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Algebras of Diagonal Operators of the Form Scalar-Plus-Compact Are Calkin Algebras

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Abstract. For every Banach space X with a Schauder basis, consider the Banach algebra $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$ of all diagonal operators that are of the form $\lambda I + K$. We prove that $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$ is a Calkin algebra, that is, there exists a Banach space $\mathcal{Y}_X$ such that the Calkin algebra of $\mathcal{Y}_X$ is isomorphic as a Banach algebra to $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$. Among other applications of this theorem, we obtain that certain hereditarily indecomposable spaces and the James spaces J_p and their duals endowed with natural multiplications are Calkin algebras; that all non-reflexive Banach spaces with unconditional bases are isomorphic as Banach spaces to Calkin algebras; and that sums of reflexive spaces with unconditional bases with certain James–Tsirelson type spaces are isomorphic as Banach spaces to Calkin algebras.

Introduction

This paper is aiming to contribute to the ongoing effort of understanding the types of unital Banach algebras A that may occur as Calkin algebras, that is, those for which there exists a Banach space X so that A is isomorphic as a Banach algebra to the Calkin algebra of X. This is the quotient algebra $\mathcal{C}al(X) = \mathcal{L}(X)/\mathcal{K}(X)$, where $\mathcal{L}(X)$ denotes the unital Banach algebra of all bounded linear operators on X and $\mathcal{K}(X)$ denotes the ideal of all compact ones. This unital Banach algebra was introduced by Calkin in [Cal] in the 1940s, at first only for X being the Hilbert space. The topic of Calkin algebras of general Banach spaces later gathered attention as well. A classical result from the 1950s, due to Atkinson (see [At]), is that a bounded linear operator on a Banach space is Fredholm precisely when its equivalence class in the Calkin algebra is invertible. Another example from the same era is the observation of Yood in [Y] that, unlike the algebra $\mathcal{L}(X)$, in certain cases the Calkin algebra of a Banach space may fail to be semi-simple. This is true in particular for the space L_1.

The origins of the topic studied herein can be traced to only much later, namely to 2011, when the first example of a Banach space $\mathfrak{X}_{\mathrm{AH}}$ with the scalar-plus-compact property was presented by Argyros and Haydon in [AH]. On this space every bounded linear operator is a scalar multiple of the identity plus a compact operator, which means that the Calkin algebra of $\mathfrak{X}_{\mathrm{AH}}$ is one-dimensional. This

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was in fact the first time a Calkin algebra of a Banach space could be explicitly described. Of particular importance in the construction is that $\mathfrak{X}_{\mathrm{AH}}$ is a hereditarily indecomposable (HI) $\mathscr{L}_\infty$-Bourgain–Delbaen space. In the years that followed a number of examples of Calkin algebras have appeared that can be explicitly described in terms of classical Banach algebras. All these examples are in one way or another tightly knit together with the theory of HI spaces, the theory of $\mathscr{L}_\infty$-spaces, or both. In his PhD thesis Tarbard [Ta] combined the technique of Gowers and Maurey from [GM] with the technique from [AH] to construct for every $n \in \mathbb{N}$ a Banach space the Calkin algebra of which coincides with all $n \times n$ upper triangular Toeplitz matrices and also a Banach space the Calkin algebra of which coincides isometrically with the convolution algebra of $\ell_1(\mathbb{N}_0)$. Tarbard posed the question of what unital Banach algebras can be realized as Calkin algebras. Alternatively, he proposed that one should seek for obstructions that would prevent a unital Banach algebra from being a Calkin algebra. This paper is focused on studying the first question and as of yet no results concerning the second one exist. A contribution to the first one was made by Kania and Laustsen in [KL] where they observed that carefully manipulating and taking finite direct sums of powers of appropriate versions of the space $\mathfrak{X}_{\mathrm{AH}}$ can lead to something surprising: all finite dimensional semi-simple complex algebras are Calkin algebras. In particular, for any natural numbers $m_1, \dots, m_n$, the algebra $\mathbb{M}_{m_1}(\mathbb{K}) \oplus \cdots \oplus \mathbb{M}_{m_n}(\mathbb{K})$ endowed with point-wise multiplication is a Calkin algebra. Here, $\mathbb{K}$ denotes the scalar field and $\mathbb{M}_k(\mathbb{K})$ denotes the algebra of all $k \times k$ matrices over $\mathbb{K}$. A remark worth making is that Tarbard's aforementioned finite dimensional examples of Calkin algebras are not semi-simple. In the infinite dimensional setting the first two authors and Zisimopoulou proved in [MPZ] that for every countable compactum K the space $C(K)$ is a Calkin algebra. A noteworthy reason for which one may be interested in these examples is that many of them provide insight into ideals of $\mathcal{L}(X)$. Indeed, all aforementioned algebras are Calkin algebras of spaces with the bounded approximation property and information about the ideal structure of the Calkin algebra can be lifted to study the ideals of the corresponding $\mathcal{L}(X)$ space. For a detailed exposition of this topic, we refer the interested reader to the introduction of [KL].

The motivation for the present paper stems from [MPZ, Question 1, p. 66] of the existence of a Banach space with an infinite dimensional and reflexive Calkin algebra. This is indeed interesting as all infinite dimensional aforementioned examples are either isomorphic to ℓ_1 or c_0-saturated and thus on the far opposite side of being reflexive. This question is difficult to answer and a space with a reflexive Calkin algebra cannot have too many complemented subspaces. Instead, we were interested in investigating whether we could find a quasi-reflexive Calkin algebra. Recall that a Banach space is called quasi-reflexive (of order one) if its canonical image in its second dual is of codimension one. While affirmatively answering this question, we were able to identify a rather large variety of explicitly described spaces that can be realized as Calkin algebras that are, from a Banach spaces perspective, truly different to the previously understood examples. Although many of them admit unconditional bases, one example is HI. The main result of the present paper has the following statement.

THEOREM I. *Let X be a Banach space with a Schauder basis. Then there exists a Banach space $\mathcal{Y}_X$ such that the Calkin algebra of $\mathcal{Y}_X$ is isomorphic as a Banach algebra to $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$.*

In fact, for every $\varepsilon > 0$, the space $\mathcal{Y}_X$ can be constructed so that the corresponding Banach algebra isomorphism $\Phi : \mathcal{Y}_X \to \mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$ satisfies $\|\Phi\|\|\Phi^{-1}\| \leq 1 + \varepsilon$.

Using a Theorem of Argyros, Deliyanni, and the third author, that in special cases explicitly describes the diagonal operators of a Banach space with a basis in terms of its dual (see [ADT]), we describe several examples of spaces $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$ which are, by virtue of Theorem I, examples of Calkin algebras as well. The unital Banach algebra $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$ is always commutative and semi-simple. The first example is a hereditarily indecomposable Banach algebra $\mathfrak{X}_{\text{ADT}}$ from [ADT] that is additionally quasi-reflexive of order one.

THEOREM II. *There exists a hereditarily indecomposable Calkin algebra that is quasi-reflexive of order one.*

The possibility of such extreme behavior of the quotient $\mathcal{L}(X)/\mathcal{K}(X)$ contrasts with the more canonical one of $\mathcal{L}(X)$. The latter space is always decomposable, containing complemented copies of both X and X^*.

James' classical space J_p from [J] is, for each $1 < p < \infty$, the Banach space consisting of all scalar sequences $(a_i)_i$ for which the quantity

$$\|(a_i)_i\|^p = \sup_{(E_k)_k} \sum_k \left| \sum_{i \in E_k} a_i \right|^p,$$

where the supremum is taken over all disjoint collections of intervals of $\mathbb{N}$, is finite. The spaces J_p are quasi-reflexive of order one. It was first observed by Andrew and Green in [AG] that J_p, after appropriate renorming, becomes a nonunital Banach algebra when endowed with coordinate-wise multiplication with respect to the basis $e_1, e_2 - e_1, e_3 - e_2, \ldots$. We denote the unitization of James space by $\mathbb{R}e_\omega \oplus J_p$ for $1 < p < \infty$.

THEOREM III. *The spaces $\mathbb{R}e_\omega \oplus J_p$, $1 < p < \infty$ are Calkin algebras.*

Bellenot, Haydon, and Odell in [BHO] extended the definition of James, based on the unit vector basis of ℓ_p, to an arbitrary space X with an unconditional basis to define the space $J(X)$, the "jamesification" of X. The space $J(X)$ is quasi-reflexive of order one whenever X is reflexive. As it so happens this space is a nonunital Banach algebra as well, the unitization of which we denote by $\mathbb{R}e_\omega \oplus J(X)$. Furthermore, a special subspace of $J(X)^*$, which we denote by $\mathcal{J}_*(X)$ and which coincides with $J(X)^*$ when X does not contain ℓ_1, is a separable unital Banach algebra.

THEOREM IV. *For any space X with a normalized unconditional basis, the spaces $\mathbb{R}e_\omega \oplus J(X)$ and $\mathcal{J}_*(X)$ are Calkin algebras.*

Next we turn our attention to spaces with unconditional bases endowed with coordinate-wise multiplication. We study these spaces themselves as Banach algebras. We do not prove that they are Calkin algebras but they always embed in them as complemented ideals. In what follows "$\oplus$" denotes the direct sum of Banach spaces.

THEOREM V. *Let $\mathcal{A}$ be a Banach space with a normalized unconditional basis endowed with coordinate-wise multiplication and $\mathcal{B}$ be a Banach algebra. In the cases described in the list below, there exists a Banach space $\mathcal{Y}$ such that the Calkin algebra $\mathcal{C}al(\mathcal{Y})$ contains an ideal $\tilde{\mathcal{A}}$ isomorphic as a Banach algebra to $\mathcal{A}$ and a subalgebra $\tilde{\mathcal{B}}$ isomorphic as a Banach algebra to $\mathcal{B}$ such that $\mathcal{C}al(\mathcal{Y}) = \tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}$.*

(a) *The space $\mathcal{B}$ is $C(\omega)$, the space of convergent scalar sequences with point-wise multiplication.*
(b) *The space $\mathcal{B}$ is* bv_1, *the space of all scalar sequences of bounded variation with point-wise multiplication.*
(c) *The space $\mathcal{A}$ does not contain an isomorphic copy of c_0 and $\mathcal{B}$ is the space $J(T_{\mathcal{M}}^{1/2})^*$, where $T_{\mathcal{M}}^{1/2}$ is the Tsirelson space over an appropriate regular family $\mathcal{M}$.*
(d) *The space $\mathcal{A}$ does not contain an isomorphic copy of ℓ_1 and $\mathcal{B}$ is the space $\mathbb{R}e_\omega \oplus J(T_{\mathcal{M}}^{1/2})$, where $T_{\mathcal{M}}^{1/2}$ is the Tsirelson space over an appropriate regular family $\mathcal{M}$.*

In statement (c) the complexity, that is, the Cantor–Bendixson index, of $\mathcal{M}$ depends on the Szlenk index of the natural predual of $\mathcal{A}$ and in statement (d) it depends on the Szlenk index of $\mathcal{A}$. Of course, the spaces $C(\omega)$ and bv_1 are isomorphic as Banach spaces to c_0 and to ℓ_1 respectively. By a well-known theorem of James, any nonreflexive Banach space X with an unconditional basis is either isomorphic to $X \oplus c_0$ or to $X \oplus \ell_1$. Consequently statements (a) and (b) yield something interesting.

THEOREM VI. *Every nonreflexive Banach space X with a normalized unconditional basis is isomorphic as a Banach space to a Calkin algebra* (*that contains a complemented ideal isomorphic as a Banach algebra to X endowed with coordinate-wise multiplication*).

For reflexive Banach spaces, we do not obtain the same result; however, since the space $J(T_{\mathcal{M}}^{1/2})$ is quasi-reflexive of order one from statement (c) or (d), we may deduce the following. In the statement that follows multiplication is also coordinate-wise with respect to the given unconditional basis.

THEOREM VII. *Every reflexive Banach space with a normalized unconditional basis is isomorphic as a Banach algebra to a complemented ideal of a separable quasi-reflexive Calkin algebra.*

It is worth mentioning that in certain cases, for example, when the space X is super-reflexive, in statements (c) and (d) of Theorem V the space $J(T_{\mathcal{M}}^{1/2})$ may be replaced with the space J_p for appropriate $1 < p < \infty$. Going back to our initial question of the existence of quasi-reflexive Calkin algebras, we also observe that these exist for any finite order. That is, for any $n \in \mathbb{N}$, there exists a Calkin algebra that is quasi-reflexive of order n. There are a few more examples that are mentioned throughout Section 1.

The technically most challenging part of this paper is the proof of Theorem I, that is, given a Banach space X with a Schauder basis, the construction of a space $\mathcal{Y}_X$ the Calkin algebra of which is isomorphic as a Banach algebra to $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$. The definition of the space $\mathcal{Y}_X$ is based on a method of Zisimopoulou from [Z] for defining direct sums $\mathcal{Z} = (\sum_k \oplus X_k)_{\text{AH}}$ of a sequence of Banach spaces $(X_k)_k$, where the outside norm is in turn based on the Argyros–Haydon space from [AH]. The main feature of the construction from [Z] is that, under certain assumptions, every bounded linear operator $T : \mathcal{Z} \to \mathcal{Z}$ is a scalar operator plus an operator that vanishes on block sequences, namely a horizontally compact operator. A variation of this method was iterated transfinitely in [MPZ] to show that for a countable compactum K the space $C(K)$ is a Calkin algebra. In this paper we define an X-Bourgain–Delbaen–Argyros–Haydon direct sum $\mathcal{Y}_X = (\sum_k \oplus X_k)_{\text{AH}}^X$ of a sequence of Argyros–Haydon Banach spaces $(X_k)_k$. This direct sum is designed so that the space X is crudely finitely representable in an appropriate block way. This is performed in such a manner that the diagonal operators on X can be viewed as compact perturbations of diagonal operators with respect to the decomposition $(X_k)_k$ of $\mathcal{Y}_X$. The result is that the space $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$ embeds into $\mathcal{C}al(\mathcal{Y}_X)$. The proof that this embedding is onto goes through all the delicate intricacies of the Argyros–Haydon construction, suitably modified to the context of the present setting. The main difference is that so-called rapidly increasing block sequences have to be defined so as to take into consideration the "X-part" of the direct sum $(\sum_k \oplus X_k)_{\text{AH}}^X$. In the end, the main theorem about the operators on the space is that they are of the form

$$T = a_0 I + \lim_n \left(\sum_{k=1}^{n} a_k I_k + K_n \right), \tag{$\mathfrak{a}$}$$

where the I_k are projections onto the spaces X_k and K_n are compact operators. The definition of the space $\mathcal{Y}_X$ is presented comprehensively, and most proofs are explained thoroughly. We have chosen to leave out a small number of details that are nearly exactly identical to proofs from other papers, for which we provide references.

