \in \mathbb{N}$, applying our inductive assumption, we conclude that $X^*_{s_n} \simeq^{1+\delta} \ell_1$.

The result follows by the choice of δ. □

Remark 2.4. The above, in conjunction with the principle of local reflexivity [9], yields that the space $X_{(\mathcal{T},L,\varepsilon)}$ is a $\mathcal{L}_{\infty,(1+\varepsilon')}$-space for every $\varepsilon' > \varepsilon$.

At this point, we shall make a few observations that follow from Definition 2.1 and the discussion made in Section 1. First, we mention that each space $X_{(\mathcal{T},L,\varepsilon)}$ is associated with a sequence $(\Delta_n)_{n\in\mathbb{N}}$ that is involved in its construction, and we denote the union $\bigcup_n \Delta_n$ by $\Gamma(\mathcal{T},L,\varepsilon)$. In particular, if $o(T) = 0$, the space $X_{(\mathcal{T},L,\varepsilon)}$ has an FDD $(M_n)_{n\in\mathbb{N}}$ such that each M_n is isometric to $\ell_\infty(\Delta_n)$, while in the case that $o(\mathcal{T}) > 0$ and $\{s_n \mid n \in \mathbb{N}\}$ is the enumeration of the set $\mathrm{succ}(\varnothing_{\mathcal{T}})$ (as in the definition of the space $X_{(\mathcal{T},L,\varepsilon)}$), the space admits a Schauder Decomposition $(Z_n)_{n\in\mathbb{N}}$ such that each Z_n is isometric to $(X_{s_n} \oplus \ell_\infty(\Delta_n))_\infty$.

Using the already-introduced terminology of Section 1, for the sequel we denote by P_n the projections associated with the FDD or the Schauder Decomposition of the space $X_{(\mathcal{T},L,\varepsilon)}$, where the image $\operatorname{Im} P_n = Z_n$ by Lemma 1.9 is isometric to $(X_{s_n} \oplus \ell_\infty(\Delta_n))_\infty$. Also, for $n \in \mathbb{N}$, we denote by

$$j_n : X_{s_n} \to (\sum_{k=1}^{n} \oplus (X_{s_n} \oplus \ell_\infty(\Delta_n))_\infty)_\infty$$

the natural embedding, as well as by $\pi_n : (\sum_{k=1}^{n} \oplus (X_{s_n} \oplus \ell_\infty(\Delta_n))_\infty)_\infty \to X_{s_n}$ the natural coordinate projection, and by $R_{[1,n]} : \mathcal{Z} \to (\sum_{k=1}^{n} \oplus (X_{s_n} \oplus \ell_\infty(\Delta_n))_\infty)_\infty$ the natural restriction mappings. We shall define the projections $I_n : \mathcal{Z} \to \mathcal{Z}$ as $I_n = i_n \circ j_n \circ \pi_n \circ R_{[1,n]}$. Lemma 1.10 yields that $\|I_n\| = 1$ for every $n \in \mathbb{N}$, and by Corollary 1.9 we obtain that the image $\operatorname{Im} I_n$ is isometric to X_{s_n}.

In a similar manner as above, for each $t \in \mathcal{T}$ non-maximal, following the notation in Proposition 2.5(iv), we denote by P^t_n the projections defined upon

X_t, such that for each n, the image $\operatorname{Im} P_n^t$ is isometric to $(X_{s_n} \oplus \ell_\infty(\Delta_n^t))_\infty$ and $\bigcup_n \Delta_n^t = \Gamma(\mathcal{T}_t, L_t, \delta_t)$. Note that by the above, we have $P_n = P_n^{\varnothing_{\mathcal{T}}}$. Moreover, for $I = (n, m]$ interval of natural numbers and $t \in \mathcal{T}$, we define $P_I^t = \sum_{i=n+1}^m P_i^t$.

Finally, the notion of a block sequence in $X_{(\mathcal{T},L,\varepsilon)}$ is defined as in Section 1 using the FDD (if $o(T) = 0$) or the Schauder Decomposition (if $o(T) \neq 0$). In the second case, we call the block sequence *horizontally block*.

Proposition 2.5. *Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1. Then, there exist $(L_s)_{s\in\mathcal{T}}$ infinite subsets of the natural numbers, $(\varepsilon_s)_{s\in\mathcal{T}}$ positive real numbers, $(I_s)_{s\in\mathcal{T}}$ norm-one projections defined on $X_{(\mathcal{T},L,\varepsilon)}$, and $(X_s)_{s\in\mathcal{T}}$ infinitely dimensional subspaces of $X_{(\mathcal{T},L,\varepsilon)}$ such that the following are satisfied:*

(i) *The sets $L_{\varnothing_{\mathcal{T}}}$ and L are equal; if s, t are nodes in $\mathcal{T}$, and if t and s are incomparable, we have that $L_s \cap L_t = \emptyset$, while if $s \leq t$, then we have that $L_t \subset L_s$.*

(ii) *The projection $I_{\varnothing_{\mathcal{T}}}$ is the identity map; if s, t are nodes in $\mathcal{T}$, and if t and s are incomparable, we have that $\operatorname{Im} I_s \subset \ker I_t$, and if $s \leq t$ then $I_s \circ I_t = I_t \circ I_s = I_t$.*

(iii) *The image of the operator I_s is the space X_s, and X_s is isometric to $X_{(\mathcal{T}_s,L_s,\varepsilon_s)}$.*

(iv) *If t is a non-maximal node of $\mathcal{T}$, then there is an enumeration $\{s_n \mid n \in \mathbb{N}\}$ of $\operatorname{succ}(t)$, and L_t^0 an infinite subset of $L_t \setminus \bigcup_{s\in\operatorname{succ}(t)} L_s$ such that the following is satisfied:*

$$X_t = \Big(\sum_{n=1}^{\infty} \oplus X_{s_n}\Big)_{\mathrm{AH}(L_t^0)}.$$

Proof. We use induction on the order of the tree $\mathcal{T}$, $o(\mathcal{T})$. In the case that $o(\mathcal{T}) = 0$, let L' be such that $X_{(\mathcal{T},L,\varepsilon)} = X_{\mathrm{AH}(L')}$. We set $L_{\varnothing_{\mathcal{T}}} = L$, $I_{\varnothing_{\mathcal{T}}} = I$, where I denotes the identity map on $X_{\mathrm{AH}(L')}$ and $\delta_{\varnothing_{\mathcal{T}}} = \varepsilon$. Assume that $o(\mathcal{T}) = \alpha > 0$ and that the statement has been proved for every S, M, δ such that $o(S) < \alpha$, with $M \subseteq \mathbb{N}$ infinite and $\delta > 0$.

Let $\{s_n \mid n \in \mathbb{N}\}$, $(L_n)_{n=0}^{\infty}$, $\delta > 0$, and X_{s_n} as in Definition 2.1(ii) such that

$$X_{(\mathcal{T},L,\varepsilon)} = \Big(\sum_{n=1}^{\infty} \oplus X_{s_n}\Big)_{\mathrm{AH}(L_0)}.$$

Let also $I_n : X_{(\mathcal{T},L,\varepsilon)} \to X_{(\mathcal{T},L,\varepsilon)}$ be the norm-one projections for every $n \in \mathbb{N}$ such that $\operatorname{Im} I_n$ is isometric to X_{s_n}. We recall that $X_{s_n} = X_{(\mathcal{T}_{s_n},L_n,\delta)}$, and since $o(\mathcal{T}_{s_n}) < \alpha$, applying our inductive assumption for every $n \in \mathbb{N}$, we obtain $(L_s)_{s\in\mathcal{T}_{s_n}}$ infinite subsets of the natural numbers, $(\varepsilon_s)_{s\in\mathcal{T}_{s_n}}$ positive real numbers, and $(I_s)_{s\in\mathcal{T}_{s_n}}$ norm-one projections defined on $X_{(\mathcal{T}_{s_n},L_n,\delta)}$ satisfying the conditions (i)–(iv) above.

We set $L_{\varnothing_{\mathcal{T}}} = L$, $I_{\varnothing_{\mathcal{T}}} = I$, where I denotes the identity map on $X_{(\mathcal{T},L,\varepsilon)}$, $\varepsilon_{\varnothing_{\mathcal{T}}} = \varepsilon$. For $t \in \mathcal{T}$ with $t \neq \varnothing_{\mathcal{T}}$, note that there exists $n_0 \in \mathbb{N}$ such that $t \in \mathcal{T}_{s_{n_0}}$; hence, L_t, ε_t, and $I_t : X_{s_{n_0}} \to X_{s_{n_0}}$ (where $\operatorname{Im} I_t$ is isometric to X_t) have already been defined by our inductive assumption.

We extend $I_t : X_{(\mathcal{T},L,\varepsilon)} \to X_{(\mathcal{T},L,\varepsilon)}$ as $I_t = I_t \circ I_{n_0}$, and we set $X_t = I_t[X_{(\mathcal{T},L,\varepsilon)}]$. It is not hard to check that $(L_s)_{s\in\mathcal{T}}$, $(\varepsilon_s)_{s\in\mathcal{T}}$, $(I_s)_{s\in\mathcal{T}}$, and $(X_s)_{s\in\mathcal{T}}$ satisfy the thesis. ◻

3. OPERATORS ON THE SPACES $X_{(\mathcal{T},L,\varepsilon)}$

We study the properties of the operators defined on the spaces $X_{(\mathcal{T},L,\varepsilon)}$. The purpose is to show that every bounded linear operator is approximated by a sequence of operators, each one of which is a linear combination of the projections I_s, $s \in \mathcal{T}$ plus a compact operator. We start by giving the definition of Rapidly Increasing Sequences (RIS) in the spaces $X_{(\mathcal{T},L,\varepsilon)}$.

Definition 3.1. Let L be an infinite subset of $\mathbb{N}$, $(X_n)_n$ be a sequence of separable Banach spaces, and $X = (\sum_n \oplus X_n)_{\mathrm{AH}(L)}$. We say that a block sequence (respectively, a horizontally block sequence) $(z_k)_{k\in\mathbb{N}}$ in $X_{\mathrm{AH}(L)}$ (respectively, in X) is a C-rapidly increasing sequence (RIS) if there exists a constant $C > 0$ and a strictly increasing sequence $(j_k)_{k\in\mathbb{N}}$ in L such that the following hold:

(i) $\|z_k\| \leq C$ for all $k \in \mathbb{N}$.
(ii) $j_{k+1} > \max \operatorname{ran} z_k$.
(iii) $|\vec{e}^{\,*}_\gamma(z_k)| \leq C/m_i$ for every $\gamma \in \Gamma$ with $w(\gamma) < m_{j_k}$.

Note that in $X_{\mathrm{AH}(L)}$, the definition of a C-RIS essentially coincides with the corresponding one presented in [1]. Furthermore, the existence of C-RIS in $X = (\sum_n \oplus X_n)_{\mathrm{AH}(L)}$ is proved in a manner similar to what is described in [12], and makes use of Proposition 1.16 as well as an analogue of [1, Lemma 8.4].

The next result follows readily from [1, Proposition 5.4; 12, Proposition 5.12].

Proposition 3.2. *Let L be an infinite subset of the natural numbers, and X be either $X_{\mathrm{AH}(L)}$ or $X = (\sum_{n=1}^\infty \oplus X_n)_{\mathrm{AH}(L)}$, where $(X_n)_n$ is a sequence of separable Banach spaces. Let also $(z_k)_k$ be a C-RIS in X and $j_0 \in \mathbb{N}$. Then,*

$$\Big\|n_{j_0}^{-1}\sum_{k=1}^{n_j} z_k\Big\| \leq \begin{cases} \dfrac{10C}{m_{j_0}} & \text{if } j_0 \in L,\\[2ex] \dfrac{10C}{m_{j_0}^2} & \text{if } j_0 \notin L.\end{cases}$$

The following result is proved in [12, Proposition 5.14]

Lemma 3.3. *Let L be an infinite subset of the natural numbers, $(X_n)_n$ be a sequence of separable Banach spaces, and $X = (\sum_{n=1}^\infty \oplus X_n)_{\mathrm{AH}(L)}$. Let also Y be a Banach space, and $T : X \to Y$ be a bounded linear operator; and assume there exists a seminormalized horizontally block sequence $(x_k)_k$ in X so that* $\limsup_k \|Tx_k\| > 0$.

Then, there exists a RIS $(y_k)_k$ *in X such that* $\limsup_k \|Ty_k\| > 0$.

We adapt the following definition given in [12].

Definition 3.4. Let $(X_n)_n$ be a sequence of separable Banach spaces, L be an infinite subset of the natural numbers, $X = (\sum_{n=1}^\infty \oplus X_n)_{\mathrm{AH}(L)}$, and Y be a Banach space. We say that an operator $K : X \to Y$ is horizontally compact if, for every

$\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|K - K \circ P_{[1,n_0]}\| < \delta$. Equivalently, K is horizontally compact if $\lim_k \|K(x_k)\| = 0$ for every horizontally block sequence $(x_k)_k$ in X.

Proposition 3.5. *Let $(X_n)_n$ be a sequence of separable Banach spaces, L be an infinite subset of $\mathbb{N}$, and Y be a Banach space. The following hold:*

(i) *If a bounded linear operator $T : X_{\mathrm{AH}(L)} \to Y$ is not compact, then there exists a* RIS $(x_k)_k$ *in $X_{\mathrm{AH}(L)}$ such that* $\limsup_k \|Tx_k\| > 0$.

(ii) *If a bounded linear operator $T : (\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(L)} \to Y$ is not horizontally compact, then there exists a* RIS $(x_k)_k$ *in $(\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(L)}$ such that* $\limsup_k \|Tx_k\| > 0$.

Proof. For (i), we first observe that, since T is not compact, there exists $\delta > 0$ and a bounded block sequence $(x_k)_k$ in $X_{\mathrm{AH}(L)}$ such that $\|T(x_k)\| \geq \delta$. By [1, Proposition 5.11] the result follows. For (ii), we assume, towards a contradiction, that $\limsup_k \|Tx_k\| = 0$ for every RIS sequence $(x_k)_k$ in X. By Lemma 3.3, again we conclude that $\limsup_k \|Tx_k\| = 0$ for every bounded horizontally block sequence in X. It follows that T is horizontally compact, yielding a contradiction. ∎

Lemma 3.6. *Let $(X_k)_k$, $(Y_k)_k$ be sequences of separable Banach spaces, and L, M be infinite subsets of $\mathbb{N}$ such that $L \cap M$ is finite. Let, moreover, $(x_k)_k$ be a C-*RIS *in $(\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(L)}$ and $(y_k)_k$ be a seminormalized horizontally block sequence in $(\sum_{n=1}^{\infty} \oplus Y_n)_{\mathrm{AH}(M)}$. Then, $(x_k)_k$ does not dominate $(y_k)_k$; that is, the map $x_k \to y_k$ does not extend to a bounded linear operator.*

Proof. By Proposition 1.16, there exists $\tilde{M}$ subset of M with the property that, for every $j \in \tilde{M}$ passing to a subsequence, we have that

$$\Big\|\sum_{k=1}^{n_j} y_k\Big\| \geq \frac{1}{2m_j}\sum_{k=1}^{n_j}\|y_k\| \geq \frac{1}{m_{\min M}}\frac{n_j}{4m_j}.$$

Moreover, by Proposition 3.2, for $j \notin L$ we obtain that $\|\sum_{k=1}^{n_j} x_k\| \leq 10Cn_j/m_j^2$.