The paper can be viewed as being divided into two main parts. The first part consists of Section 1. In this section the basics around the space $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$

are discussed. Several examples of these spaces are presented, namely those mentioned in the introduction and a few others. As a result of Theorem I, they are all Calkin algebras. Several results concerning James spaces and Tsirelson spaces are proved and utilized. To the most part the tools used are relatively elementary and a reading of this section should not be too challenging to the reader who is familiar with classical Banach space theory. With the exception of concluding Section 8 the remaining Sections 2–7 focus on defining the space $\mathcal{Y}_X$ and proving the necessary properties to achieve the desired conclusion about its Calkin algebra. Section 2 concerns the definition of a "first stage" $\mathcal{X}$ of the final direct sum $\mathcal{Y}_X = (\sum_k \oplus X_k)_{\mathrm{AH}}^X$ that does not involve the Bourgain–Delbaen–Argyros–Haydon part. Although X is finitely block represented in $\mathcal{X}$, every normalized block sequence has a subsequence that is equivalent to the unit vector basis of c_0. We state most properties of $\mathcal{X}$ without proof because we do not evoke them directly. Section 3 is devoted to precisely defining the space $\mathcal{Y}_X = (\sum_k \oplus X_k)_{\mathrm{AH}}^X$ and determining its most fundamental properties. In Section 4 we prove that the Calkin algebra of $\mathcal{Y}_X$ is $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$. This is done by assuming that ($\mathfrak{a}$) holds, the proof of which is the objective of Sections 5–7. In Section 5 rapidly increasing sequences (RIS) are defined. Here, RIS have the additional property that the X part in the X-Bourgain–Delbaen–Argyros–Haydon sum completely vanishes on them allowing them to be treated in the same manner as in [AH] and [Z]. In the same section the basic inequality is proved and it is shown that operators that vanish on RIS are horizontally compact. In Section 6 it is proved that bounded operators on the space are scalar multiples of the identity plus a horizontally compact operator, and in Section 7 ($\mathfrak{a}$) is finally proved. Section 8 contains several remarks and open problems.

1. The Spaces $\mathcal{L}_{\mathrm{diag}}(X)$ and $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$

A sequence $(e_i)_i$ in a Banach space X is called a Schauder basis of X if every element can be represented uniquely as $x = \sum_{i=1}^\infty a_i e_i$, where the convergence is in the norm topology. Then the natural projections $P_n(\sum_{i=1}^\infty a_i e_i) = \sum_{i=1}^n a_i e_i$, $n \in \mathbb{N}$, are uniformly bounded by some $C \geq 1$, which is called the monotone constant of $(e_i)_i$. The sequence $(e_i^*)_i$ in X^* defined by $e_i^*(e_j) = \delta_{i,j}$ is called the biorthogonal sequence of $(e_i)_i$. For each $x \in X$, we define the support of x to be the subset of $\mathbb{N}$, $\mathrm{supp}(x) = \{i : e_i^*(x) \neq 0\}$ and the range of x to be the smallest interval of $\mathbb{N}$ containing the support of x. A sequence $(x_k)_k$ in X is called a block sequence if the supports of the corresponding vectors are successive subsets of $\mathbb{N}$.

Given a Banach space X with a Schauder basis $(e_i)_i$, a bounded linear operator $T : X \to X$ is called diagonal if, for every $i \neq j \in \mathbb{N}$, we have $e_j^*(Te_i) = 0$. We denote the subspace of $\mathcal{L}(X)$ consisting of all diagonal operators by $\mathcal{L}_{\mathrm{diag}}(X)$. The simplest diagonal operator is the identity I. The "building blocks of diagonal operators" may be considered the operators $(e_i \otimes e_i)_i$ defined by $e_i^* \otimes e_i(x) =$

$e_i^*(x)e_i$. Every diagonal operator T then can be written as

$$T = \mathrm{SOT} - \sum_{i=1}^{\infty} e_i^*(Te_i)e_i^* \otimes e_i. \tag{1}$$

We denote the space of compact diagonal operators by $\mathcal{K}_{\mathrm{diag}}(X)$. That is, $\mathcal{K}_{\mathrm{diag}}(X) = \mathcal{L}_{\mathrm{diag}}(X) \cap \mathcal{K}(X)$. It is straightforward to check that a diagonal operator T is compact if and only if the convergence of the sum in (1) is in the norm topology. It follows that $(e_i^* \otimes e_i)_i$ is a Schauder basis for $\mathcal{K}_{\mathrm{diag}}(X)$. It is worth pointing out that $\mathcal{L}_{\mathrm{diag}}(X)$ is isomorphic to the dual of the space V spanned by the biorthogonal sequence of $(e_i^* \otimes e_i)_i$ in $(\mathcal{K}_{\mathrm{diag}}(X))^*$.

We are particularly interested in the subspace of $\mathcal{L}_{\mathrm{diag}}(X)$ consisting of all operators of the form $T = \lambda I + K$ with $K \in \mathcal{K}_{\mathrm{diag}}(X)$. We naturally denote this space by $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$. This space endowed with operator composition is a commutative unital Banach algebra. An operator T in $\mathcal{L}_{\mathrm{diag}}(X)$ is in $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$ if and only if

$$\lim_m \lim_n \left\| \sum_{i=m}^{n} (e_i^*(Te_i) - e_m^*(Te_m))e_i^* \otimes e_i \right\| = 0. \tag{2}$$

It is a well-known and easy to prove fact that if $(e_i)_i$ is unconditional, then $\mathcal{L}_{\mathrm{diag}}(X)$ is naturally isomorphic to ℓ_∞. Then $\mathcal{K}_{\mathrm{diag}}(X)$ is naturally isomorphic to c_0 (endowed with the unit vector basis) and $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$ is naturally isomorphic to c, the space of convergent real sequences. As we will see in several examples, if the basis is not unconditional, then more interesting things may occur.

1.1. Ideals of $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$

We observe that the ideals in $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$ are the same as in the space $c = C(\omega)$. Here, ω is the first infinite ordinal number. For an ordinal number α, we follow the common convention by which the space of all continuous functions on the compact set $[1, \alpha]$ is denoted by $C(\alpha)$.

PROPOSITION 1.1. *Let X be a Banach space with a Schauder basis. For every $T \in \mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$ and $i \in \mathbb{N}$, define $\lambda_{T,i} = e_i^*(Te_i)$ and also define $\lambda_{T,\omega} = \lim_i \lambda_{T,i}$ (which exists by (2)). For every closed subset L of $[1, \omega]$, define*

$$\mathcal{A}_L = \{T \in \mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X) : \lambda_{T,\kappa} = 0 \text{ for all } \kappa \in L\}.$$

Then the closed ideals of $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$ are precisely the spaces $\mathcal{A}_L$ for all closed subsets L of $[1, \omega]$.

Proof. We start by making some elementary observations that follow from the fact that $(e_i^* \otimes e_i)_i$ is a Schauder basis of $\mathcal{K}_{\mathrm{diag}}(X)$. If $\omega \in L$, then $\mathcal{A}_L$ is the closed linear span of $(e_i^* \otimes e_i^*)_{i \in \mathbb{N} \setminus L}$. If $\omega \notin L$, then L has an upper bound and $\mathcal{A}_L$ is the closed linear span of $P_{\mathbb{N} \setminus L} = I - \sum_{i \in L} e_i^* \otimes e_i$ together with $(e_i^* \otimes e_i^*)_{i \in \mathbb{N} \setminus L}$.

Given a closed ideal $\mathcal{A}$ define $L = \{\kappa \in [1, \omega] : \lambda_{T,\kappa} = 0$ for all $T \in \mathcal{A}\}$. Clearly, L is closed and $\mathcal{A} \subset \mathcal{A}_L$. By the fact that $\mathcal{A}$ is an ideal, it is easy to see that $e_i^* \otimes e_i$ is in $\mathcal{A}$ for all $i \in \mathbb{N} \setminus L$. In the case $\omega \in L$ this easily yields

$\mathcal{A}_L = [(e_i^* \otimes e_i)_{i\in\mathbb{N}\setminus L}] \subset \mathcal{A}$, that is, $\mathcal{A} = \mathcal{A}_L$. For the case $\omega \notin L$, it is sufficient to show that $P_{\mathbb{N}\setminus L} = I - \sum_{i\in L} e_i^* \otimes e_i$ is in $\mathcal{A}$. To that end, let T be any element in $\mathcal{A}$ with $\lambda_{T,\omega} \neq 0$. Then, by (2),

$$\begin{aligned} T &= \lambda_{T,\omega} I + \sum_{i=1}^{\infty} (\lambda_{T,i} - \lambda_{T,\omega}) e_i^* \otimes e_i \\ &= \lambda_{T,\omega} I - \sum_{i\in L} \lambda_{T,\omega} e_i^* \otimes e_i + \sum_{i\in\mathbb{N}\setminus L}^{\infty} (\lambda_{T,i} - \lambda_{T,\omega}) e_i^* \otimes e_i \\ &= \lambda_{T,\omega} P_{\mathbb{N}\setminus L} + S, \end{aligned}$$

where $S = \sum_{i\in\mathbb{N}\setminus L}^{\infty} (\lambda_{T,i} - \lambda_{T,\omega}) e_i^* \otimes e_i$. As $e_i^* \otimes e_i$ is in $\mathcal{A}$ for all $i \in \mathbb{N} \setminus L$, it follows that $S \in \mathcal{A}$, which yields the desired conclusion. □

1.2. Initial Examples of Spaces $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$

An important result for explicitly describing the space $\mathcal{L}_{\mathrm{diag}}(X)$ is the following.

THEOREM 1.2 ([ADT, Theorem 1.1]). *Let X be a Banach space with a Schauder basis $(e_i)_i$. The following are equivalent*:

(i) *The map $e_i^* \to e_i^* \otimes e_i$ extends to an isomorphism between X^* and $\mathcal{L}_{\mathrm{diag}}(X)$.*
(ii) (a) *The basis $(e_i)_i$ dominates the summing basis of c_0.*
 (b) *The space X^* is submultiplicative, that is, there exists C such that, for all sequences of scalars $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$, we have*

$$\Big\| \sum_{i=1}^n a_i b_i e_i^* \Big\| \le C \Big\| \sum_{i=1}^n a_i e_i^* \Big\| \Big\| \sum_{i=1}^n b_i e_i^* \Big\|.$$

The following, fairly immediate, corollary is sometimes more convenient.

COROLLARY 1.3. *Let X be a Banach space with a Schauder basis, for which there exist positive constants C_1, C_2 such that*

(i) *for all $n \in \mathbb{N}$ $\| \sum_{i=1}^n e_i \| \le C_1$ and*
(ii) *for all sequences of scalars $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n$*

$$\Big\| \sum_{i=1}^n a_i b_i e_i \Big\| \le C_2 \Big\| \sum_{i=1}^n a_i e_i \Big\| \Big\| \sum_{i=1}^n b_i e_i \Big\|.$$

Then, if $Y = [(e_i^*)_i]$, the space $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(Y)$ is isomorphic as a Banach algebra to the unitization $\mathbb{R}e_\omega \oplus X$ of X via the map $I \mapsto e_\omega$ and for all $i \in \mathbb{N}$ $e_i^{**} \otimes e_i^* \to e_i$.

Clearly, the sequence $(e_\omega, e_1, e_2, \ldots)$ forms a Schauder basis of $\mathbb{R}e_\omega \oplus X$. The following result can be easily obtained by combining Proposition 1.1 and Corollary 1.3.

COROLLARY 1.4. *Let X be a Banach space with a Schauder basis that satisfies the assumptions of Corollary 1.3. The ideals of $\mathbb{R}e_\omega \oplus X$ are precisely the spaces $\mathcal{A}_L = \bigcap_{\kappa \in L} \ker e_\kappa^*$ for all closed subsets L of* $[1, \omega]$.

We mention now some examples of spaces to which this theorem can be applied without the requirement of a lot of other theory. Some of these examples are from [ADT]. The subsequent subsections of this section are devoted to providing more examples to which this theorem can be applied.

EXAMPLE 1.5. As it is explained in [ADT, Example 2.7], if $X = c$ endowed with the monotone summing basis $(s_n)_n$, then $\mathcal{L}_{\mathrm{diag}}(X)$ is isometric to $\ell_1(\mathbb{N})$. In this case $(s_n^*)_n$ spans a space of codimension one in X^* which, by Theorem 1.2, implies that $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X) = \mathcal{L}_{\mathrm{diag}}(X) \equiv \ell_1$. To be more precise, $\mathcal{L}_{\mathrm{diag}}(X)$ isometrically coincides with the Banach algebra bv_1 of all scalar sequences of bounded variation equipped with coordinate-wise multiplication and the norm $\|(a_i)_i\| = \sum_i |a_i - a_{i+1}| + \lim_i |a_i|$.

EXAMPLE 1.6. One of the main results of [ADT] is Theorem 1.4 which states that there exists a hereditarily indecomposable Banach space X_{ADT} that is quasi-reflexive of order one, with a hereditarily indecomposable dual, and a Schauder basis $(e_i)_i$ that satisfies the assumptions of Theorem 1.2. This means that $\mathcal{L}_{\mathrm{diag}}(X_{\mathrm{ADT}}) \equiv X_{\mathrm{ADT}}^*$, which is hereditarily indecomposable (in fact, as it is stated in [ADT], this identification is isometric). Also, by [ADT, Theorem 14(iii)] we directly conclude $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X_{\mathrm{ADT}}) = \mathcal{L}_{\mathrm{diag}}(X_{\mathrm{ADT}}) \equiv X_{\mathrm{ADT}}^*$ (this follows from the quasi-reflexivity of X_{ADT}).

Recall that every Banach space X with a 1-unconditional basis is a Banach algebra when endowed with coordinate-wise multiplication.

PROPOSITION 1.7. *Let X be a Banach space with a normalized 1-unconditional basis $(x_i)_i$. Then there exists a Banach space Y with a Schauder basis such that the space $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(Y)$ contains an ideal $\widetilde{X}$ that is isomorphic as a Banach algebra to X and a subalgebra $\mathcal{A}$ that is isomorphic as a Banach algebra to $C(\omega)$ so that $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(Y) = \widetilde{X} \oplus \mathcal{A}$.*

Proof. Let $(s_i)_i$ denote the monotone basis of c, that is, $\|\sum_i a_i s_i\| = \sup_n |\sum_{i=1}^n a_i|$.We define a norm on $c_{00}(\mathbb{N})$ so that, for any scalar sequence $(a_i)_i$ that is eventually zero, we have

$$\left\|\sum_{i=1}^{\infty} a_i e_i\right\| = \max\left\{\left\|\sum_{i=1}^{\infty} a_{2i} x_i\right\|, \left\|\sum_{i=1}^{\infty} a_i s_i\right\|\right\}, \tag{3}$$

and let W denote its completion. The sequence $(d_i)_i$ defined by $d_1 = e_1$ and for all $i \in \mathbb{N}$ $d_{i+1} = e_{i+1} - e_i$ forms a Schauder basis for W. The norm on the sequence

$(d_i)_i$ is given by

$$\left\|\sum_{i=1}^{\infty} a_i d_i\right\| = \max\left\{\left\|\sum_{i=1}^{\infty}(a_{2i} - a_{2i+1})x_i\right\|, \sup_i |a_1 - a_i|\right\}.$$

A standard argument using the triangle inequality yields that W endowed with $(d_i)_i$ satisfies the assumptions of Corollary 1.3. The formula also immediately yields that the sequence $(d_{2i})_i$ is equivalent to $(x_i)_i$ and by Corollary 1.4 the space $\widetilde{X} = [(d_{2i})_i]$ is an ideal of W. It is also straightforward to check that the map $Q : W \to W$ defined by $Q(\sum_i a_i d_i) = \sum_i (a_{2i} - a_{2i+1})d_{2i}$ defines a bounded linear projection onto $\widetilde{X}$. Note that the sequence $(d_1)^\frown(d_{2i} + d_{2i+1})_i$ is equivalent to the unit vector basis of c_0, the space $\bar{\mathcal{A}} = \mathbb{R}d_1 \oplus [(d_{2i} + d_{2i+1})_i]$ is closed under multiplication, and it is isomorphic as a Banach algebra to c_0 endowed with coordinate-wise multiplication with respect to its unit vector basis. It is also easy to see that $\bar{\mathcal{A}}$ is complementary to $\widetilde{X}$. Finally, take $\widetilde{X}$ and $\mathcal{A} = \mathbb{R}e_\omega \oplus \bar{\mathcal{A}}$ as subspaces of $\mathbb{R}e_\omega \oplus W$, which, by Corollary 1.3, coincides with $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(Y)$ for $Y = [(d_i^*)_i]$. □

The following lemma allows to dualize certain spaces and obtain Banach algerbas.

LEMMA 1.8. *Let X and Y be Banach spaces with normalized Schauder bases $(x_i)_i$ and (y_i) respectively, and assume that $(x_i)_i$ is unconditional and that $(y_i^*)_i$ is submultiplicative. If $M = \{m_1 < m_2 < \cdots\}$, $N = \{n_1 < n_2 < \cdots\}$ are subsets of $\mathbb{N}$ with $M \cup N = \mathbb{N}$, define a norm on $c_{00}(\mathbb{N})$ so that, for any sequence of scalars $(a_i)_i$ that is eventually zero,*

$$\left\|\sum_{i=1}^{\infty} a_i e_i\right\| = \max\left\{\left\|\sum_{i=1}^{\infty} a_{m_i} x_i\right\|, \left\|\sum_{i=1}^{\infty} a_{n_i} y_i\right\|\right\},$$

and let W denote the completion of $c_{00}(\mathbb{N})$ with this norm. Then the sequence $(e_i^*)_i$ in W^* is submultiplicative.

Proof. By renorming our spaces we may assume that $(y_i^*)_i$ is bimonotone and 1-submultiplicative and that $(x_i)_i$ is 1-unconditional. We will show that $(e_i^*)_i$ is 1-submultiplicative. Define the isometric embedding $T : W \to (X \oplus Y)_\infty$ given by

$$Te_i = (x, y) \quad \text{where } x = \begin{cases} x_j : & \text{if } i = m_j \\ 0 : & \text{otherwise} \end{cases} \quad \text{and} \quad y = \begin{cases} y_j : & \text{if } i = n_j \\ 0 : & \text{otherwise} \end{cases}$$

and observe that $T^* : (X^* \oplus Y^*)_1 \to W^*$ is the w^*-continuous map given by $T^*(x_i^*, 0) = e_{m_i}^*$ and $T^*(0, y_i^*) = e_{n_i}^*$ for all $n \in \mathbb{N}$.

Let u_1^* and ${}_2^*$ be normalized elements in W^*. We will show that the element $u_1^* u_2^* = w^*\text{-}\sum_i u_1^*(e_1)u_2^*(e_i)e_i^*$ is well defined and it has norm at most one. By the Hahn–Banach theorem, there exist elements $f_1^* = w^*\text{-}\sum_i \lambda_i x_i^*$ and $g_1^* = w^*\text{-}\sum_i \mu_i y_i^*$ with $\|f_1^*\| + \|g_1^*\| = 1$ such that $u_1^* = T^*(f_1^*, g_1^*) = w^*\text{-}\sum_i \lambda_i e_{m_i}^* + w^*\text{-}\sum_i \mu_i e_{n_i}^*$, and similarly there exist $f_2^* = w^*\text{-}\sum_i \xi_i x_i^*$ and $g_2^* = w^*\text{-}\sum_i \zeta_i y_i^*$ with $\|f_2^*\| + \|g_2^*\| = 1$ such that $u_2^* = T^*(f_2^*, g_2^*) = w^*\text{-}\sum_i \xi_i e_{m_i}^* + w^*\text{-}\sum_i \zeta_i e_{n_i}^*$.

Define, for each $i \in \mathbb{N}$, the scalars $\tilde{\zeta}_i = \begin{cases} \zeta_j : \text{if } m_i = n_j \text{ for some } j \\ 0 : \text{ otherwise} \end{cases}$ and $\tilde{\mu}_i = \begin{cases} \mu_j : \text{if } m_i = n_j \text{ for some } j \\ 0 : \text{ otherwise} \end{cases}$. Since $\sup_i |\tilde{\zeta}_i| \le \|g_2^*\|$ and $\sup_i |\tilde{\mu}_i| \le \|g_1^*\|$, by unconditionality we obtain that $f_3^* = w^*\text{-}\sum_i \lambda_i \tilde{\zeta}_i x_i^*$ and $f_4^* = w^*\text{-}\sum_i \xi_i \tilde{\mu}_i x_i^*$ are well defined and $\|f_3^*\| \le \|f_1^*\|\|g_2^*\|$, $\|f_4^*\| \le \|f_2^*\|\|g_1^*\|$. A straightforward calculation yields that if $f^* = f_1^* f_2^* + f_3^* + f_4^*$ and $g^* = g_1^* g_2^*$, then

$$\begin{aligned} T^*(f^*, g^*) &= \left(w^*\text{-}\sum_i \lambda_i e_{m_i}^* + w^*\text{-}\sum_i \mu_i e_{n_i}^*\right)\left(w^*\text{-}\sum_i \xi_i e_{m_i}^* + w^*\text{-}\sum_i \zeta_i e_{n_i}^*\right) \\ &= u_1^* u_2^* \end{aligned}$$

and hence $\|u_1^* u_2^*\| \le \|f_1^*\|\|f_2^*\| + \|f_1^*\|\|g_2^*\| + \|f_2^*\|\|g_1^*\| + \|g_1^*\|\|g_2^*\| = 1$. □

PROPOSITION 1.9. *Let X be a Banach space with a normalized 1-unconditional basis $(x_i)_i$. Then there exists a Banach space Y with a Schauder basis such that the space $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(Y)$ contains an ideal $\widetilde{X}$ that is isomorphic as a Banach algebra to X and a subalgebra $\mathcal{A}$ that is isomorphic as a Banach algebra to* bv_1 (*see Example 1.5*) *so that $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(Y) = \widetilde{X} \oplus \mathcal{A}$.*

Proof. For notational purposes we prove the result for the space $Z = [(x_i^*)_i]$ instead of X. This is clearly sufficient by duality. Take $c_{00}(\mathbb{N})$ with the norm described by (3) from the proof of Proposition 1.7 and its completion W. By Lemma 1.8 the sequence $(e_i^*)_i$ is submultiplicative, and since $(e_i)_i$ dominates the summing basis of c_0 by Theorem 1.2, the space $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(W)$ can be naturally identified with $\mathbb{R}e_\omega^* \oplus [(e_i^*)]$, where $e_\omega^* = w^*\text{-}\sum_i e_i^*$.

By the proof of Proposition 1.7, the operator $Q : W \to W$ given by $Q(\sum_i a_i e_i) = \sum_i a_{2i}(e_{2i} - e_{2i-1})$ is a bounded projection, $(e_{2i} - e_{2i-1})_i$ onto $\tilde{X} = [(e_{2i} - e_{2i-1})_i]$, $\ker(Q) = [(e_{2i-1})_i]$, $(e_{2i} - e_{2i-1})_i$ is equivalent to $(x_i)_i$, and $(e_{2i-1})_i$ is equivalent to the summing basis if c_0. Then by duality $Q^*|_{\mathbb{R}e_\omega^* \oplus [(e_i^*)]}$ is a projection onto $(e_{2i}^*)_i$, $\ker(Q^*|_{\mathbb{R}e_\omega^* \oplus [(e_i^*)]}) = \mathbb{R}e_\omega^* \oplus [(e_{2i-1}^* + e_{2i}^*)_i]$, $(e_{2i}^*)_i$ is equivalent to $(x_i^*)_i$, and $(e_{2i-1}^* + e_{2i}^*)_i$ is equivalent to the difference basis of ℓ_1. Setting $\tilde{Z} = [(e_{2i}^*)_i]$ and $\mathcal{A} = \mathbb{R}e_\omega^* \oplus [((e_{2i-1}^* + e_{2i}^*)_i)]$, the conclusion follows. □

1.3. Jamesifying Unconditional Sequences

We discuss the jamesification of a Banach space with an unconditional basis and its dual. The classical example is the jamesification J of ℓ_2 by James in [J] (hence also the term jamesification). The purpose is to study these spaces and their duals as Banach algebras of diagonal operators. We recall the definition of the jamesification of a Schauder basic sequence from [BHO, p. 21].

DEFINITION 1.10 ([BHO]). *Let X be a Banach space with a normalized Schauder basis $(x_i)_i$. We define a norm on $c_{00}(\mathbb{N})$ as follows: for every sequence of scalars*

$(a_i)_i$ that is eventually zero, we set

$$\Big\|\sum_{i=1}^{\infty} a_i e_i\Big\| = \sup\Big\{\Big\|\sum_{n=1}^{\infty}\Big(\sum_{i=k_n}^{m_n} a_i\Big)x_{k_n}\Big\| : 1 \le k_1 \le m_1 < k_2 \le m_2 < \cdots\Big\}. \tag{4}$$

We denote the completion of $c_{00}(\mathbb{N})$ with this norm by $J(X)$, and we call it the "jamesification" of $(x_i)_i$. We call the sequence $(e_i)_i$ the unit vector basis of $J(X)$.

Of course, the space $J(X)$ depends on the basis $(x_i)_i$ of X. In the sequel the basis $(x_i)_i$ used to define $J(X)$ will be specified or will be clear from the context.

1.4. The Spaces $J(X)^*$ and $J(X)$ as Banach Algebras of Diagonal Operators

It was observed by Andrew and Green in [AG] that J can be viewed as a Banach algebra. We apply Theorem 1.2 to them to show that, for any space X with an unconditional basis, the spaces $J(X)$ and $J(X)^*$ may be viewed as Banach algebras of diagonal operators.

PROPOSITION 1.11. *Let X be a Banach space with a normalized and 1-unconditional basis $(x_i)_i$, and let $(e_i)_i$ denote the Schauder basis of its jamesification $J(X)$. The following hold*:

(i) *The sequence $(e_i)_i$ is a normalized and monotone Schauder basis of $J(X)$.*
(ii) *For any sequence of scalars $(a_i)_{i=1}^n$, we have $|\sum_{i=1}^n a_i| \le \|\sum_{i=1}^n a_i e_i\|$.*
(iii) *For any sequences of scalars $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n$, we have*

$$\Big\|\sum_{i=1}^{n} a_i b_i e_i^*\Big\| \le 2\Big\|\sum_{i=1}^{n} a_i e_i^*\Big\|\Big\|\sum_{i=1}^{n} b_i e_i^*\Big\|.$$

Proof. Statements (i) and (ii) easily follow from (4) and unconditionality. We will use the equivalence of statements (2) and (3) of [ADT, Theorem 2.4]. For (iii) it is sufficient to find a 1-norming set K in $J(X)^*$ that is contained in the linear span of $(e_i^*)_i$ and satisfies $K \cdot K \subset 2B_{J(X)^*}$. We will prove that the desired set is

$$K = \Big\{\sum_{n=1}^{\infty} c_n\Big(\sum_{i=k_n}^{m_n} e_i^*\Big) : (c_n)_{n=1}^{\infty} \in c_{00}(\mathbb{N}), k_1 \le m_1 < k_2 \le m_2 < \cdots$$
$$\text{and } \Big\|\sum_{n=1}^{\infty} c_n x_{k_n}^*\Big\| \le 1\Big\}.$$

By (4) it is easy to see that K is a 1-norming set. To show that $K \cdot K \subset 2B_{J(X)^*}$, let $f = \sum_{n=1}^{\infty} a_n(\sum_{i=k_n}^{d_n} e_i^*)$ and $f = \sum_{m=1}^{\infty} b_n(\sum_{i=l_m}^{t_m} e_i^*)$ be in K. Then we can check that fg has the form

$$fg = \sum_{r=1}^{\infty} c_r\Big(\sum_{i=q_r}^{p_r} e_i^*\Big),$$

where each q_r is either some k_n or some l_m. Define the sets $R_f = \{r \in \mathbb{N} : q_r = k_{n_r}$ for some $n_r \in \mathbb{N}\}$ and $R_g = \{r \in \mathbb{N} : q_r = m_{l_r}$ for some $l_r \in \mathbb{N}\} \setminus R_f$ and the functionals

$$fg|_1 = \sum_{r\in R_f} c_r\Bigg(\sum_{i=q_r=k_{n_r}}^{p_r} e_i^*\Bigg) \quad\text{and}\quad fg|_2 = \sum_{r\in R_g} c_r\Bigg(\sum_{i=q_r=l_{m_r}}^{p_r} e_i^*\Bigg).$$

Clearly, $fg = fg|_1 + fg|_2$. Observe that, for each $r \in R_f$, there is $\bar{m}_r \in \mathbb{N}$ such that $c_r = a_{k_{n_r}} b_{l_{\bar{m}_r}}$. Unconditionality of the basis $(x_i)_i$ yields that

$$\Bigg\|\sum_{r\in R_f} c_r x^*_{q_r}\Bigg\| = \Bigg\|\sum_{r\in R_f} a_{k_{n_r}} b_{l_{\bar{m}_r}} x^*_{k_{n_r}}\Bigg\| \le \sup_n |b_n| \Bigg\|\sum_{n=1}^{\infty} a_k x^*_{n_k}\Bigg\| \le 1.$$

This implies that $fg|_1 \in K$, and similarly it follows that $fg|_2$ is in K as well, which yields the conclusion. □

A space that displays similar qualities to the jamesification of a space is a certain subspace of its dual defined in what follows.

Definition 1.12. *Let X be Banach space with a normalized 1-unconditional basis $(x_i)_i$. Denote by $s : J(X) \to \mathbb{R}$ the norm-one linear functional defined by $s(\sum_i a_i e_i) = \sum_i a_i$, and denote by $\mathcal{J}_*(X)$ the subspace of $J(X)^*$*

$$\mathcal{J}_*(X) = \mathbb{R}s \oplus [(e_i^*)_i].$$

Denote by $(v_i)_i$ the sequence in $\mathcal{J}_*(X)$ given by $v_1 = s = w^*\text{-}\sum_{j=1}^{\infty} e_i^*$, and for all $i \in \mathbb{N}$, $v_{i+1} = s - \sum_{j=1}^{i} e_j^* = w^*\text{-}\sum_{j=i+1}^{\infty} e_j^*$.