Assume there is a constant $c > 0$ such that $\|\sum_{k=1}^{n} a_k x_k\| \geq c\|\sum_{k=1}^{n} a_k y_k\|$ for every $n \in \mathbb{N}$ and every sequence of scalars $(a_k)_k$. Since $M \cap L$ is finite, we can choose $j \in \tilde{M}$, $j \notin L$ such that $m_j > m_{\min M} \cdot 40C/c$. Combining the above and passing to subsequences, we conclude that

$$\frac{1}{m_{\min M}}\frac{cn_j}{4m_j} \leq c\Big\|\sum_{k=1}^{n_j} y_k\Big\| \leq \Big\|\sum_{k=1}^{n_j} x_k\Big\| \leq \frac{10Cn_j}{m_j^2}.$$

The choice of j yields a contradiction. ∎

Remark 3.7. In a similar manner, for $L, M, (Y_k)_k$ as above, note that a C-RIS in $X_{\mathrm{AH}(L)}$ cannot dominate a seminormalized horizontally block sequence in the space $(\sum_{n=1}^{\infty} \oplus Y_n)_{\mathrm{AH}(M)}$.

Lemma 3.8. *Let $(X_n)_n$ be a sequence of separable Banach spaces and M be an infinite subset of the natural numbers. Let also $\mathcal{T}$, L, ε be as in Definition 2.1 such that $L \cap M$ is finite. Then, every bounded linear operator $T : (\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(M)} \to X_{(\mathcal{T},L,\varepsilon)}$ is horizontally compact.*

Proof. We use transfinite induction on the rank of the tree $o(\mathcal{T})$. Suppose that $o(\mathcal{T}) = 0$, and assume, towards a contradiction, that T is not horizontally compact. By Proposition 3.5, there exists a RIS sequence $(x_k)_k$ and $\delta > 0$ such that $\|T(x_k)\| \geq \delta$. We may assume that $(x_k)_k$ is normalized and note that $(T(x_k))_k$ is weakly null.

Since $X_{(\mathcal{T},L,\varepsilon)} = X_{\mathrm{AH}(L')}$ for some $L' \subseteq L$, we may assume $(T(x_k))_k$ is a block sequence, and by Proposition 1.14 we obtain $\tilde{L} \subseteq L'$ infinite such that, for every $j \in \tilde{L}$,

$$\Big\| \sum_{k=1}^{n_j} T(x_k) \Big\| \geq \frac{1}{m_{\min L'}} \frac{\delta n_j}{4 m_j},$$

passing to a subsequence. Let $j \in \tilde{L}$ such that $j \notin M$ and $m_j > m_{\min L'} \cdot 10\|T\|/\delta$. Proposition 3.2 yields that $\|\sum_{k=1}^{n_j} x_k\| \leq 10 n_j / m_j^2$, yielding a contradiction by the choice of j.

Suppose now that $o(\mathcal{T}) = \alpha > 0$, and assume that, for every S, M', ε' as in Definition 2.1 (such that $o(S) < \alpha$, $M' \cap M$ is finite), every bounded linear operator $T : (\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(M)} \to X_{(S,M',\varepsilon')}$ is horizontally compact. We will prove that the same holds for $T : (\sum_{n=1}^{\infty} \oplus X_n)_{\mathrm{AH}(M)} \to X_{(\mathcal{T},L,\varepsilon)}$, and let $\{s_n \mid n \in \mathbb{N}\}$, $(L_n)_{n=0}^{\infty}$, $\delta > 0$ as in Definition 2.1(ii) such that $X_{(\mathcal{T},L,\varepsilon)} = (\sum_{n=1}^{\infty} \oplus X_{s_n})_{\mathrm{AH}(L_0)}$ where $X_{s_n} = X_{(\mathcal{T}_{s_n},L_n,\delta)}$. Assume towards a contradiction that this is not the case, and by Proposition 3.5, let $(x_k)_k$ be a normalized RIS sequence and $\delta > 0$ such that $\|T(x_k)\| \geq \delta$, for every $k \in \mathbb{N}$. For $m \in \mathbb{N}$, we consider the operator

$$R_{[1,m]} \circ T : \Big(\sum_{n=1}^{\infty} \oplus X_n \Big)_{\mathrm{AH}(M)} \to \Big(\sum_{n=1}^{m} \oplus (X_{s_n} \oplus \ell_\infty(\Delta_n))_\infty \Big)_\infty,$$

where $\bigcup_n \Delta_n = \Gamma(\mathcal{T}, L, \varepsilon)$. Using our inductive assumption, we deduce that the operator $R_{[1,m]} \circ T$ is horizontally compact, and hence $\lim_k \|P_{[1,m]}(T(x_k))\| = \lim_k \|i_m \circ R_{[1,m]}(T(x_k))\| = 0$.

By a sliding hump argument, it follows that $(T(x_k))_k$ is equivalent to a horizontally block sequence in $X_{(\mathcal{T},L,\varepsilon)}$, and, since T is bounded, by Lemma 3.6 we arrive at a contradiction. ◻

Remark 3.9. For $M, \mathcal{T}, L, \varepsilon$ as above, we obtain that every bounded linear operator $T : X_{\mathrm{AH}(M)} \to X_{(\mathcal{T},L,\varepsilon)}$ is compact.

The following result is similar to [1, Theorem 10.4].

Proposition 3.10. *Let $\mathcal{T}, L, \varepsilon$ and S, M, δ be as in Definition 2.1 such that $L \cap M$ is finite. Then, every bounded linear operator $T : X_{(\mathcal{T},L,\varepsilon)} \to X_{(S,M,\delta)}$ is compact.*

Proof. We use a transfinite induction on $o(\mathcal{T})$. In the case that $o(\mathcal{T})$ is zero, let L' such that $X_{(\mathcal{T},L,\varepsilon)} = X_{\mathrm{AH}(L')}$, and observe that the statement follows by Remark 3.9.

Assume now $o(\mathcal{T}) = \alpha$, and suppose that for every tree $\mathcal{T}'$ with $o(\mathcal{T}') < \alpha$, every operator $T : X_{(\mathcal{T}',L',\varepsilon)} \to X_{(S,M,\delta)}$ is compact for every S, M, δ that satisfy Definition 2.1. Fix S, M, δ, and let also $\{s_n \mid n \in \mathbb{N}\}$, $(L_n)_{n=0}^{\infty}$, $\delta > 0$ as in Definition 2.1(ii) such that we have $X_{(\mathcal{T},L,\varepsilon)} = (\sum_{n=1}^{\infty} \oplus X_{s_n})_{\mathrm{AH}(L_0)}$ where $X_{s_n} = X_{(\mathcal{T}_{s_n},L_n,\delta)}$. In order to show that $T : X_{(\mathcal{T},L,\varepsilon)} \to X_{(S,M,\delta)}$ is compact, we first need to observe that, by Lemma 3.8, the operator T is horizontally compact. Hence, $\lim_k \|T - TP_{[1,k]}\| = 0$.

Let $I_{s_n} : X_{(\mathcal{T},L,\varepsilon)} \to X_{s_n}$ be the projections defined in Proposition 2.5. By Remark 1.11, the operator $T \circ P_{[1,k]} - \sum_{n=1}^{k} T \circ I_{s_n}$ is compact. Consider the bounded operators $T \circ I_{s_n} : X_{s_n} \to X_{(S,M,\delta)}$. Since $o(\mathcal{T}_{s_n}) < \alpha$, applying our inductive assumption, we conclude that $T \circ I_{s_n}$ is compact for every $n = 1, \dots, k$. It follows that $T \circ P_{[1,k]}$ is compact, and therefore $T = \lim_k T \circ P_{[1,k]}$ is compact as well. ◻

The statement of the next lemma is proved as in [12, Lemma 7.7] by using Proposition 3.10.

Lemma 3.11. *Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1, $(x_k)_k$ be a* RIS *in $X_{(\mathcal{T},L,\varepsilon)}$, and $T : X_{(\mathcal{T},L,\varepsilon)} \to X_{(\mathcal{T},L,\varepsilon)}$ be a bounded linear operator. Then, we have that* $\lim_k \mathrm{dist}(Tx_k, \mathbb{R}x_k) = 0$.

The next proposition shares arguments similar to those in [1, Theorem 7.4] and [12, Proposition 7.8].