The closed linear span of $(v_i)_i$ is clearly $\mathcal{J}_*(X)$ and as we shall see later it is a normalized and monotone Schauder basis as well (see Proposition 1.15(i)). The space $\mathcal{J}_*(X)$ is qualitatively similar to the space $J([(x_i^*)_i])$; however, they do not coincide in general.

Remark 1.13. If X has a normalized 1-unconditional basis, then it is implied by [BHO, Theorem 2.2(2) and Theorem 4.1(2)] that if ℓ_1 does not embed into X, then $\mathcal{J}_*(X) = J(X)^*$.

Remark 1.14. If X is a Banach space with a 1-unconditional basis $(x_i)_i$, then the sequence $(e_i^*)_i$ in $J(X)^*$ is submultiplicative with the element $s = w^*\text{-}\sum_{i=1}^{\infty} e_i^*$ acting as a multiplicative identity. Hence the space $\mathcal{J}_*(X)$ is submultiplicative (not with $(v_i)_i$) and s acts as an identity on it. In addition, $v_i v_j = v_{\max\{i,j\}}$ for all $i, j \in \mathbb{N}$.

Proposition 1.15. *If X has a normalized 1-unconditional basis $(x_i)_i$, then the basis $(v_i)_i$ of $\mathcal{J}_*(X)$ satisfies the following properties*:

(i) *The basis $(v_i)_i$ is normalized and monotone.*
(ii) *For any scalars $(a_i)_{i=1}^n$, we have $|\sum_{i=1}^n a_i| \le \|\sum_{i=1}^n a_i v_i\|$.*
(iii) *The unit ball of $\mathcal{J}_*(X)$ is a 1-norming set for $J(X)$, hence $J(X)$ is naturally isometric to a subspace of $(\mathcal{J}_*(X))^*$ with the identification $v_1^* = e_1$, and for all $i \in \mathbb{N}$, $v_{i+1}^* = e_{i+1} - e_i$. In particular, the closed linear span of $[(v_i^*)_i]$ is isometrically isomorphic to $J(X)$.*

(iv) *For any sequence of scalars* $(a_i)_i$ *that is eventually zero, we have*

$$\left\|\sum_{i=1}^{\infty} a_i v_i^*\right\| = \sup\left\{\left\|\sum_{n=1}^{\infty}(a_{k_n} - a_{m_n})x_{k_n}\right\| : 1 \le k_1 < m_1 \le k_2 < m_2 \le \cdots\right\}. \quad (5)$$

(v) *For any sequences of scalars* $(a_i)_{i=1}^n$ *and* $(b_i)_{i=1}^n$, *we have*

$$\left\|\sum_{i=1}^{\infty} a_i b_i v_i^*\right\| \le 2\left\|\sum_{i=1}^{\infty} a_i v_i^*\right\|\left\|\sum_{i=1}^{\infty} b_i v_i^*\right\|. \quad (6)$$

Proof. The second statement is a consequence of the fact that $(e_i)_i$ is normalized. For (iii) observe that the linear span of $(e_i^*)_i$ is a subset of $\mathcal{J}_*(X)$ and use the monotonicity of $(e_i)_i$. Then (iv) follows from (iii) and (4). Furthermore, (iv) implies that $(v_i^*)_i$ is monotone and (4) yields that $\|v_i\| = 1$ for all $i \in \mathbb{N}$, that is (i) holds. Finally, (v) follows from 1-unconditionality of $(x_k)_k$ and the triangle inequality applied to (iv). $\square$

REMARK 1.16. If X is a Banach space with a normalized 1-unconditional basis $(x_i)_i$, then the sequence $(v_i^*)_i$ is also a monotone (but not necessarily normalized) basis of $J(X)$ that satisfies (5). Also, $J(X)$ with this basis satisfies (6). We call $(v_i^*)_i$ the difference basis of $J(X)$. Hence, $J(X)$ is submultiplicative when endowed with point-wise multiplication with respect to this basis. Note that $e_i e_j = e_{\min\{i,j\}}$ for all $i, j \in \mathbb{N}$ and hence $(e_i)_i$ is an approximate identity; however, $J(X)$ does not contain a multiplicative identity. If we identify $J(X)$ with a subspace of $(\mathcal{J}_*(X))^*$, then the element $e_\omega = w^*\text{-}\sum_{i=1}^{\infty} v_i^* = w^*\text{-}\lim_i e_i$ acts as a multiplicative identity on $J(X)$. We view the subspace $\mathbb{R}e_\omega \oplus J(X)$ of $(\mathcal{J}_*(X))^*$ as the unitization of $J(X)$.

PROPOSITION 1.17. *Let X be a Banach space with a 1-unconditional basis $(x_i)_i$. The following hold*:

(i) $\mathcal{L}_{\text{diag}}(J(X)) \equiv J(X)^*$ *and* $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(J(X)) \equiv \mathcal{J}_*(X) = \mathbb{R}s \oplus [(e_i^*)_i]$, *and*
(ii) $\mathcal{L}_{\text{diag}}(\mathcal{J}_*(X)) \equiv \mathcal{J}_*(X)^*$ *and* $\mathcal{K}_{\text{diag}}(\mathcal{J}_*(X)) \equiv J(X)$.

More precisely, the map $A : \mathcal{L}_{\text{diag}}(J(X)) \to J(X)^*$ *defined by*

$$A\left(\text{SOT-}\sum_{i=1}^{\infty} \lambda_i e_i^* \otimes e_i\right) = w^*\text{-}\sum_{i=1}^{\infty} \lambda_i e_i^*$$

is an onto isomorphism with $\|A\|\|A^{-1}\| \le 2$ *and the image of* $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(J(X))$ *under* A *is* $\mathcal{J}_*(X)$. *Also, the map* $B : \mathcal{L}_{\text{diag}}(\mathcal{J}_*(X)) \to \mathcal{J}_*(X)^*$ *defined by*

$$B\left(\text{SOT-}\sum_{i=1}^{\infty} \lambda_i v_i^* \otimes v_i\right) = w^*\text{-}\sum_{i=1}^{\infty} \lambda_i v_i^*$$

is an onto isomorphism with $\|B\|\|B^{-1}\| \le 2$ *and the image of* $\mathbb{R} \oplus \mathcal{K}_{\text{diag}}(\mathcal{J}_*(X))$ *is* $\mathbb{R}e_\omega \oplus J(X)$.

Proof. Item (i) follows readily from Proposition 1.11 and [ADT, Theorem 2.4] and the definition of $\mathcal{J}_*(X)$. Item (ii) follows from Proposition 1.15 and [ADT, Theorem 2.4] as well. $\square$

REMARK 1.18. Taking ℓ_p, $1 < p < \infty$, with the unit vector basis yields that the spaces $\mathbb{R}e_\omega \oplus J_p$ and J_p^*, $1 < p < \infty$, are of the form $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(Y)$.

COROLLARY 1.19. *For every $n \in \mathbb{N}$, there is a space Y with a basis such that $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(Y)$ is quasi-reflexive of order n.*

Proof. For $n = 2$ take, for example, the space $J = J(\ell_2)$ with the basis $(e_i)_i$ and $Y = J \oplus J$ endowed with the basis $(w_i)_i$ such that $w_{2i-1} = (v_i^*, 0)$ and $w_{2i} = (0, v_i^*)$. It is straightforward to check that this basis satisfied the assumptions of Corollary 1.3 and hence $\mathcal{K}_{\text{diag}}(Y)$ is isomorphic to $J \oplus J$, which is quasi-reflexive of order two. Adding one dimension does not alter this fact. Of course, this works for any $n \in \mathbb{N}$. $\square$

The following demonstrates some additional examples of spaces and their duals that can be viewed as spaces of the form $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(Y)$. In particular, it applies to a large class of spaces with spreading bases, for example, to the diagonal of $\ell_q \oplus J_p$ for $1 < q < p < \infty$. It follows from [AMS] that this corollary does not apply to all spaces with conditional spreading bases.

COROLLARY 1.20. *Let X and Y be Banach spaces with normalized 1-unconditional bases $(x_i)_i$ and $(y_i)_i$ respectively. Denote by $(\tilde{e}_i)_i$ the unit vector basis of $J(X)$, define a norm on $c_{00}(\mathbb{N})$ so that, for any sequence of scalars $(a_i)_i$ that is eventually zero,*

$$\left\|\sum_{i=1}^{\infty} a_i e_i\right\| = \max\left\{\left\|\sum_{i=1}^{\infty} a_i y_i\right\|, \left\|\sum_{i=1}^{\infty} a_i \tilde{e}_i\right\|\right\},$$

and let Z denote the completion of $c_{00}(\mathbb{N})$ with respect to this norm. Then the sequence $(e_i^*)_i$ in Z^* is submultiplicative, $\mathcal{L}_{\text{diag}}(Z) \equiv Z^*$ and $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(Z) \equiv \mathbb{R}s \oplus [(e_i^*)_i]$. Also, if we define the basis $(d_i)_i$ of Z by setting $d_1 = e_1$ and $d_{i+1} = e_{i+1} - e_i$ for $i \in \mathbb{N}$, then Z endowed with this basis is submultiplicative and $\mathcal{K}_{\text{diag}}(\mathbb{R}s \oplus [(e_i^*)_i]) \equiv Z$.

Proof. The part about $\mathcal{L}_{\text{diag}}(Z)$ and Z^* can be obtained by combining Lemma 1.8 with Proposition 1.17. The fact that $(d_i)_i$ is submultiplicative follows from the triangle inequality and Proposition 1.15(v) and the fact that $d_i^* = w^*\text{-}\sum_{j=i}^{\infty} e_i^*$ and hence $\mathbb{R}s \oplus [(e_i^*)] = [(d_i^*)_i]$. $\square$

1.5. Spaces with Right (or Left) Dominant Unconditional Bases as Complemented Ideals of Spaces of Compact Diagonal Operators

We prove that if X has a right dominant unconditional basis, then X embeds as a complemented ideal in the space $\mathcal{L}_{\text{diag}}(\mathcal{J}_*(X \oplus X))$. We also show that if X has a left dominant unconditional basis $(x_i)_i$, then X embeds as a complemented ideal in the space $\mathcal{K}_{\mathrm{diag}}(J([(x_i^*)_i]\oplus[(x_i^*)_i]))$. This will be used in the next subsection to embed reflexive spaces with unconditional bases as ideals into quasi-reflexive algebras of diagonal operators.

We recall the following definition from [BHO, p. 22].

Definition 1.21. *An unconditional basis $(x_i)_i$ of a Banach space X is said to be C-right dominant for some constant $C>0$ if, for all $1\le k_1\le m_1<k_2\le m_2<\cdots$ and any sequence of scalars $(a_i)_i$ that is eventually zero, we have*

$$\Big\|\sum_{i=1}^{\infty}a_{m_i}x_{k_i}\Big\|\le C\Big\|\sum_{i=1}^{\infty}a_{m_i}x_{m_i}\Big\|. \tag{7}$$

We say that $(x_i)_i$ is right dominant if it is C-right dominant for some $C>0$. Reversing the inequality in (7), we obtain the definition of a $1/C$-left dominant sequence.

Examples of right dominant sequences are the bases of Tsirelson space from [FJ], Tsirelson's dual from [Ta], Schreier space from [Schr], and any subsymmetric sequence such as ℓ_p spaces or Schlumprecht space [Schl]. These examples are left dominant as well and the only case in which this is not straightforward is Tsirelson space where it follows from [CJS, Proposition 6].

The property of being right dominant can be reformulated as follows. A sequence $(x_i)_i$ is right dominant if there exists a constant C such that, for every strictly increasing sequence of natural numbers $(k_i)_i$ and every sequence $(m_i)_i$ with $m_i\in[k_i,k_{i+1})$, we have that $(x_{k_i})_i$ is C-dominated by $(x_{m_i})_i$.

Remark 1.22. If $(x_i)_i$ is suppression unconditional and C-right dominant, $(a_i)_i$ is a sequence of scalars that is eventually zero and $1\le k_1<m_1\le k_2<m_2\le k_3<m_3\le\cdots$, then by splitting the even and odd terms of the sequence we get

$$\begin{aligned}\Big\|\sum_{i=1}^{\infty}a_ix_{k_i}\Big\| &\le \Big\|\sum_{i=1}^{\infty}a_{2i}x_{k_{2i}}\Big\|+\Big\|\sum_{i=1}^{\infty}a_{2i-1}x_{k_{2i-1}}\Big\|\\ &\le C\Big\|\sum_{i=1}^{\infty}a_{2i}x_{m_{2i}}\Big\|+C\Big\|\sum_{i=1}^{\infty}a_{2i-1}x_{m_{2i-1}}\Big\|\le 2C\Big\|\sum_{i=1}^{\infty}a_ix_{m_i}\Big\|.\end{aligned}$$

The following result builds on [BHO, Proposition 2.3] and, in a sense, it is a slight generalization of it by removing an extra assumption.

Lemma 1.23. *Let X be a Banach space with a normalized 1-unconditional basis $(x_i)_i$ that is right dominant. The sequence $(e_{2i}-e_{2i-1})_i$ in $J(X)$ is equivalent to $(x_{2i})_i$ and it spans a complemented subspace $\widetilde{X}_{2\mathbb{N}}$ of $J(X)$ via the map $Q: J(X)\to J(X)$ defined by $Qx=\sum_{i=1}^{\infty}e_{2i}^*(x)(e_{2i}-e_{2i-1})$. Furthermore, we have that $(I-Q)(x)=\sum_{i=1}^{\infty}(e_{2i-1}^*+e_{2i}^*)(x)e_{2i-1}$ and the space $J(X)_{2\mathbb{N}-1}=(I-Q)[J(X)]$ is a subalgebra of $J(X)$.*

Proof. Let C be the constant for which (7) holds. We first show the equivalence. Note that the basis under consideration is in fact $(v_{2i}^*)_i$ and it satisfies (5), which immediately yields that it dominates $(x_{2i-1})_i$ with constant one. For the inverse domination, let $(a_i)_i$ be a sequence of scalars that are eventually zero and all odd entries are zero as well. Then there exist $k_1 < m_1 \leq k_2 < m_2 \leq \cdots$ such that

$$\left\| \sum_{i=1}^{\infty} a_{2i} v_{2i}^* \right\| = \left\| \sum_{i=1}^{\infty} a_i v_i^* \right\| = \left\| \sum_{n=1}^{\infty} (a_{k_n} - a_{m_n}) x_{k_n} \right\|.$$

Denote the quantity in the equation by λ and define

$$K = \{n : k_n \text{ is even}\}, \qquad M = \{n : m_n \text{ is even}\}.$$

We then calculate

$$\begin{aligned} \lambda &= \left\| \sum_{n \in K \cap M} (a_{k_n} - a_{m_n}) x_{k_n} + \sum_{n \in K \setminus M} a_{k_n} x_{k_n} - \sum_{n \in M \setminus K} a_{m_n} x_{k_n} \right\| \\ &= \left\| \sum_{n \in K} a_{k_n} x_{k_n} - \sum_{n \in M} a_{m_n} x_{k_n} \right\| \leq \left\| \sum_{n \in K} a_{k_n} x_{k_n} \right\| + \left\| \sum_{n \in M} a_{m_n} x_{k_n} \right\| \\ &\leq \left\| \sum_{n \in K} a_{k_n} x_{k_n} \right\| + 2C \left\| \sum_{n \in M} a_{m_n} x_{m_n} \right\| \quad \text{(by Remark 1.22).} \end{aligned} \tag{8}$$

By unconditionality we obtain $\lambda \leq (1 + 2C) \| \sum_{i=1}^{\infty} a_{2i} x_{2i} \|$.