Proposition 3.12. *Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1 and $T : X_{(\mathcal{T},L,\varepsilon)} \to X_{(\mathcal{T},L,\varepsilon)}$ be a bounded linear operator. Then, there exists a real number λ such that the operator $\lambda I - T$ is horizontally compact.*

Lemma 3.13. *Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1, and let also s be a node of $\mathcal{T}$. Then, for every bounded linear operator $T : X_{(\mathcal{T},L,\varepsilon)} \to X_{(\mathcal{T},L,\varepsilon)}$, we have that the operator $T \circ I_s - I_s \circ T \circ I_s$ is a compact one.*

Proof. We prove this lemma using transfinite induction on the rank of $\mathcal{T}$. If $o(\mathcal{T}) = 0$, then $\mathcal{T} = \{\varnothing_{\mathcal{T}}\}$, and therefore $s = \varnothing_{\mathcal{T}}$; that is, I_s is the identity map. We conclude that $T \circ I_s - I_s \circ T \circ I_s$ is the zero operator, which is compact.

Assume now that α is a countable cardinal number such that the statement holds for every S, M, δ as in Definition 2.1 with $o(S) < \alpha$, and let $\mathcal{T}$ be a tree with $o(\mathcal{T}) = \alpha$. Assume that $X_{(\mathcal{T},L,\varepsilon)} = (\sum_{n=1}^{\infty} \oplus X_{s_n})_{\mathrm{AH}(L_0)}$, where $\{s_n \mid n \in \mathbb{N}\}$ is an enumeration of $\mathrm{succ}(\varnothing_{\mathcal{T}})$ and $L_t \cap L_0 = \emptyset$ for every $t \in \mathcal{T}$ with $t \neq \varnothing_{\mathcal{T}}$.

We shall use transfinite induction once more, this time on the rank $\rho_{\mathcal{T}}(s)$ of the node t. Assume that $\rho(s) = 0$, that is, $X_s = X_{\mathrm{AH}(L_s)}$. Let $S : X_s \to X_{(\mathcal{T},L,\varepsilon)}$ be the restriction of $T \circ I_s - I_s \circ T \circ I_s$ onto X_s. As I_s is a projection onto X_s, it is enough to show that S is compact. Assume that it is not; then, by Proposition 3.5(1), there exists a RIS $(x_k)_k$ in X_s and a positive real number θ such that $\|Sx_k\| > \theta$ for all $k \in \mathbb{N}$.

Let n_0 denote the unique natural number such that $s \in \mathcal{T}_{s_{n_0}}$, and let P_n denote the natural projections on the components $Z_n = i_n[(X_{s_n} \oplus \ell_\infty(\Delta_n))_\infty]$. Recall that, from Remark 1.11, $P_n - I_{s_n}$ is a finite-rank operator, and therefore it is compact. For $n \neq n_0$, since $L_{s_n} \cap L_{s_{n_0}} = \emptyset$ and $L_s \subset L_{s_{n_0}}$, Proposition 3.10 yields that $P_n \circ S$ is compact. Moreover, for $n = n_0$, the inductive assumption implies that the map $P_n \circ S$ is compact, and therefore we have that $\lim_k P_n \circ Sx_k = 0$ for all $n \in \mathbb{N}$. We conclude that the sequence $(Sx_k)_k$ has a subsequence equivalent to a horizontally block sequence of $X_{(\mathcal{T},L,\varepsilon)}$. Since $L_0 \cap L_s = \emptyset$, Lemma 3.6 yields a contradiction.

Assume now that $0 < \beta \leq o(\mathcal{T})$ is an ordinal number such that the statement holds for every $s \in \mathcal{T}$ with $\rho_{\mathcal{T}}(s) < \beta$, and let s be a node of $\mathcal{T}$ with $\rho_{\mathcal{T}}(s) = \beta$. If s is the root of the tree, then by the fact that $I_{\varnothing_{\mathcal{T}}}$ is the identity map, one can easily deduce the desired result. It is therefore sufficient to check the case in which $\rho_{\mathcal{T}}(s) < \rho_{\mathcal{T}}(\varnothing_{\mathcal{T}}) = o(\mathcal{T}) = \alpha$.

Since s is a non-maximal node, by Proposition 2.5 there exists an enumeration $\{t_n \mid n \in \mathbb{N}\}$ of $\mathrm{succ}(s)$ and L_s^0 an infinite subset of $L_s \setminus \bigcup_{t \in \mathrm{succ}(s)} L_t$ such that $X_s = (\sum_{n=1}^{\infty} \oplus X_{t_n})_{\mathrm{AH}(L_s^0)}$.

By setting $S = I_s \circ T \circ I_s$, since $o(\mathcal{T}_s) < \alpha$, the inductive assumption yields that the operators $S \circ I_{t_n} - I_{t_n} \circ S \circ I_{t_n}$ are compact. In other words, the operators $I_s \circ T \circ I_{t_n} - I_{t_n} \circ T \circ I_{t_n}$ are compact for all $n \in \mathbb{N}$. Moreover, since $\rho_{\mathcal{T}}(t_n) < \beta$ for all $n \in \mathbb{N}$, the second inductive assumption yields that the operators $T \circ I_{t_n} - I_{t_n} \circ T \circ I_{t_n}$ are compact for all $n \in \mathbb{N}$. We conclude that the operators $T \circ I_{t_n} - I_s \circ T \circ I_{t_n}$ are compact for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, we recall that P_n^s denotes the natural projections defined on X_s onto the component $X_{t_n} \oplus \ell_\infty(\Delta_n^s)$, and we denote by P_{t_n} the operators $P_n^s \circ I_s$. As before, the operators $I_{t_n} - P_{t_n}$ are compact, which yields the following:

(3.1) For every $n \in \mathbb{N}$, the operator $T \circ P_{t_n} - I_s \circ T \circ P_{t_n}$ is compact.

Observe, moreover, that $I_s = SOT - \sum_{i=1}^{\infty} P_{t_i}$, and hence the following holds:

$$T \circ I_s - I_s \circ T \circ I_s = SOT - \sum_{i=1}^{\infty} (T \circ P_{t_i} - I_s \circ T \circ P_{t_i}). \tag{3.2}$$

To conclude that the operator $T \circ I_s - I_s \circ T \circ I_s$ is compact, (3.1) implies it is enough to show that the series on the right-hand side of (3.2) converges in operator norm. In other words, it is sufficient to show that, if we set $R : X_s \to X_{(\mathcal{T},L,\varepsilon)}$ to be the restriction of $T \circ I_s - I_s \circ T \circ I_s$ onto X_s, then R is horizontally compact.

Towards a contradiction, assume that this is not the case. Lemma 3.3 yields that there exists a RIS $(x_k)_k$ in X_s and a positive real number θ with $\|Rx_k\| > \theta$ for all $k \in \mathbb{N}$. Arguing exactly as in the case $\rho_{\mathcal{T}}(s) = 0$, we conclude that the sequence $(Rx_k)_k$ has a subsequence equivalent to a horizontally block sequence of $X_{(\mathcal{T},L,\varepsilon)}$. Since $L_0 \cap L_s = \emptyset$, once more Lemma 3.6 yields a contradiction. □

Lemma 3.14. *Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1 with $o(\mathcal{T}) > 0$. Let also s be a non-maximal node of $\mathcal{T}$, let $\{s_n \mid n \in \mathbb{N}\}$ be the enumeration of* $\mathrm{succ}(s)$*, and let L_s^0 be an infinite subset of L provided by Proposition 2.5(iv). Let also $T : X_{(\mathcal{T},L,\varepsilon)} \to X_{(\mathcal{T},L,\varepsilon)}$ be a bounded linear operator. Then, there exists a unique real number λ_s and a sequence of compact operators $(C_n)_n$ defined on $X_{(\mathcal{T},L,\varepsilon)}$, such that if $T_s = I_s \circ T \circ I_s$, then $|\lambda_s| \leq \|T_s\|$ and the following is satisfied:*

$$\lim_n \Big\| T_s - \Big(\lambda_s I_s + \sum_{i=1}^{n} (I_{s_i} \circ (T_s - \lambda_s I_s) \circ I_{s_i})\Big) - C_n \Big\| = 0. \tag{3.3}$$

Proof. We consider the projections $P^s_{[1,n]} : X_s \to (\sum_{k=1}^n \oplus (X_{s_k} \oplus \ell_\infty(\Delta_k))_\infty)_\infty$ where $\bigcup_n \Delta_n = \Gamma(\mathcal{T}_s, L_s, \delta_s)$. By Proposition 3.12, there exists a real number λ_s such that $K_s = T_s - \lambda_s I_s$ is horizontally compact. We will show that λ_s is the desired scalar.