For the complementation of the sequence $(v_{2i}^*)_i$, it is enough to show that the map $Q = \mathrm{SOT} - \sum_{i=1}^{\infty} (v_{2i} - v_{2i+1}) \otimes v_{2i}^*$ defines a bounded linear map on $J(X)$. To that end, let $(a_i)_i$ be a sequence of scalars that is eventually zero and set $\mu = \| \sum_{i=1}^{\infty} a_i u_i^* \|$ as well as $\nu = \| \sum_{i=1}^{\infty} (a_{2i} - a_{2i+1}) v_{2i}^* \|$. Then there are $1 \leq k_1 < m_1 \leq k_2 < m_2 \leq \cdots$ such that if $K = \{n : k_n \text{ is even }\}$ and $M = \{n : m_n \text{ is even}\}$ then

$$\begin{aligned} \nu &= \Bigg\| \sum_{n \in K \cap M} (a_{k_n} - a_{k_n+1} - a_{m_n} + a_{m_n+1}) x_{k_n} \\ &\quad + \sum_{n \in K \setminus M} (a_{k_n} - a_{k_n+1}) x_{k_n} - \sum_{n \in M \setminus K} (a_{m_n} - a_{m_n+1}) x_{k_n} \Bigg\| \\ &= \left\| \sum_{n \in K} (a_{k_n} - a_{k_n+1}) x_{k_n} - \sum_{n \in M} (a_{m_n} - a_{m_n+1}) x_{k_n} \right\|. \end{aligned}$$

Repeating the same argument as earlier yields $\nu \leq (1 + 2C) \| \sum_{i=1}^{\infty} (a_i - a_{i+1}) x_i \| \leq (1 + 2C) \mu$.

To see that the space $J(X)_{2\mathbb{N}-1}$ is a subalgebra, note that it is spanned by the vectors $(e_{2i-1})_i$. It follows that the sequence $v_1^*, v_2^* + v_3^*, v_4^* + v_5^*, \ldots$ is a basis of $J(X)_{2\mathbb{N}-1}$, which clearly yields that $J(X)_{2\mathbb{N}-1}$ is closed under multiplication. $\square$

LEMMA 1.24. *Let X be a Banach space with a normalized 1-unconditional C-right dominant Schauder basis $(x_i)_i$, and let $W = (X \oplus X)_\infty$ endowed with the* *basis $(w_i)_i$, where $(x_i,0)=w_{2i-1}$ and $(0,x_i)=w_{2i}$ for all $i\in\mathbb{N}$. Then $(w_i)_i$ $5C$-is right dominant.*

Proof. Let $(k_i)_i$ be strictly increasing, let $(m_i)_i$ be a sequence with $k_i\le m_i<k_{i+1}$, and let $(a_i)_i$ be a sequence of scalars that is eventually zero. Define

$$\lambda=\Big\|\sum_i a_i w_{k_i}\Big\|,\qquad \mu=\Big\|\sum_i a_i w_{m_i}\Big\|,$$
$$K=\{i:k_i \text{ even}\}\quad\text{and}\quad M=\{i:m_i\text{ even}\}.$$

If we set $A=(K\cap M)\cup(K^c\cap M^c)$, then the sets A, $K\setminus M$, and $M\setminus K$ form a partition of $\mathbb{N}$. By using that $(w_{2i-1})_i$ and $(w_{2i})_i$ are isometrically equivalent to each other and that they are both C-right dominant, we obtain the following:

$$\Big\|\sum_{i\in A}a_i w_{k_i}\Big\|\le C\Big\|\sum_{i\in A}a_i w_{m_i}\Big\|,$$
$$\Big\|\sum_{i\in K\setminus M}a_i w_{k_i}\Big\|=\Big\|\sum_{i\in K\setminus M}a_i w_{k_i-1}\Big\|\le 2C\Big\|\sum_{i\in K\setminus M}a_i w_{m_i}\Big\|\quad\text{(by Remark 1.22)},$$
$$\Big\|\sum_{i\in M\setminus K}a_i w_{k_i}\Big\|=\Big\|\sum_{i\in M\setminus K}a_i w_{k_i+1}\Big\|\le C\Big\|\sum_{i\in M\setminus K}a_i w_{m_i}\Big\|,$$

where in the last inequality we used that for $i\in M\setminus K$ we have $k_i+1\le m_i<k_{i+1}+1$. We use unconditionality to conclude $\lambda\le 5C\mu$. □

The statement of the following result is somewhat lengthy, but most of the details are necessary in the sequel.

PROPOSITION 1.25. *Let X be a Banach space with a right dominant normalized 1-unconditional basis $(x_i)_i$. Let Y denote the space $[(x_i^*)_i]$, and let W denote the space $(X\oplus X)_\infty$ endowed with the basis $(w_i)_i$, where $(x_i,0)=w_{2i-1}$ and $(0,x_i)=w_{2i}$ for all $i\in\mathbb{N}$. Denote by $(e_i)_i$ the unit vector basis of $J(X)$ and by $(\bar e_i)_i$ the unit vector basis of $J(W)$. Also denote by $s:J(X)\to\mathbb{R}$ and $\bar s:J(W)\to\mathbb{R}$ the corresponding summing functionals. The following hold.*

(i) *The sequence $(\bar e_{2i}-\bar e_{2i-1})_i$ is equivalent to $(x_i)_i$ and complemented in $J(W)$ via the projection $Qx=\sum_{i=1}^\infty \bar e_{2i}^*(x)(\bar e_{2i}-\bar e_{2i-1})$.*

(ii) *The sequence $(\bar e_{2i-1})_i$ is equivalent to $(e_i)_i$ and complemented in $J(W)$ via the projection $(I-Q)(x)=\sum_{i=1}^\infty(\bar e_{2i-1}^*(x)+\bar e_{2i}^*(x))\bar e_{2i-1}$.*

In particular, the space $\widetilde X=[(\bar e_{2i}-\bar e_{2i-1})_i]$ is an ideal of $J(W)$ that is isomorphic as a Banach algebra to X, the space $\widetilde{J(X)}=[(\bar e_{2i-1})_i]$ is a subalgebra of $J(W)$ that is isomorphic as a Banach algebra to $J(X)$, and $J(W)=\widetilde X\oplus\widetilde{J(X)}$.

(iii) *The sequence $(\bar e_{2i}^*)_i$ is equivalent to $(x_i^*)_i$ and complemented in $\mathcal{J}_*(W)$ via the projection $R(f)=\sum_{i=1}^\infty(f(\bar e_{2i})-f(\bar e_{2i-1}))\bar e_{2i}^*$.*

(iv) *The sequence $(\bar s)^\frown(\bar e_{2i}^*+\bar e_{2i-1}^*)_i$ is equivalent to $(s)^\frown(e_i^*)_i$ and complemented in $\mathcal{J}_*(W)$ via the map $S:\mathcal{J}_*(W)\to\mathcal{J}_*(W)$ that is defined as follows: $S(\bar s)=\bar s$ and $S|_{[(\bar e_i^*)_i]}(f)=\sum_{i=1}^{*}f(\bar e_{2i-1})(\bar e_{2i-1}^*+\bar e_{2i}^*)$.*

In particular, the space $\widetilde{Y} = [(\bar{e}^*_{2i})_i]$ *is an ideal of* $\mathcal{J}_*(W)$ *that is isomorphic as a Banach algebra to* Y*, the space* $\widetilde{\mathcal{J}_*(Y)} = \mathbb{R}\bar{s} \oplus [(\bar{e}^*_{2i-1} + \bar{e}^*_{2i})_i]$ *is a subalgebra of* $\mathcal{J}_*(W)$ *that is isomorphic as a Banach algebra to* $\mathcal{J}_*(Y)$*, and* $\mathcal{J}_*(W) = \widetilde{Y} \oplus \widetilde{\mathcal{J}_*(Y)}$*.*

Proof. By Lemma 1.24 the sequence $(w_i)_i$ is right dominant. Then Lemma 1.23 basically contains statement (i), whereas to obtain statement (ii) the only thing missing is that $(\bar{e}_{2i-1})_i$ is equivalent to $(e_i)_i$. This follows easily from (4) and the right dominance of $(w_i)_i$. The "in particular" part under statement (ii) is also contained in Lemma 1.23. For statement (iii), merely observe that $R = Q^*|_{\mathcal{J}_*(W)}$, whereas for statement (iv) observe that $S = I - R = (I - Q)^*|_{\mathcal{J}_*(W)}$. The remaining part of the statement is fairly straightforward. □

We can tidy up the statement of Proposition 1.25 by combining it with Proposition 1.17 to obtain the following neat corollaries, which apply to a large class of spaces, for example, those mentioned after Definition 1.21 (spaces with subsymmetric bases, Schreier space, Tsirelson space and its dual). Here, $E = \mathcal{J}_*(X \oplus X)$.

COROLLARY 1.26. *Let* X *be a Banach space with a right dominant normalized 1-unconditional basis* $(x_i)_i$*. Then there exists a Banach space* E *with a Schauder basis such that the space* $\mathcal{K}_{\text{diag}}(E)$ *contains an ideal* $\widetilde{X}$ *that is isomorphic to* X *as a Banach algebra and a subalgebra* $\widetilde{J(X)}$ *that is isomorphic to* $J(X)$ *so that* $\mathcal{K}_{\text{diag}}(E) = \widetilde{X} \oplus \widetilde{J(X)}$*.*

Here, $F = [(x^*_i)] \oplus [(x^*_i)_i]$, which by duality and Lemma 1.24 is right dominant.

COROLLARY 1.27. *Let* X *be a Banach space with a left dominant normalized 1-unconditional basis* $(x_i)_i$*, and let* $Y = [(x^*_i)_i]$*. Then there exists a Banach space* F *with a Schauder basis such that the space* $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(F)$ *contains an ideal* $\widetilde{X}$ *that is isomorphic to* X *as a Banach algebra and a subalgebra* $\widetilde{\mathcal{J}_*(Y)}$ *that is isomorphic to* $\mathcal{J}_*(Y)$ *as a Banach algebra so that* $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(F) = \widetilde{X} \oplus \widetilde{\mathcal{J}_*(Y)}$*.*

REMARK 1.28. Note that in Corollary 1.27 if X is additionally reflexive, then $Y = X^*$ and by Remark 1.13 $\mathcal{J}_*(Y) = \mathcal{J}_*(X^*) = J(X^*)^*$, and hence $\widetilde{\mathcal{J}_*(Y)}$ is isomorphic as a Banach algebra to $J(X^*)^*$.

1.6. Spaces with Unconditional Bases Plus Jamesified Tsirelson Spaces as Spaces with Compact Diagonal Operators

We will show that whenever a space X with an unconditional basis does not contain c_0, there exists an appropriate Tsirelson space $T^{1/2}_{\mathcal{M}}$ such that $X \oplus J(T^{1/2}_{\mathcal{M}})^*$ is the Calkin algebra of some space, and that whenever a space X with an unconditional basis does not contain ℓ_1, there exists an appropriate Tsirelson space $T^{1/2}_{\mathcal{M}}$ such that $X \oplus J(T^{1/2}_{\mathcal{M}})$ is the Calkin algebra of some space. We first discuss some basic notions around Tsirelson spaces.

A a collection $\mathcal{M}$ of finite subsets of $\mathbb{N}$ is called compact if it is compact when naturally identified with a subset of $2^{\mathbb{N}}$. We say that a collection $(E_i)_{i=1}^n$ of finite subsets of $\mathbb{N}$ is $\mathcal{M}$-admissible if there is a set $\{m_1,\dots,m_n\}\in\mathcal{M}$ such that $m_1\le E_1<m_2\le E_2<\cdots<m_n\le E_n$. Given a compact collection $\mathcal{M}$ and $0<\theta<1$, the Tsirelson space $T_{\mathcal{M}}^{\theta}$, defined in [AD1], is the completion of $c_{00}(\mathbb{N})$ with the uniquely defined norm satisfying the implicit formula

$$\left\|\sum_{i=1}^{\infty}a_it_i\right\|=\max\left\{\sup_i|a_i|,\theta\sup\sum_{k=1}^{n}\left\|\sum_{i\in E_k}a_it_i\right\|:(E_k)_{k=1}^n\text{ is }\mathcal{M}\text{-admissible}\right\}.$$

The sequence $(t_i)_i$ forms a 1-unconditional Schauder basis for $T_{\mathcal{M}}^{\theta}$. The classical Tsirelson space from [Ts] (or to be more precise, its dual described in [FJ]) is the space $T=T_{\mathcal{S}}^{1/2}$, where $\mathcal{S}=\{F\subset\mathbb{N}:\#F\le\min(F)\}\cup\{\emptyset\}$. It is stated in [AD1, Theorem 1] that the space $T_{\mathcal{M}}^{\theta}$ is reflexive whenever $\mathrm{CB}(\mathcal{M})(\theta/(1+\theta))>1$, where $\mathrm{CB}(\mathcal{M})$ denotes the Cantor–Bendixson index of $\mathcal{M}$. In particular, if $\mathrm{CB}(\mathcal{M})$ is infinite, then the space $T_{\mathcal{M}}^{\theta}$ is reflexive (see also [AD2, 1.1. Proposition]). A special type of collections of finite subsets of $\mathbb{N}$ are the so-called regular families. A collection $\mathcal{M}$ is called regular if it is compact, for every $A\in\mathcal{M}$ it contains all $B\subset A$, and whenever $\{k_1,\dots,k_n\}\in\mathcal{M}$ and $k_1\le m_1,\dots,k_n\le m_n$, then $\{m_1,\dots,m_n\}\in\mathcal{M}$. Whenever the family $\mathcal{M}$ is regular, it is not very hard to see that the basis $(t_i)_i$ of $T_{\mathcal{M}}^{\theta}$ is right dominant.

There are a series of results, starting with [OSZ], about controlling the norm of certain spaces via norms of Tsirelson spaces. We use some estimates from [OSZ] to prove one such result that we need later. We point out that we will not require the full statement. Here, $Sz(X)$ denotes the Szlenk index of X (see [Sz]).