To find a sequence of operators $(C_n)_n$ satisfying (3.3), it is enough to show the following: for every $\delta > 0$, there is $n_0 \in \mathbb{N}$ so that, for all $n \geq n_0$, there exists a compact operator C with

$$\Big\| T_s - \Big(\lambda_s I_s + \sum_{i=1}^{n} (I_{s_i} \circ (T_s - \lambda_s I_s) \circ I_{s_i})\Big) - C \Big\| < \delta. \tag{3.4}$$

Fix $\delta > 0$, and let $n_0 \in \mathbb{N}$ such that $\|K_s - K_s \circ P^s_{[1,n]}\| < \delta$ for all $n \geq n_0$. Observe that the operator $C' = K_s \circ P^s_{[1,n_0]} - \sum_{n=1}^{n_0} K_s \circ I_{s_n}$ is compact, and by Lemma 3.13 we have that the operator $\tilde{C} = \sum_{n=1}^{n_0} K_s \circ I_{s_n} - \sum_{n=1}^{n_0} I_{s_n} \circ K_s \circ I_{s_n}$ is compact as well. Setting $C = C' + \tilde{C}$, it is easy to check that C is the desired operator.

In order to show that λ_s is unique, let $\tilde{\lambda}_s$ be a scalar so that there exists a sequence of compact operators $(\tilde{C}_n)_n$ with

$$\lim_n \Big\| T_s - \Big(\tilde{\lambda}_s I_s + \sum_{i=1}^{n} (I_{s_i} \circ (T_s - \tilde{\lambda}_s I_s) \circ I_{s_i})\Big) - \tilde{C}_n \Big\| = 0. \tag{3.5}$$

Assume that $\tilde{\lambda}_s \neq \lambda_s$ and choose a sequence of compact operators $(C_n)_n$ so that (3.3) is satisfied. Combining (3.3) and (3.5), we conclude that

$$\lim_n \Big\| I_s - \Big(\sum_{i=1}^{n} I_{s_i} + \frac{1}{\tilde{\lambda}_s - \lambda_s} (C_n - \tilde{C}_n) \Big) \Big\| = 0.$$

This implies that the identity operator on X_s is horizontally compact, which is absurd.

In order to prove that $|\lambda_s| \leq \|T_s\|$, fix $\delta > 0$, and choose $n_0 \in \mathbb{N}$ and a compact operator C so that (3.4) is satisfied for $n = n_0$. As C is compact, we may choose $x \in P^s_{(n_0,\infty)}(X_s)$ with $\|x\| = 1$ such that $\|Cx\| < \delta$. Considering the above, we have that

$$|\lambda_s| = \|\lambda_s x\| \leq \|T_s(x) - \lambda_s(x) - C(x)\| + \|T_s(x)\| + \|C(x)\| \leq 2\delta + \|T_s\|.$$

Since δ was chosen arbitrarily, the proof is complete. ◻

Corollary 3.15. *Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1. Then, every strictly singular operator $T : X_{(\mathcal{T},L,\varepsilon)} \to X_{(\mathcal{T},L,\varepsilon)}$ is compact.*

Proof. We shall use induction on the rank $o(\mathcal{T})$ of the tree $\mathcal{T}$. The case where $o(\mathcal{T}) = 0$ (i.e., $X_{(\mathcal{T},L,\varepsilon)} = X_{\mathrm{AH}(L)}$ for some $L \subseteq \mathbb{N}$ infinite) follows by the Argyros-Haydon method of construction in [1].

Suppose now that $o(\mathcal{T}) = \alpha$ and that the thesis is true for every S, M, δ such that $o(S) < \alpha$. Let $T : X_{(\mathcal{T},L,\varepsilon)} \to X_{(\mathcal{T},L,\varepsilon)}$ be a strictly singular operator, and let also $\{s_n \mid n \in \mathbb{N}\}$, $(L_n)_{n=0}^{\infty}$, $\delta > 0$ as in Definition 2.1(ii) such that $X_{(\mathcal{T},L,\varepsilon)} = (\sum_{n=1}^{\infty} \oplus X_{s_n})_{\mathrm{AH}(L_0)}$ where $X_{s_n} = X_{(\mathcal{T}_{s_n},L_n,\delta)}$.

Since T is strictly singular, it is not hard to see that T is horizontally compact. Indeed, it follows that, for every closed subspace generated by a bounded horizontally block sequence $(x_n)_n$, there exists a further block subspace Y generated by a block sequence $(y_n)_n$ of $(x_n)_n$ such that the operator $T|_Y$ is compact, and thus horizontally compact. By Proposition 3.12, let λ be a scalar such that $T - \lambda I$ is horizontally compact. It follows that $\lambda = 0$.

Lemma 3.14 gives us that

$$\lim_n \left\| T - \sum_{i=1}^{n} (I_{s_i} \circ T \circ I_{s_i}) - C_n \right\| = 0.$$

Since $o(\mathcal{T}_{s_i}) < \alpha$, the inductive assumption applied on $\mathcal{T}_{s_i}, L_{s_i}, \delta$ yields that, for each i, the strictly singular operator $I_{s_i} \circ T \circ I_{s_i} : X_{s_i} \to X_{s_i}$ is compact; and by the above, the result follows. ◻

Recall that a tree $\mathcal{T}$ becomes a Hausdorff compact topological space if it is equipped with the topology having the sets $\mathcal{T}_t$, $t \in \mathcal{T}$ as a subbase. We are now finally ready to prove the main result of this section, which states that every operator defined on the space $X_{(\mathcal{T},L,\varepsilon)}$ is approximated by a sequence of operators, each one of which is a linear combination of the projections $I_s, s \in \mathcal{T}$ plus a compact operator.

Theorem 3.16. *Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1 and $T : X_{(\mathcal{T},L,\varepsilon)} \to X_{(\mathcal{T},L,\varepsilon)}$ be a bounded linear operator. Then, there exists a unique function $f : \mathcal{T} \to \mathbb{R}$ such that $\|f\|_\infty \leq \|T\|$, and it satisfies the following: if we set $\mu_{\varnothing_\mathcal{T}} = f(\varnothing_\mathcal{T})$, and for* *every node $s \neq \varnothing_{\mathcal{T}}$ we set $\mu_s = f(s) - f(s^-)$, then there exists an increasing sequence $(S_n)_n$ of finite downwards closed subtrees of $\mathcal{T}$ with $\mathcal{T} = \bigcup_n S_n$, and a sequence of compact operators $(C_n)_n$ such that*

$$\lim_n \Big\| T - \sum_{s \in S_n} \mu_s I_s - C_n \Big\| = 0. \tag{3.6}$$

Moreover, the function f is continuous.

Proof. We define the function f as follows. For every non-maximal node s, set $f(s) = \lambda_s$ to be the real number provided by Lemma 3.14, and for every maximal node s, $f(s) = \lambda_s$ to be the unique real number such that $I_s \circ T \circ I_s - \lambda_s I_s$ is a compact operator. By the definition of the function f, and since $\|I_s\| = 1$ (see Proposition 2.5), it immediately follows that $\|f\|_\infty \leq \|T\|$.

For the rest of the proof, we use induction on $o(\mathcal{T})$. If $o(\mathcal{T}) = 0$, then as stated above, $f(\varnothing_{\mathcal{T}})$ is the unique real number λ such that $K = T - \lambda I_{\varnothing_{\mathcal{T}}}$ is compact and (3.6) holds.