PROPOSITION 1.29. *Let X be a Banach space with a normalized monotone shrinking basis $(x_i)_i$. Then there exists a regular family $\mathcal{M}$ such that $(x_i)_i$ satisfies subsequential 15-$T_{\mathcal{M}}^{1/2}$ estimates. That is, every normalized block sequence $(y_i)_i$ of $(x_i)_i$ with $k_i=\min\operatorname{supp}(y_i)$ for all $i\in\mathbb{N}$ is 15-dominated by $(t_{k_i})_i$. Here, $(t_i)_i$ denotes the basis of $T_{\mathcal{M}}^{1/2}$.*

Proof. By [OSZ, Theorem 18] (we use the statement of the theorem for $X=Z$) there exist $1=m_0<m_1<m_2<\cdots$ and an ordinal number $\alpha<Sz(X)$ such that, for any $1\le s_0<s_1<\cdots$, every block sequence $(y_i)_i$ of $(x_i)_i$ with $\operatorname{supp}(y_i)\in(m_{s_{i-1}},m_{s_i}]$ is 5-dominated by $(t_{m_{s_{i-1}}})_i$, where $(t_i)_i$ is the basis of the space $T_{\mathcal{F}_\alpha}^{1/2}$. Here, $\mathcal{F}_\alpha$ denotes the fine Schreier family that has Cantor–Bendixson index $\alpha+1$ (see, e.g., [OSZ, p. 71]). By passing to an appropriate subsequence of $(m_i)_i$, we may assume that $(m_i-m_{i-1})_i$ is nondecreasing. Define

$$\mathcal{M}=\Bigg\{\bigcup_{i=1}^{n}A_i:A_1<\cdots<A_n,\{\min(A_i):1\le i\le n\}\in\mathcal{F}_\alpha$$
$$\text{and }\#A_i\le m_{d_i}-m_{d_{i-1}},\text{ where }d_i=\min\{i:\min(A_i)\le m_{d_i}\}\Bigg\}.$$

The family $\mathcal{M}$ is regular. This follows from the regularity of $\mathcal{F}_\alpha$ and the fact that $(m_i - m_{i-1})_i$ is nondecreasing. It also follows that $\mathrm{CB}(\mathcal{M}) \leq \omega\alpha + 1$.

We will show that $\mathcal{M}$ is the desired family. For any sequence of scalars $(a_j)_j$ that is eventually zero, we have

$$\Big\|\sum_{i=1}^{\infty}\Big(\sum_{j=m_{i-1}+1}^{m_i}|a_j|\Big)t_{m_{i-1}}\Big\|_{T^{1/2}_{\mathcal{F}_\alpha}} \leq \Big\|\sum_{i=1}^{\infty}\sum_{j=m_{i-1}+1}^{m_i}a_j t_j\Big\|_{T^{1/2}_{\mathcal{M}}}. \tag{9}$$

A way to see this is to use norming sets. Take the standard norming set K of $T^{1/2}_{\mathcal{F}_\alpha}$ and write it as $K = \bigcup_{s=0}^{\infty} K_s$ (see, e.g., the proof of [AD2, 1.1. Proposition]) and show by induction on s that, for every f in K_s with $\mathrm{supp}(f) \subset \{m_j : j \in \mathbb{N}\cup\{0\}\}$, there is $g \in B^*_{T^{1/2}_{\mathcal{M}}}$, for any $i \in \mathbb{N}$ and $j \in (m_{i-1}, m_i]$, we have $g(e_j) = f(e_{m_{i-1}})$. We leave the details to the reader.

Let $(x_j)_{j=1}^n$ be a block sequence in X and for each j set $k_j = \min\mathrm{supp}(x_j)$, and also set $B_j = \{i : \mathrm{range}(x_j)\cap(m_{i-1}, m_i] \neq \emptyset\}$. The sets B_j are intervals of $\mathbb{N}$ clearly satisfying $\max(B_j) \leq \min(B_{j+1})$. This easily implies that we may define a partition C_0, C_1, C_2 of $\{1,\dots,n\}$ so that for that, for each $\varepsilon = 0, 1, 2$ and each $j_1, j_2 \in C_\varepsilon$, we have that B_{j_1} and B_{j_2} either are singletons that coincide or they are disjoint. For $i \in \mathbb{N}$ and $\varepsilon = 0, 1, 2$, the set $A_i^\varepsilon = \{j \in C_\varepsilon : \min(B_j) = m_i\}$. Set $D_\varepsilon = \{i : A_i^\varepsilon \neq \emptyset\}$ and for $i \in D_\varepsilon$ define $y_i^\varepsilon = \sum_{j\in A_i^\varepsilon} x_j$. Because of the property of C_ε, we obtain that $(y_i^\varepsilon/\|y_i^\varepsilon\|)_{i\in D_\varepsilon}$ is 5-dominated by $(t_{m_{i-1}})_{i\in D_\varepsilon}$. We calculate

$$\begin{aligned}
\Big\|\sum_{j\in C_\varepsilon} x_j\Big\| &= \Big\|\sum_{i\in D_\varepsilon} y_i^\varepsilon\Big\| \leq 5\Big\|\sum_{i\in D_\varepsilon}\|y_i^\varepsilon\|t_{m_{i-1}}\Big\|_{T^{1/2}_{\mathcal{F}_\alpha}} = 5\Big\|\sum_{i\in D_\varepsilon}\Big\|\sum_{j\in A_i^\varepsilon}x_j\Big\|t_{m_{i-1}}\Big\|_{T^{1/2}_{\mathcal{F}_\alpha}}\\
&\leq 5\Big\|\sum_{i\in D_\varepsilon}\Big(\sum_{j\in A_i^\varepsilon}\|x_j\|\Big)t_{m_{i-1}}\Big\|_{T^{1/2}_{\mathcal{F}_\alpha}} \quad \text{(triangle ineq. and uncond.)}\\
&\leq 5\Big\|\sum_{i\in D_\varepsilon}\sum_{j\in A_i^\varepsilon}\|x_j\|t_{k_j}\Big\|_{T^{1/2}_{\mathcal{M}}} \quad \text{(by (9))}\\
&= 5\Big\|\sum_{j\in C_\varepsilon}\|x_j\|t_{k_j}\Big\|_{T^{1/2}_{\mathcal{M}}} \leq 5\Big\|\sum_{j=1}^{n}\|x_j\|t_{k_j}\Big\|_{T^{1/2}_{\mathcal{M}}} \quad \text{(by uncond.).}
\end{aligned}$$

The conclusion follows by adding the estimates for C_0, C_1, and C_2. □

REMARK 1.30. The proof of Proposition 1.29 and [Cau, Theorem 6.2] yield the following. If $\mathrm{Sz}(X) = \omega$, then $\mathrm{CB}(\mathcal{M}) < \omega^2$, whereas if $\mathrm{Sz}(X) > \omega$, then $\mathrm{CB}(\mathcal{M}) < \mathrm{Sz}(X)$.

The following says that spaces with an unconditional basis that do not contain c_0 are embedded as complemented ideals into quasi-reflexive spaces of the form $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(Y)$.

PROPOSITION 1.31. *Let X be a Banach space with a normalized and 1-unconditional basis $(x_i)_i$ that does not contain an isomorphic copy of c_0. Then* *there exists a regular family $\mathcal{M}$ and a Banach space Y with a Schauder basis such that the space $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(Y)$ contains a complemented ideal $\widetilde{X}$ that is isomorphic as a Banach algebra to X and a subalgebra $\mathcal{A}$ that is isomorphic as a Banach algebra to $J(T_{\mathcal{M}}^{1/2})^*$ so that $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(Y) = \widetilde{X} \oplus \mathcal{A}$.*

Proof. Take $\mathcal{M}$ given by Proposition 1.29 applied to the space $[(x_i^*)_i]$. This means that, for any sequence of scalars $(a_i)_i$ that is eventually zero, we have

$$\left\|\sum_{i=1}^{\infty} a_i x_i\right\| \geq \frac{1}{15}\left\|\sum_{i=1}^{\infty} a_i t_i^*\right\|, \tag{10}$$

where $(t_i^*)_i$ is the basis of $(T_{\mathcal{M}}^{1/2})^*$. Let $Z = [(t_i)_i] \oplus [(t_i)_i] = T_{\mathcal{M}}^{1/2} \oplus T_{\mathcal{M}}^{1/2}$ endowed with the basis $(z_i)_i$ as in the statement of Lemma 1.24, which is right dominant, let $(e_i)_i$ denote the unit vector basis of $J(Z)$, and let $(v_i)_i$ denote the basis of the space $\mathcal{J}_*(Z) = J(Z)^*$ (equality follows from Remark 1.13). Also denote by $(\bar{e}_i)_i$ the unit vector basis of $J(T_{\mathcal{M}}^{1/2})$,

Set $w_i = v_1 - v_{i+1} = \sum_{j=1}^{i} e_i^*$. We define a norm on $c_{00}(\mathbb{N})$ given by

$$\left\|\sum_{i=1}^{\infty} a_i \tilde{e}_i\right\| = \max\left\{\left\|\sum_{i=1}^{\infty} a_{2i-1} x_i\right\|, \left\|\sum_{i=1}^{\infty} a_i w_i\right\|\right\} \tag{11}$$

and denote by W the completion of $c_{00}(\mathbb{N})$ with respect to this norm. The sequence $(d_i)_i$ given by $d_1 = \tilde{e}_1$ and for $i \in \mathbb{N}$ $d_{i+1} = \tilde{e}_{i+1} - \tilde{e}_i$ is a Schauder basis of W and it satisfies the formula

$$\left\|\sum_{i=1}^{\infty} a_i d_i\right\| = \max\left\{\left\|\sum_{i=1}^{\infty} (a_{2i-1} - a_{2i}) x_i\right\|, \left\|\sum_{i=1}^{\infty} a_i e_i^*\right\|\right\}.$$

It follows that W endowed with $(d_i)_i$ satisfies the assumptions of Corollary 1.3. Hence, it is enough to show the desired decomposition for the unitization of W. Furthermore

$$\left\|\sum_{i=1}^{\infty} a_i d_{2i}\right\| = \max\left\{\left\|\sum_{i=1}^{\infty} a_i x_i\right\|, \left\|\sum_{i=1}^{\infty} a_i e_{2i}^*\right\|\right\}.$$

By Proposition 1.25(iii) the sequence $(e_{2i}^*)_i$ is equivalent to $(t_i^*)_i$, and by (10) we are able to conclude that $(d_{2i})_i$ is equivalent to $(x_i)_i$. In fact, by Proposition 1.25(iii) the map $R(\sum_i a_i e_i^*) = \sum_i (a_{2i} - a_{2i-1}) e_{2i}^*$ is a bounded linear projection which easily implies that $\tilde{R}(\sum_i a_i d_i) = \sum_i (a_{2i} - a_{2i-1}) d_{2i}$ is a bounded linear projection onto the ideal $\widetilde{X} = [(d_{2i})]$, the kernel of which is the subalgebra $\tilde{\mathcal{A}} = [(d_{2i} + d_{2i-1})_i] = [(\tilde{e}_{2i})_i]$. Clearly, by (11), the sequence $(\tilde{e}_{2i})_i$ is equivalent to the sequence $(w_{2i})_i$, which implies that $(d_{2i} + d_{2i-1})_i$ is equivalent to $(e_{2i}^* + e_{2i-1}^*)$, which by Proposition 1.25(iv) is equivalent $(\bar{e}_i^*)_i$. Setting $\mathcal{A} = \mathbb{R}e_\omega \oplus \tilde{\mathcal{A}}$ in the unitization of W concludes the proof. □

The proof of the following result uses the proof of Proposition 1.31 and Lemma 1.8. We omit its proof because it is very similar to the proof of Proposition 1.9.

PROPOSITION 1.32. *Let X be a Banach space with a normalized 1-unconditional basis that does not contain an isomorphic copy of ℓ_1. Then there exists a regular family $\mathcal{M}$ and a Banach space Y with a Schauder basis such that the space $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(Y)$ contains a complemented ideal $\widetilde{X}$ that is isomorphic as a Banach algebra to X and a subalgebra $\mathcal{A}$ that is isomorphic as a Banach algebra to $\mathbb{R}e_\omega \oplus J(T_{\mathcal{M}}^{1/2})$, so that $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(Y) = \widetilde{X} \oplus \mathcal{A}$.*

REMARK 1.33. As the proof of Proposition 1.31 clearly indicates, if the basis of X dominates the unit vector basis of ℓ_q for some $1 < q < \infty$, then in the conclusion of Proposition 1.31 $\mathcal{A}$ may be taken to be isomorphic as a Banach algebra to $J_p^* = J(\ell_p)^*$, where $1/p + 1/q = 1$. This happens, for example, if the space X has nontrivial cotype. Similarly, if the basis of X is dominated by the unit vector basis of ℓ_p for some $1 < p < \infty$, then in the conclusion of Proposition 1.32 $\mathcal{A}$ may be taken to be isomorphic as a Banach algebra to $J_p = J(\ell_p)$. This happens when, for example, the space X has nontrivial type.

2. Control on Diagonal Operators via Horizontally Block Finite Representability

Given a Banach space X with a normalized Schauder basis $(e_k)_k$ and a sequence of Banach spaces $(X_k)_{k=1}^\infty$, all having normalized Schauder bases, we shall define a type of direct sum $\mathcal{X} = (\sum \oplus X_k)_{\mathrm{utc}}^X$, the outside norm of which is a mixture of c_0 alongside finite, but arbitrarily large, pieces of the basis of the space X. This sum is fairly simply defined and it has the property that the sequence of projection operators $(I_k)_k$, each on the space X_k within the sum, is equivalent (in the operator norm), to the sequence of diagonal operators $(e_k^* \otimes e_k)_k$ in $\mathcal{L}_{\mathrm{diag}}(X)$. We will use this space $\mathcal{X}$ in the sequel and it exhibits certain key properties given to it by its components. We will use these components in Section 3 to define the more complicated space.

2.1. *The Definition of $\mathcal{X} = (\sum \oplus X_k)_{\mathrm{utc}}^X$*

Let us fix a Banach space X with a normalized Schauder basis $(e_i)_{i=1}^\infty$ with bimonotone constant A_0. That is, for each interval E of $\mathbb{N}$, the natural projection on the vectors $(e_k)_{k\in E}$ has norm at most A_0. Let us also fix a sequence of Banach spaces $(X_k)_{k=1}^\infty$, each one of which has a normalized Schauder basis $(t_{k,i})_{i=1}^\infty$. We do not, yet, make any additional assumption on the bases $(t_{k,i})_{i=1}^\infty$.

Let us denote by $c_{00}(X_k)_k$ the vector space of all sequence $(x_k)_k$, where $x_k \in X_k$ for all $k \in \mathbb{N}$ and only finitely many entries are nonzero. For each $k \in \mathbb{N}$, we naturally identify a vector $x \in X_k$ with the sequence $(x_k)_k$, the kth entry of which is x and all other entries are 0. Similarly define $c_{00}(X_k^*)_k$ and make the same identification. We define two subset of $c_{00}(X_k^*)_k$, namely

$$\mathcal{G}^{\mathrm{utc}} = \left\{ \sum_{k=1}^{i_0} a_k t_{k,i_0}^* : i_0 \in \mathbb{N} \text{ and } \exists (a_k)_{k=1}^\infty \text{ such that } \left\| w^* \text{-} \sum_{k=1}^\infty a_k e_k^* \right\|_{X^*} \le 1 \right\} \quad \text{and}$$

$$\mathcal{G}_0^{\text{utc}} = \mathcal{G}^{\text{utc}} \cup \left(\bigcup_{k=1}^{\infty} \left(\frac{1}{A_0} B_{X_k^*} \right) \right) \quad \text{(each } B_{X_k^*} \text{ is viewed in } c_{00}(Y_k^*)_k\text{)}.$$

REMARK 2.1. The term utc stems from the fact that if we view each element $x^* = (x_k^*)_k$ of $\mathcal{G}^{\text{utc}}$ as a matrix $((a_{k,i})_{k=1}^{\infty})_{i=1}^{\infty}$, where each $x_k^* = w^*\text{-}\sum_i a_{k,i} t_{k,i}^*$, then this matrix has nonzero entries in only one column i_0 and only above the diagonal.

For each $x = (x_k)_k$ and $x^* = (x_k^*)_k$ in $c_{00}(X_k)_k$ and $c_{00}(X_k^*)_k$, respectively, we define $x^*(x) = \sum_{k=1}^{\infty} x_k^*(x_k)$. We now define a norm for $x = (x_k)_k$ in $c_{00}(X_k)$:

$$\|x\|_{\mathcal{X}} = \sup\{x^*(x) : x^* \in \mathcal{G}_0^{\text{utc}}\}.$$

We set $\mathcal{X} = (\sum \oplus X_k)_{\text{utc}}^X$ to be the completion of $c_{00}(X_k)$ endowed with this norm. We also denote for each $n \in \mathbb{N}$, for later use, by $\mathcal{X}_n = (\sum_{k=1}^{n} \oplus X_k)_{\text{utc}}^X$ the subspace of $\mathcal{X}$ that consists of all $x = (x_k)_k$ with $x_k = 0$ for all $k > n$.

For each vector $x = (x_k)_k$ in $\mathcal{X}$, we define $\text{supp}(x) = \{k : x_k \neq 0\}$. We also define, for each $k \in \mathbb{N}$,

$$\operatorname*{supp}_k(x) = \{i \in \mathbb{N} : t_{k,i}^*(x_k) \neq 0\} = \operatorname*{supp}_{(t_{k,i})_i}(x_k). \tag{12}$$

We list some facts about the space $\mathcal{X}$.

PROPOSITION 2.2. *The space $\mathcal{X} = (\sum \oplus X_k)_{\text{utc}}^X$ satisfies the following*:

(i) *The sequence $(X_k)_k$ forms a shrinking Schauder decomposition of the space with bimonotone constant A_0. Hence, for each $k \in \mathbb{N}$, we may define the natural projection I_k, the image of which is X_k.*

(ii) *For every $i_0 \in \mathbb{N}$ and a sequence of scalars $(a_k)_{k=1}^{i_0}$, we have*

$$\left\| \sum_{k=1}^{i_0} a_k t_{k,i_0} \right\|_{\mathcal{X}} = \left\| \sum_{k=1}^{i_0} a_k e_k \right\|_X.$$

In particular, X is finitely representable in $\mathcal{X}$.

(iii) *For any vectors y, w in $\mathcal{X}$ such that* $\max \text{supp}(y) < \min \text{supp}(w)$ *with the property that for all $k \in \mathbb{N}$ the set* $\text{supp}_k(y)$ *is finite* (*i.e.*, *if $y = (x_k)_k$, then each x_k has finite support*) *and satisfy the condition*

$$\max_{k \in \mathbb{N}}\{\max(\operatorname*{supp}_k(y))\} < \min \text{supp}(w),$$

we have $\|y + w\|_{\mathcal{X}} = \max\{\|y\|_{\mathcal{X}}, \|w\|_{\mathcal{X}}\}$. In particular, every normalized block sequence $(w_k)_k$ in $\mathcal{X}$ has, for every $\varepsilon > 0$, a subsequence that is $(1 + \varepsilon)$-equivalent to the unit vector basis of c_0.

(iv) *For all sequences of scalars $(a_k)_{k=1}^{\infty}$, we have*

$$\left\| \text{SOT-}\sum_{k=1}^{\infty} a_k I_k \right\| = \left\| \text{SOT-}\sum_{k=1}^{\infty} a_k e_k^* \otimes e_k \right\|.$$

In particular, $(I_k)_k$ is isometrically equivalent to $(e_k^* \otimes e_k)_k$.

The first two statements are fairly straightforward (except perhaps the shrinking property in (i), which follows from (iii)). We do not explicitly use statements (iii) or (iv), so we do not include a proof of any of them. We note that (iii) follows from the "utc" condition in $\mathcal{G}^{\mathrm{utc}}$. The proof of (iv) is a simplified version of the proof of [ADT, Theorem 2.4], and later on we prove something similar to this for a more complicated space (see Propositions 4.4 and 4.6).

REMARK 2.3. If, for fixed $k_0 \in \mathbb{N}$, we take the natural identification id_{k_0} of X_{k_0} with a subspace of $\mathcal{X} = (\sum_k \oplus X_k)_X^{\mathrm{utc}}$, then id_{k_0} is not necessarily an isometry. In fact, for each $x \in X_{k_0}$, we have $(1/A_0)\|x\| \le \|\mathrm{id}_{k_0}(x)\| \le \sup_i \|t^*_{k_0,i}\|\|x\|$. The upper bound comes from the set $\mathcal{G}^{\mathrm{utc}}$.

REMARK 2.4. The assumption that $(t_{k,i})_i$ is a Schauder basis of X_k is not entirely necessary. It is, for example, sufficient if for each k there is a complemented subspace W_k of X_k such that $(t_{k,i})_i$ is a Schauder basis of W_k. In general, what is required is that for each X_k there is a meaningful notion of support with respect to $(t_{k,i})_i$ in the sense that there exists a bounded sequence $(t^*_{k,i})_i$ in X^*_k that is orthogonal to $(t_{k,i})_i$ so that each vector w can be approximated by a sequence of vectors $(w_j)_j$ so that, for each $j \in \mathbb{N}$, the set $\{i : t^*_{k,i}(w_j) \neq 0\}$ is finite. A bounded Markushevich basis is sufficient as well.

3. An X-Bourgain–Delbaen–Argyros–Haydon Direct Sum of Spaces

In [Z] Zisimopoulou defined a Bourgain–Delbaen direct sum $(\sum \oplus X_n)_{\mathrm{AH}}$ of a sequence of separable Banach spaces where the outside norm is based on the Argyros–Haydon construction from [AH]. One of the most important features of this construction is that, under certain assumptions, every bounded linear operator on this space is a multiple of the identity plus a horizontally compact operator (see Definition 6.1). This is used in [MPZ] where an appropriate choice of the sequence $(X_n)_n$ yields a space with a $C(\omega)$ Calkin algebra. A careful iteration of this procedure is also implemented in that paper, and this leads, for each countable compactum K, to a space having $C(K)$ as a Calkin algebra. In this section we modify the construction of Zisimopoulou by adding the space $(\sum \oplus X_k)^X_{\mathrm{utc}}$ as an ingredient. More precisely, we will define a direct sum of the spaces X_k, $k \in \mathbb{N}$, where the outside norm is a mixture of a Bourgain–Delbaen–Argyros–Haydon sum with a "utc" sum. The purpose of this is to obtain a space $\mathcal{Y}_X$ that has the space $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$ as a Calkin algebra instead of $C(\omega)$. The definition of this space $\mathcal{Y}_X$ is long and technical; however, it is similar to [Z], and we present it rather comprehensively. We shall shorten the proof of some of the properties of the resulting space, whenever they are almost word for word applicable in the present case, by referring the reader to the appropriate proof in the appropriate paper.

3.1. Determining the Shape of the Space $\mathcal{Y}_X$

We fix for the rest of this paper a Banach space X with a normalized Schauder basis $(e_i)_{i=1}^\infty$ with bimonotone constant A_0 and a sequence of Banach spaces $(X_k)_{k=1}^\infty$, each one with a Schauder basis $(t_{k,i})_{i=1}^\infty$, and we set $\mathcal{X} = (\sum \oplus X_k)_{\mathrm{utc}}^X$. We will specify the spaces X_k eventually but this is not yet important. The space $\mathcal{Y}_X$ is going to be a subspace of the following "large" space:

$$\begin{aligned}\mathcal{Z}_X^\infty &= \left(\left(\sum_{k=1}^\infty \oplus X_k\right)_{\mathrm{utc}}^X \oplus \ell_\infty(\Gamma)\right)_\infty \\ &\equiv \left(\left(\sum_{k=1}^\infty \oplus X_k\right)_{\mathrm{utc}}^X \oplus \left(\sum_{k=1}^\infty \oplus \ell_\infty(\Delta_k)\right)_\infty\right)_\infty, \qquad (13)\end{aligned}$$

where Γ is a countable set that is the union of a collection of pairwise disjoint finite sets Δ_k, $k \in \mathbb{N}$, that will be determined later. The identification in (13) is done in the obvious way, and each element z in $\mathcal{Z}_X^\infty$ can be represented in the form $z = (x_k, y_k)_{k=1}^\infty$, where $x_k \in X_k$ and $y_k \in \ell_\infty(\Delta_k)$ for each $k \in \mathbb{N}$. Furthermore, for each $n \in \mathbb{N}$ set $\Gamma_n = \bigcup_{k=1}^n \Delta_k$ and

$$\begin{aligned}\mathcal{Z}_X^n &= \left(\left(\sum_{k=1}^n \oplus X_k\right)_{\mathrm{utc}}^X \oplus \ell_\infty(\Gamma_n)\right)_\infty \\ &\equiv \left(\left(\sum_{k=1}^n \oplus X_k\right)_{\mathrm{utc}}^X \oplus \left(\sum_{k=1}^n \oplus \ell_\infty(\Delta_k)\right)_\infty\right)_\infty, \qquad (14)\end{aligned}$$

that is, the space of all $z = (x_k, y_k)_{k=1}^\infty \in \mathcal{Z}_X^\infty$ with $x_k = 0$ and $y_k = 0$ for all $k > n$. We naturally represent each such z in $\mathcal{Z}_X^n$ by $(x_k, y_k)_{k=1}^n$. For each $n \in \mathbb{N}$, we define $R_n : \mathcal{Z}_X^\infty \to \mathcal{Z}_X^n$ to be the restriction onto the n first coordinates, that is, $R_n(x_k, y_k)_{k=1}^\infty = (x_k, y_k)_{k=1}^n$ (this vector may also be represented as an infinite sequence by adding zeros to the tail). We point out that $\|R_n\| \le A_0$ (R_n may fail to have norm one) because of Proposition 2.2(i).

Before defining the embedding of $\mathcal{Y}_X$ into $\mathcal{Z}_X^\infty$, we discuss certain ingredients on the latter space. We assume the existence of these ingredient and we do not define them precisely until later; however, in the end this reduces to the definition of the sets Δ_k, $k \in \mathbb{N}$. Let us assume that, for each $n \in \mathbb{N}$ and $\gamma \in \Delta_{n+1}$, we have fixed a bounded linear functional $c_\gamma^* : \mathcal{Z}_X^n \to \mathbb{R}$. Define, for $n \in \mathbb{N}$, the bounded linear extension operator $i_{n,n+1} : \mathcal{Z}_X^n \to \mathcal{Z}_X^{n+1}$ given by

$$\begin{aligned}&i_{n,n+1}(x_k, y_k)_{k=1}^n = (\tilde{x}_k, \tilde{y}_k)_{k=1}^{n+1}, \\ &\quad \text{where } \tilde{x}_k = x_k,\ \tilde{y}_k = y_k \text{ for } 1 \le k \le n \text{ and } \tilde{x}_{n+1} = 0, \\ &\quad \tilde{y}_{n+1} = (c_\gamma^*((x_k, y_k)_{k=1}^n))_{\gamma \in \Delta_{n+1}}.\end{aligned}$$

This naturally defines for $m < n \in \mathbb{N}$ the bounded linear extension operator $i_{m,n} : \mathcal{Z}_X^m \to \mathcal{Z}_X^n$ given by $i_{m,n} = i_{n-1,n} \circ i_{n-2,n-1} \circ \cdots \circ i_{m,m+1}$. For $n = m$, we take $i_{n,n}$ to be the identity on $\mathcal{Z}_X^n$. The following are easy to see.

REMARK 3.1. For all $l \le m \le n$, we have

$$i_{l,n} = i_{m,n} \circ i_{l,m} = i_{m,n} \circ R_m \circ i_{l,n}.$$

Moreover, for $m \le n \in \mathbb{N}$ and $z = (x_k, y_k)_{k=1}^m \in \mathcal{Z}_X^m$, if $i_{m,n}(z) = (\tilde{x}_k, \tilde{y}_k)_{k=1}^n$, then

(i) $\tilde{x}_k = x_k$ and $\tilde{y}_k = y_k$ for $1 \le k \le m$,
(ii) $\tilde{x}_k = 0$ for $m < k \le n$, and
(iii) for each $m < k \le n$ and $\gamma \in \Delta_k$, we have $e_\gamma^*(i_{m,n}(\tilde{y}_k)) = c_\gamma^*(i_{m,k-1}(z))$,

where e_γ^* denote the coordinate functionals on $\ell_\infty(\Delta_k)$.

For each $m \le n$, we define the bounded linear operator $P_m^{(n)} : \mathcal{Z}_X^\infty \to \mathcal{Z}_X^n$

$$P_m^{(n)} = i_{m,n} \circ R_m,$$

which may be also viewed as an operator on $\mathcal{Z}_X^n$. One can easily check the following.

REMARK 3.2. Let $l, m \le n \in \mathbb{N}$. Then

(i) $P_m^{(n)}$ is a projection,
(ii) $P_l^{(n)} P_m^{(n)} = P_{\min\{l,m\}}^{(n)}$, and
(iii) $P_n^n[\mathcal{Z}_X^\infty] = R_n[\mathcal{Z}_X^\infty] = \mathcal{Z}_X^n$.