Assume $o(\mathcal{T}) = \alpha$ and that the statement is true for every tree $\mathcal{T}', L', \varepsilon'$ as in Definition 2.1 with $o(\mathcal{T}') < \alpha$. Let also $\{t_n \mid n \in \mathbb{N}\}$, $(L_n)_{n=0}^\infty$, $\delta > 0$ as in Definition 2.1(ii) such that $X_{(\mathcal{T},L,\varepsilon)} = (\sum_{n=1}^\infty \oplus X_{t_n})_{\mathrm{AH}(L_0)}$ where we have $X_{t_n} = X_{(\mathcal{T}_{t_n}, L_n, \delta)}$.

Since $o(\mathcal{T}_{t_n}) < \alpha$, we apply the inductive assumption to each $\mathcal{T}_{t_n}, L_n, \delta$, $K_{t_n} = I_{t_n} \circ (T - \mu_{\varnothing_{\mathcal{T}}} I) \circ I_{t_n}$, and we thus obtain a unique continuous function $f_n : \mathcal{T}_{t_n} \to \mathbb{R}$ with $\|f_n\|_\infty \leq \|K_{t_n}\|$, an increasing sequence $(S_i^{t_n})_i$ of finite downwards closed subtrees of $\mathcal{T}_{t_n}$ with $\mathcal{T}_{t_n} = \bigcup_i S_i^{t_n}$, and a sequence of compact operators $(C_i^{t_n})_i$ such that if $\tilde{\mu}_{\varnothing_{\mathcal{T}_{t_n}}} = f_n(t_n)$ and $\tilde{\mu}_s = f_n(s) - f_n(s^-)$ for every $s \in \mathcal{T}_{t_n}$, $s \neq \varnothing_{\mathcal{T}_{t_n}} = t_n$, then the following holds:

$$\lim_i \Big\| T_{t_n} - \sum_{s \in S_i^{t_n}} \tilde{\mu}_s I_s - C_i^{t_n} \Big\| = 0.$$

Observe that, by Lemma 3.14 and the definition of the functions f and f_n, it follows that $f(s) - \mu_{\varnothing_{\mathcal{T}}} = f_n(s)$ and therefore $\mu_s = \tilde{\mu}_s$ for every $s \in \mathcal{T}_{t_n}$. Since $f(\varnothing_{\mathcal{T}}) = \mu_{\varnothing_{\mathcal{T}}}$ such that $T - \mu_{\varnothing_{\mathcal{T}}} I$ is horizontally compact, the uniqueness of f_n implies that f is unique. Moreover, by Lemma 3.14 there exists a sequence of compact operators $(C_n')_n$ such that

$$\lim_n \Big\| T - \Big(\mu_{\varnothing_{\mathcal{T}}} I_{\varnothing_{\mathcal{T}}} + \sum_{i=1}^n K_{t_i} \Big) - C_n' \Big\| = 0.$$

Here, $\mathcal{T} = \bigcup_n \mathcal{T}_{t_n}$, and for each n we set $S_n = \bigcup_{i=1}^n S_n^{t_i}$, $C_n = \sum_{i=1}^n C_i^{t_n} + C_n'$. It follows that $\mathcal{T} = \bigcup_n S_n$, where $(S_n)_n$ is an increasing sequence of finite downwards closed subtrees, and $(C_n)_n$ is a sequence of compact operators defined on

$X_{(\mathcal{T},L,\varepsilon)}$; using a diagonalization argument, we conclude that

$$\lim_n \Big\| T - \sum_{s \in S_n} \mu_s I_s - C_n \Big\| = 0.$$

It remains to show that f is continuous. Observe that it is enough to show that f is continuous on $\varnothing_{\mathcal{T}}$, or equivalently, that $(f(s_n))_n$ converges to $f(\varnothing_{\mathcal{T}})$. We recall by the above that $\|f_n\|_\infty \leq \|I_{t_n} \circ (T - \mu_{\varnothing_{\mathcal{T}}}) \circ I_{t_n}\|$. Since $T - \mu_{\varnothing_T}$ is horizontally compact, it follows that $(f_n)_n$ converges to zero in norm. Since we have $|f_n(s_n)| \leq \|f_n\|_\infty$ and $f_n(s_n) = f(s_n) - \mu_{\varnothing_{\mathcal{T}}}$, the proof is complete. ◻

4. The Calkin Algebras of the Spaces $X_{(\mathcal{T},L,\varepsilon)}$

As was proved in the previous section, every bounded linear operator defined on $X_{(\mathcal{T},L,\varepsilon)}$ is approximated by a sequence of operators, each one of which is a linear combination of the projections I_s, $s \in \mathcal{T}$ plus a compact operator. For a given operator, these linear combinations define a continuous function with domain the tree $\mathcal{T}$, which is used to define a map $\Phi_{(\mathcal{T},L,\varepsilon)} : \mathcal{C}al(X_{(\mathcal{T},L,\varepsilon)}) \to C(\mathcal{T})$ that is an onto bounded algebra homomorphism.

Remark 4.1. By Remark 2.4 and [7, Theorem 5.1], we conclude that the space $X_{(\mathcal{T},L,\varepsilon)}$ has a basis. Also, by Proposition 2.3, the dual $X^*_{(\mathcal{T},L,\varepsilon)}$ is separable, and hence the space of all compact operators on $X_{(\mathcal{T},L,\varepsilon)}$ is separable as well. Theorem 3.16 clearly yields that the space $\langle \{I_s \mid S \in \mathcal{T}\}\rangle + \mathcal{K}(X_{(\mathcal{T},L,\varepsilon)})$ is dense in $\mathcal{L}(X_{(\mathcal{T},L,\varepsilon)})$, and hence the space of all bounded linear operators on $X_{(\mathcal{T},L,\varepsilon)}$ is separable. Therefore, $\mathcal{C}al(X_{(\mathcal{T},L,\varepsilon)})$, the Calkin algebra of $X_{(\mathcal{T},L,\varepsilon)}$, is separable; in particular, the linear span of the set $\{[I_s] \mid s \in \mathcal{T}\}$ is dense in $\mathcal{C}al(X_{(\mathcal{T},L,\varepsilon)})$.

Proposition 4.2. *Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1. We define a map $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)} : \mathcal{L}(X_{(\mathcal{T},L,\varepsilon)}) \to C(\mathcal{T})$ such that, for every operator T, $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}(T)$ is the function provided by Theorem 3.16. Then, $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}$ is a norm-one algebra homomorphism with dense range, and* $\ker \tilde{\Phi}_{(\mathcal{T},L,\varepsilon)} = \mathcal{K}(X_{(\mathcal{T},L,\varepsilon)})$.

Proof. The fact that $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}$ has norm at most one follows from Theorem 3.16, in particular, from the fact that $\|\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}(T)\|_\infty \leq \|T\|$ for every bounded operator T. Also, $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}$ maps the identity map to the constant unit function, and therefore $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}$ has norm one.

We now show that $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}$ has dense range, and that it is an algebra homomorphism on the space $\langle \{I_s \mid s \in \mathcal{T}\}\rangle$. First, by Proposition 2.5(ii), observe that, for each $s \in \mathcal{T}$, the image $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}(I_s)$ coincides with the characteristic function upon the subtree $\mathcal{T}_s$, denoted as $\mathcal{X}_{\mathcal{T}_s}$. From this, it also follows that the image of $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}$ is dense in $C(\mathcal{T})$. Moreover, observe that for $S = \sum_{i=1}^n \lambda_i I_{s_i}$, $T = \sum_{i=1}^m \mu_i I_{t_i}$ we have that $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}(T \circ S) = \tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}(T) \cdot \tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}(S)$.

We now show that $\ker \tilde{\Phi}_{(\mathcal{T},L,\varepsilon)} = \mathcal{K}(X_{(\mathcal{T},L,\varepsilon)})$. First, we observe that, if $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}(T) = 0$, then by Lemma 3.16, T is the limit of a sequence of compact operators, and thus T is compact. Now, let T be a compact operator. Observe that the zero function satisfies the conclusion of Theorem 3.16, and by uniqueness, we conclude that T is in $\ker \tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}$.

Since, by Remark 4.1, the space $\langle\{I_s \mid s \in \mathcal{T}\}\rangle + \mathcal{K}(X_{(\mathcal{T},L,\varepsilon)})$ is dense in $\mathcal{L}(X_{(\mathcal{T},L,\varepsilon)})$, it also follows that $\tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}$ is an algebra homomorphism. ◻

Remark 4.3. Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1. Then, by the above, it follows that the operator, $\Phi_{(\mathcal{T},L,\varepsilon)} : Cal(X_{(\mathcal{T},L,\varepsilon)}) \to C(\mathcal{T})$, defined by the rule $\Phi_{(\mathcal{T},L,\varepsilon)}([T]) = \tilde{\Phi}_{(\mathcal{T},L,\varepsilon)}(T)$, is a 1-1 algebra homomorphism with dense range and $\|\Phi_{(\mathcal{T},L,\varepsilon)}\| = 1$.

Proposition 4.4. *Let $\mathcal{T}, L, \varepsilon$ be as in Definition 2.1. Then, $\Phi_{(\mathcal{T},L,\varepsilon)}$ is a bijection; that is, $Cal(X_{(\mathcal{T},L,\varepsilon)})$ is isomorphic, as a Banach algebra, to $C(\mathcal{T})$. More precisely, we have that $\|\Phi_{(\mathcal{T},L,\varepsilon)}\|\,\|\Phi^{-1}_{(\mathcal{T},L,\varepsilon)}\| \leq 1+\varepsilon$.*

Proof. As $\Phi_{(\mathcal{T},L,\varepsilon)}$ is a norm-one algebra homomorphism with dense range, it is enough to show that it is bounded below, and it is evidently enough to do so for a dense subset of $Cal(X_{(\mathcal{T},L,\varepsilon)})$. We will show that, for every bounded linear operator T on $X_{(\mathcal{T},L,\varepsilon)}$ that is a finite linear combination of the I_s, $s \in \mathcal{T}$, there is a compact operator K on $X_{(\mathcal{T},L,\varepsilon)}$ so that

$$\|T-K\| \leq (1+\varepsilon)\|\Phi_{(\mathcal{T},L,\varepsilon)}([T])\|, \tag{4.1}$$

which of course yields that $\|\,[T]\,\| \leq (1+\varepsilon)\|\Phi_{(\mathcal{T},L,\varepsilon)}([T])\|$.

Let us first make a few simple observations:

(i) For every $s \in \mathcal{T}$, we have that $\Phi_{(\mathcal{T},L,\varepsilon)}([I_s])$ is equal to $\mathcal{X}_{\mathcal{T}_s}$, the characteristic function of the clopen set $\mathcal{T}_s$.

(ii) For all real numbers $(\lambda_s)_{s\in\mathcal{T}}$, finitely many of which are not zero, if $f = \sum_{s\in\mathcal{T}} \lambda_s \mathcal{X}_{\mathcal{T}_s}$, we have that $\|f\| = \max_{s\in\mathcal{T}} \Big|\sum_{\varnothing_\mathcal{T} \leq t \leq s} \lambda_t\Big|$.

The first observation follows trivially from the definition of the map $\Phi_{(\mathcal{T},L,\varepsilon)}$, while the second one is an immediate consequence of the fact that $f(s) = \sum_{\varnothing_\mathcal{T}\leq t\leq s}\lambda_t$ for every $s \in \mathcal{T}$.

By observations (i) and (ii), it is enough to show that if $T = \sum_{s\in\mathcal{T}} \lambda_s I_s$, then there is a compact operator K so that

$$\|T-K\| \leq (1+\varepsilon)\max_{s\in\mathcal{T}} \Big|\sum_{\varnothing_\mathcal{T}\leq t\leq s} \lambda_t\Big|.$$

We are now ready to prove (4.1), by induction on $o(\mathcal{T})$. If $o(\mathcal{T}) = 0$, then $\mathcal{T} = \{\varnothing_\mathcal{T}\}$, and $\Phi_{(\mathcal{T},L,\varepsilon)}$ is an isometry onto the one-dimensional Banach space $C(\mathcal{T})$. Assume $o(\mathcal{T}) = \alpha$, and that the statement is true for every tree $\mathcal{T}', L', \varepsilon'$ as in Definition 2.1 with $o(\mathcal{T}') < \alpha$. Let also $\{t_k \mid k \in \mathbb{N}\}$, $(L_k)_{k=0}^{\infty}$, $\delta > 0$ be as in Definition 2.1(ii) such that $X_{(\mathcal{T},L,\varepsilon)} = (\sum_{k=1}^{\infty} \oplus X_{t_k})_{\mathrm{AH}(L_0)}$ where $X_{t_k} = X_{(\mathcal{T}_{t_k},L_k,\delta)}$.

Let $[T] = \sum_{s\in\mathcal{T}} \lambda_s[I_s]$ be a finite linear combination of the $[I_s]$, $s \in \mathcal{T}$. Choose $n \in \mathbb{N}$ so that $\lambda_s = 0$ for every $s \notin \{\varnothing_{\mathcal{T}}\} \cup (\bigcup_{k=1}^n \mathcal{T}_{t_k})$. For $k = 1, \dots, n$, define the operators $T'_k = \sum_{s\in\mathcal{T}_{t_k}} \lambda_s I_s$ and $T_k = \lambda_{\varnothing_{\mathcal{T}}} I_{t_k} + T'_k$. Observe that

$$T_k = (\lambda_{\varnothing_{\mathcal{T}}} + \lambda_{t_k}) I_{t_k} + \sum_{s>t_k} \lambda_s I_s, \tag{4.2}$$

and rewrite T as follows:

$$\begin{aligned} T &= \lambda_{\varnothing_{\mathcal{T}}} P_{[1,n]} + \sum_{k=1}^n T_k + \lambda_{\varnothing_{\mathcal{T}}} P_{(n,+\infty)} \\ &= \lambda_{\varnothing_{\mathcal{T}}} \sum_{k=1}^n (P_k - I_{t_k}) + \sum_{k=1}^n (\lambda_{\varnothing_{\mathcal{T}}} I_{t_k} + T_k) + \lambda_{\varnothing_{\mathcal{T}}} P_{(n,+\infty)} \\ &= \sum_{k=1}^n T_k + \lambda_{\varnothing_{\mathcal{T}}} P_{(n,+\infty)} + \lambda_{\varnothing_{\mathcal{T}}} \sum_{k=1}^n (P_k - I_{t_k}). \end{aligned} \tag{4.3}$$

For $k = 1, \dots, n$, we write $T_k = \tilde{S}_k$ where S_k is an operator on X_{t_k} (see Remark 1.12). The inductive assumption and (4.2) yield that there is a compact operator C_k on X_{t_k} so that

$$\|T_k - \tilde{C}_k\| \le (1+\delta) \max_{s\in\mathcal{T}_{t_k}} \Big|\lambda_{\varnothing_{\mathcal{T}}} + \sum_{t_k\le t\le s} \lambda_t\Big|. \tag{4.4}$$

By using the above and applying Proposition 1.17, there is a compact operator K' on $X_{(\mathcal{T},L,\varepsilon)}$ so that if

$$S = \sum_{k=1}^n (T_k - \tilde{C}_k) + \lambda_{\varnothing_{\mathcal{T}}} P_{(n,+\infty)} - K',$$

then

$$\begin{aligned} \|S\| &\le (1+\delta) \max\Big\{ \max_{1\le k\le n} (1+\delta) \max_{s\in\mathcal{T}_{t_k}} |\lambda_{\varnothing_{\mathcal{T}}} + \sum_{t_k\le t\le s} \lambda_t|, |\lambda_{\varnothing_{\mathcal{T}}}|\Big\} \\ &\le (1+\delta)^2 \max_{s\in\mathcal{T}} \Big| \sum_{\varnothing_{\mathcal{T}}\le t\le s} \lambda_t \Big|. \end{aligned} \tag{4.5}$$

Finally, set

$$K = \sum_{k=1}^n \tilde{C}_k + K' - \lambda_{\varnothing_{\mathcal{T}}} \sum_{k=1}^n (P_k - I_{t_k}).$$

By (4.3), we have that $T - K = S$. By Remark 1.11, the choice of δ, and (4.5), we conclude that K is the desired compact operator. ◻

We conclude this section with some remarks concerning the space of bounded operators on $X_{(\mathcal{T},L,\varepsilon)}$.

Remark 4.5. The space $\mathcal{L}(X_{(\mathcal{T},L,\varepsilon)})$ does not contain an isomorphic copy of c_0. Indeed, towards a contradiction, assume there is a sequence of operators $(T_k)_k$ on $X_{(\mathcal{T},L,\varepsilon)}$ equivalent to the unit vector basis of c_0. It follows that, for every $x \in X_{(\mathcal{T},L,\varepsilon)}$ and $x^* \in X^*_{(\mathcal{T},L,\varepsilon)}$, the series $\sum_k x^* T_k x$ converges absolutely. By Remark 2.2, c_0 does not embed into $X_{(\mathcal{T},L,\varepsilon)}$. A well-known theorem by Bessaga and Pełczyński yields that, for every $x \in X_{(\mathcal{T},L,\varepsilon)}$, the series $\sum_k T_k x$ converges unconditionally, which implies that the operator $R : \ell_\infty(\mathbb{N}) \to \mathcal{L}(X_{(\mathcal{T},L,\varepsilon)})$ with $R(a_k)_k = SOT - \sum_k a_k T_k$ is well defined and bounded. By [10, Proposition 1.2], there is an infinite subset L of $\mathbb{N}$ so that R restricted onto $\ell_\infty(L)$ is an isomorphic embedding. Remark 4.1 yields a contradiction.

Remark 4.6. The quotient map $Q : \mathcal{L}(X_{(\mathcal{T},L,\varepsilon)}) \to \mathcal{C}al(X_{(\mathcal{T},L,\varepsilon)})$ is strictly singular. Indeed, by Proposition 4.4, $\mathcal{C}al(X_{(\mathcal{T},L,\varepsilon)})$ is isomorphic to $C(\mathcal{T})$. If $o(\mathcal{T}) = 0$, then $\mathcal{C}al(X_{(\mathcal{T},L,\varepsilon)})$ is one-dimensional, and the result trivially holds. Otherwise, $\mathcal{T}$ is an infinitely countable compact metric space: that is, $C(\mathcal{T})$ is c_0 saturated, and hence so is $\mathcal{C}al(X_{(\mathcal{T},L,\varepsilon)})$. Remark 4.5 yields that the quotient map Q is strictly singular.

Remark 4.7. In [11], a space $\mathfrak{X}_\infty$ is presented whose Calkin algebra is ℓ_1. It follows that the space of compact operators on $\mathfrak{X}_\infty$ is complemented in the space of bounded operators. This is also the case for $X_{(\mathcal{T},L,\varepsilon)}$ if $o(\mathcal{T}) = 0$, as the compact operators are of co-dimension one in the space of bounded operators. However, if $o(\mathcal{T}) > 0$, this is no longer the case (i.e., $\mathcal{K}(X_{(\mathcal{T},L,\varepsilon)})$ is not complemented in $\mathcal{L}(X_{(\mathcal{T},L,\varepsilon)})$). Indeed, if we assume there is a subspace Y of $\mathcal{L}(X_{(\mathcal{T},L,\varepsilon)})$ so that $\mathcal{L}(X_{(\mathcal{T},L,\varepsilon)}) = \mathcal{K}(X_{(\mathcal{T},L,\varepsilon)}) \oplus Y$, the open mapping theorem implies that $Q|_Y : Y \to \mathcal{C}al(X_{(\mathcal{T},L,\varepsilon)})$ is an onto isomorphism. Since $o(\mathcal{T}) > 0$, we conclude that Y is necessarily infinite dimensional, which contradicts Remark 4.6.

Remark 4.8. By Remark 4.5, $\mathcal{K}(X_{(\mathcal{T},L,\varepsilon)})$ does not contain c_0. If, moreover, $o(\mathcal{T}) > 0$, then by Remark 4.7, $\mathcal{K}(X_{(\mathcal{T},L,\varepsilon)})$ is not complemented in $\mathcal{L}(X_{(\mathcal{T},L,\varepsilon)})$. This is related to Question B from [4], and is, to our knowledge, the first known example of a Banach space where the space of compact operators does not contain c_0 and is at the same time not complemented in the space of bounded operators.

Remark 4.9. Corollary 3.15 and Remark 4.6 imply that, if $o(\mathcal{T}) > 1$, then for every $\delta > 0$ there is a non-strictly singular operator defined on $X_{(\mathcal{T},L,\varepsilon)}$ that is δ-close to a compact one. For example, if $\{s_n \mid n \in \mathbb{N}\}$ are the immediate successors of the root of $\mathcal{T}$, then Remark 4.5 implies there is a finite subset F of $\mathbb{N}$ so that $\|\sum_{k\in F} I_{s_k}\| \geq 2/\delta$. Proposition 1.17 yields that $T = \|\sum_{k\in F} I_{s_k}\|^{-1}(\sum_{k\in F} I_{s_k})$ is such an operator.

5. Main Result

In this final section, we conclude that, for every countable compact metric space K, the algebra $C(K)$ is homomorphic to the Calkin algebra of some Banach space.

Theorem 5.1. *Let K be a countable compact metric space. Then, there exists a $\mathcal{L}_\infty$-space X, with X^* isomorphic to ℓ_1, and there also exists a norm-one algebra* *homomorphism* $\Phi : Cal(X) \to C(K)$ *that is one-to-one and onto. Even more, for every* $\varepsilon > 0$*, the space* X *can be chosen so that* $\|\Phi\|\,\|\Phi^{-1}\| \le 1 + \varepsilon$.

Proof. A well-known theorem by Sierpinski and Mazurkiewicz implies there exist an ordinal number α and a natural number n so that K is homeomorphic to the ordinal number $\omega^\alpha n$. Note that ω^α is homeomorphic to a tree $\mathcal{T}$ (which is well founded and has a unique root; and every non-maximal node of it has countable infinitely many immediate successors), and that this tree is of order α.

Let $L_1, \ldots, L_n$ be pairwise disjoint infinite subsets of the natural numbers, $\varepsilon > 0$, and consider the spaces $X_{(\mathcal{T},L_1,\varepsilon)}, \ldots, X_{(\mathcal{T},L_n,\varepsilon)}$. We define the space $X = (\sum_{i=1}^n \oplus X_{(\mathcal{T},L_i,\varepsilon)})_\infty$, and claim it has the desired properties. By Proposition 2.3 and Remark 2.4, it easily follows that X is a $\mathcal{L}_\infty$-space with X^* isomorphic to ℓ_1. Also, Proposition 3.10 and the fact that the sets $L_1, \ldots, L_n$ are pairwise disjoint easily yield that $Cal(X)$ is isometric, as a Banach algebra, to $(\sum_{i=1}^n \oplus Cal(X_{(\mathcal{T},L_i,\varepsilon)}))_\infty$. This, by Proposition 4.4, is $(1+\varepsilon)$-isomorphic as a Banach algebra to $(\sum_{i=1}^n \oplus C(\omega^\alpha))_\infty$, which is of course isometric as a Banach algebra to $C(\omega^\alpha n)$; thus, this yields the desired result. ◻

Discovering the variety of Banach algebras that can occur as Calkin algebras is a topic we believe should be investigated further. We point out here that all known examples of Calkin algebras of Banach spaces are either finite dimensional or non-reflexive.

Question 1. *Does there exists a Banach space whose Calkin algebra is reflexive and infinite dimensional?*

It is worth mentioning that the method used in this paper does not seem to be able to provide an example of a Banach space whose Calkin algebra is a $C(K)$ space for K uncountable.

Question 2. *Does there exists a Banach space whose Calkin algebra is isomorphic, as a Banach algebra, to* $C(K)$ *for an uncountable compact space* K*?*

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