We define for $l < m \le n$ the bounded linear projection $P_{(l,m]}^{(n)} = P_m^{(n)} - P_l^{(n)}$. We will now make an additional assumption on the form of the functionals $(c_\gamma^*)_{\gamma \in \Delta_{n+1}}$, $n \in \mathbb{N}$. Recall that the sequence $(X_k)_k$ is a Schauder decomposition of the space $(\sum \oplus X_k)_{\mathrm{utc}}^X$ with bimonotone constant A_0. Let us fix $0 < \beta_0 < 1/A_0$ and assume that, for every $n \in \mathbb{N}$ and $\gamma \in \Delta_{n+1}$, there are $\beta \in [-\beta_0, \beta_0]$ and b^* in the unit ball of $(\mathcal{Z}_X^n)^*$ such that

$$c_\gamma^* = e_\eta^* + \beta b^* \circ P_{(m,n]}^{(n)}, \quad \text{where } 1 \le m < n \text{ and } \eta \in \Delta_m, \text{ or} \tag{15a}$$

$$c_\gamma^* = \beta b^* \circ P_n^{(n)}. \tag{15b}$$

REMARK 3.3. It is important to note that whether properties (15a) and (15b) of the functionals c_γ^*, $\gamma \in \Delta_{n+1}$ are satisfied is witnessed on the space $\mathcal{Z}_X^n$ and it does not depend on the entire space $\mathcal{Z}_X^\infty$.

Although the proof of the following is identical to that of [Z, Proposition 5.1], we include a short description of it for completeness. We fix

$$C_0 = 1 + 2\beta_0 A_0/(1 - \beta_0 A_0) \tag{16}$$

throughout the rest of the paper.

REMARK 3.4. If β_0 is chosen sufficiently close to zero, then C_0 can be desirably close to one.

LEMMA 3.5. *Let us fix $n \in \mathbb{N}$ and assume that, for all $1 \le m \le n$ and $\gamma \in \Delta_{m+1}$, the functional c_γ^* satisfies* (15a) *or* (15b)*. Then $\|i_{l,m}\| \le C_0$ (see* (16)*) for all $1 \le l \le m \le n+1$.*

Proof. Fix $l \le n$ and prove the statement by induction on $l \le m \le n+1$. For $l = m$, the map $i_{l,l}$ is the identity and there is nothing to prove. Assume that the conclusion holds for all $l \le d \le m$ for some $l \le m \le n$, which also implies that $\|P_{(d,m]}^{(m)}\| \le A_0 + C_0 A_0$ for all $l \le d \le m$. Let $z = (x_k, y_k)_{k=1}^l \in \mathcal{Z}_X^l$ with $\|z\| \le 1$. If $i_{l,m+1}(z) = (\tilde{x}_k, \tilde{y}_k)_{k=1}^{m+1}$, then by Remark 3.1 we can deduce that

$$\|i_{l,m+1}(z)\| = \max\{\|i_{l,m}(z)\|, \|\tilde{y}_{m+1}\|\} \le \max\Big\{C_0, \max_{\gamma \in \Delta_{m+1}} |c_\gamma^*(i_{l,m}(z))|\Big\}.$$

To complete the proof, fix $\gamma \in \Delta_{m+1}$. If c_γ^* satisfies (15b), then one can check that $|c_\gamma^*(i_{l,m}(z))| \le \beta_0 C_0 < C_0$. Otherwise, there are $1 \le d < m$, $\eta \in \Delta_d$, $\beta \in [\beta_0, \beta_0]$, and b^* in the unit ball of $(\mathcal{Z}_X^m)^*$ with $c_\gamma^* = e_\eta^* + \beta b^* \circ P_{(d,m]}^{(m)}$. If $d \le l$, then it follows that $e_\eta^*(i_{l,m}(z)) = e_\gamma^*(z)$, therefore $|c_\gamma^*(i_{l,m}(z))| \le 1 + \beta_0(A_0 + C_0 A_0) = C_0$. If $l < d < m$, then it can be seen that $P_{(d,m]}^{(m)}(i_{l,m}(z)) = 0$, that is, $|c_\gamma^*(i_{l,m}(z))| = |e_\eta^*(i_{l,m}(z))| \le C_0$. $\square$

Lemma 3.5 and Remark 3.2 allow us to define, for each $n \in \mathbb{N}$, the extension operator $i_n : \mathcal{Z}_X^n \to \mathcal{Z}_X^\infty$ with $i_n(x) = \lim_m i_{n,m}(x)$. This extension is well defined by Remark 3.1(ii). Let us restate Remark 3.1 in the language of the new extension operators.

REMARK 3.6. For all $n \in \mathbb{N}$, we have

$$\|i_n\| \le C_0, \quad \text{and for all } m \le n, \text{ we have } i_m = i_n \circ R_n \circ i_m. \tag{17}$$

Moreover, for $n \in \mathbb{N}$ and $z = (x_k, y_k)_{k=1}^n \in \mathcal{Z}_X^n$, if $i_n(z) = (\tilde{x}_k, \tilde{y}_k)_{k=1}^\infty$, then

(i) $\tilde{x}_k = x_k$ and $\tilde{y}_k = y_k$ for $1 \le k \le n$,
(ii) $\tilde{x}_k = 0$ for all $k > n$, and
(iii) for each $k > n$ and $\gamma \in \Delta_k$, we have $e_\gamma^*(i_n(z)) = c_\gamma^*(i_{n,k-1}(z))$,

where e_γ^* denote the coordinate functionals on $\ell_\infty(\Delta_k)$. By Lemma 3.5 we also have

$$\|z\| \le \|i_n(z)\| \le C_0\|z\|. \tag{18}$$

We now define, for each $n \in \mathbb{N}$, the space $Y_X^n = i_n[\mathcal{Z}_X^n]$, and for all $n \in \mathbb{N}$, we define the bounded linear operator $P_n = i_n \circ R_n$.

REMARK 3.7. The following hold:

(i) The space Y_X^n is C_0-isomorphic to $\mathcal{Z}_X^n$ for all $n \in \mathbb{N}$ via the map i_n, the inverse of which is the map R_n (by (18));
(ii) for all $n \in \mathbb{N}$, we have that $P_n : \mathcal{Z}_X^\infty \to Y_X^n$ is a bounded linear projection with $\|P_n\| \le A_0 C_0$; and
(iii) for all $m \le n \in \mathbb{N}$, we have $P_m P_n = P_n P_m = P_m$ (this follows from (17)).

We may therefore define for $m \le n \in \mathbb{N}$ the projection $P_{(m,n]} = P_n - P_m$, which has norm at most $2C_0A_0$. We also write $P_{(0,n]} = P_n$.

It follows that the sequence of spaces $(Y_X^n)_n$ is increasing with respect to inclusion. We define $\mathcal{Y}_X$ to be the closure (in the norm topology) of $\bigcup_n Y_X^n$ in the space $\mathcal{Z}_X^\infty$. The space $\mathcal{Y}_X$ admits a Schauder decomposition $(Z_n)_n$ with associated projections $(P_n)_n$. That is, $P_1[\mathcal{Y}_X] = Z_1$ and for $n \ge 2$ $P_{(n-1,n]}[\mathcal{Y}_X] = Z_n$. Using (17), it is not hard to see that in fact

$$Z_n = i_n[(X_n \oplus \ell_\infty(\Delta_n))_\infty],$$

where $(X_n \oplus \ell_\infty(\Delta_n))_\infty$ is viewed as a subspace of $\mathcal{Z}_X^n$ in the natural way. Hence, we may write $\mathcal{Y}_X = \sum \oplus Z_n$, and if $z \in \mathcal{Y}_X$, then $z = \sum_{n=1}^\infty P_{\{n\}}z$, where $P_{\{n\}} = P_{(n-1,n]}$. We can define the set

$$\operatorname*{supp}_{\mathrm{BD}}(z) = \{n \in \mathbb{N} : P_{\{n\}}z \neq 0\},$$

and we also denote by $\mathrm{range}_{\mathrm{BD}}(z)$ the smallest interval of $\mathbb{N}$ containing $\mathrm{supp}_{\mathrm{BD}}(z)$. For $0 < m \le n$, it can be seen that

$$P_{(m,n]}[\mathcal{Y}_X] = \sum_{k=m+1}^{n} \oplus Z_k = i_n[\mathcal{Z}_X^{(m,n]}], \tag{19}$$

where

$$\mathcal{Z}_X^{(m,n]} = \left(\left(\sum_{k=m+1}^{n} \oplus X_k\right)_{\mathrm{utc}}^{X} \oplus \left(\sum_{k=m+1}^{n} \oplus \ell_\infty(\Delta_k)\right)_\infty\right)_\infty$$

is viewed as a subspace of $\mathcal{Z}_X^n$ in the natural way.

Remark 3.8. As $\mathcal{Y}_X$ is a subspace of $\mathcal{Z}_X^\infty$, every $z \in \mathcal{Y}_X$ is of the form $z = (x_k, y_k)_{k=1}^\infty$ and $\|z\| = \max\{\|(x_k)\|_{\mathcal{X}}, \|(y_k)_k\|_{\ell_\infty(\Gamma)}\}$. By setting $x = (x_k)_k \in \mathcal{X}$ and $y = (y_k)_k \in \ell_\infty(\Gamma)$, we obtain

$$\begin{aligned} \|x\| = \max\{&\sup\{|x^*(x)| : x^* \in \mathcal{G}^{\mathrm{utc}}\},\\ &\sup\{|x^*(x_k)| : x^* \in (1/A_0)B_{X_k^*} : k \in \mathbb{N}\},\\ &\sup\{|e_\gamma^*(y)| : \gamma \in \Gamma\}\}. \end{aligned} \tag{20}$$

3.2. The Precise Definition of the Space $\mathcal{Y}_X$

We have established the general form of the space $\mathcal{Y}_X$. To abide by the assumptions of Lemma (3.5), we have to fix $0 < \beta_0 < 1/A_0$ and choose a sequence of disjoint finite sets $(\Delta_n)_n$ and bounded linear functionals $(c_\gamma^*)_{n \in \Delta_{n+1}}$ defined on $\mathcal{Z}_X^n$ so that (15a) and (15b) are satisfied. Crucially, by Remark 3.3, we are allowed to choose by induction on n the set Δ_n and the corresponding functionals $(c_\gamma^*)_{\gamma \in \Delta_n}$ as they act each time only on what has been defined so far.

First, we fix a pair of increasing sequences of natural numbers $(\tilde{m}_j, \tilde{n}_j)_j$ that satisfy [AH, Assumption 2.3], namely

$$\begin{aligned} &(1) \quad \tilde{m}_1 \geq 4, \qquad (2) \quad \tilde{m}_{j+1} \geq \tilde{m}_j^2, \\ &(3) \quad \tilde{n}_1 \geq \tilde{m}_1^2, \quad \text{and} \quad (4) \quad \tilde{n}_{j+1} \geq (16\tilde{n}_j)^{\log_2(\tilde{m}_{j+1})}. \end{aligned} \tag{21}$$

Let us next choose an infinite sequence of pairwise disjoint infinite subsets of the natural numbers $(L_k)_{k=0}^{\infty}$. For each $k \in \mathbb{N}$, we define X_k to be the Argyros–Haydon space defined in [AH, Section 10.2] using the sequence $(\tilde{m}_j, \tilde{n}_j)_{j \in L_k}$. Each of these spaces has the "scalar-plus-compact" property (i.e., every $T : X_k \to X_k$ is a compact perturbation of a scalar operator), and for $k \neq m$, every bounded linear operator $T : X_k \to X_m$ is compact (see [AH, Theorem 10.4]). We will use the set $L_0 = \{\ell_1^0 < \ell_2^0 < \cdots\}$ to define the outside norm of the direct sum. Henceforth we write $\tilde{m}_{\ell_j^0} = m_j$ and $\tilde{n}_{\ell_j^0} = n_j$. We make the assumption that $\beta_0 = 1/m_1 < 1/A_0$.

Let us choose, for each $k \in \mathbb{N}$, a 1-norming countable and symmetric subset $\tilde{F}_k$ of the unit ball of X_k^*, and let F_k^n be the symmetric subset of $(1/A_0)\tilde{F}_k$ set consisting of the first n-elements of $(1/A_0)\tilde{F}_k$ and their negatives. Each set F_k^n may be naturally identified with a subset of the unit ball of $((\sum \oplus X_k)_{\mathrm{utc}}^X)^*$. For each $n \in \mathbb{N}$, define $K_n = \bigcup_{k=1}^n F_k^n$. Let us also take the set $\mathcal{G}^{\mathrm{utc}}$ (see the beginning of Section 2.1) which is a symmetric and separable subset of the unit ball of $((\sum \oplus X_k)_{\mathrm{utc}}^X)^*$. Choose an increasing sequence of finite symmetric subsets $(\mathcal{G}_n^{\mathrm{utc}})_n$, the union of which is dense in $\mathcal{G}^{\mathrm{utc}}$.

We are prepared to inductively define the sets $(\Delta_n)_n$ and the corresponding functionals. Set $\Delta_1 = \{0\}$. There is no need to define c_γ^* for $\gamma \in \Delta_1$. Assume that we have defined the sets $\Delta_1, \ldots, \Delta_n$ and the families of functionals $(c_\gamma^*)_{\gamma \in \Delta_k}$, $1 \leq k \leq n$. Having defined these elements means having defined the space $\mathcal{Z}_X^n$ (see (14)). We also assume that to each $\gamma \in \Gamma_n = \bigcup_{k=1}^n \Delta_k$ we have assigned a natural number $\sigma(\gamma)$, so that $\max_{\gamma \in \Delta_k} \sigma(\gamma) < \min_{\gamma \in \Delta_{k+1}} \sigma(\gamma)$ for $1 \leq k \leq n-1$ and the map $\sigma : \Gamma_n \to \mathbb{N}$ is injective. Furthermore, assume that to each $\gamma \in \Gamma_n \setminus \Gamma_1$ we have a assigned a positive number $\mathrm{weight}(\gamma) = 1/m_j$, for some $j \in \mathbb{N}$, and a natural number $\mathrm{age}(\gamma) = a$ with $1 \leq a \leq n_j$. For each $1 \leq k \leq n$, if $\gamma \in \Delta_k$, we also write $\mathrm{rank}(\gamma) = k$.

Set $N_{n+1} = 2^n(\#\Gamma_n)$ and let B_n be the set of all linear combination $\sum_{\eta \in \Gamma_n} a_\eta e_\eta^*$ with $\sum_{\eta \in \Gamma_n} |a_\eta| \leq 1$, and a_η is a rational number with denominator dividing $N_{n+1}!$. Set $A_n = K_n \cup \mathcal{G}_n^{\mathrm{utc}} \cup B_n$. The set Δ_{n+1} is the union of the following four finite sets consisting of triples and quadruples:

$$\Delta_{n+1}^{\mathrm{Even}_0} = \{(n+1, 1/m_{2j}, b^*) : 2j \leq n+1, b^* \in A_n\}, \tag{22a}$$

$$\begin{aligned} \Delta_{n+1}^{\mathrm{Odd}_0} = \{&(n+1, 1/m_{2j-1}, \eta) : 2j-1 \leq n+1, \eta \in \Gamma_n \text{ with} \\ &\mathrm{weight}(\eta) = 1/m_{4i-2} < (1/n_{2j-1})^2\}, \end{aligned} \tag{22b}$$

$$\begin{aligned} \Delta_{n+1}^{\mathrm{Even}_1} = \{&(n+1, \xi, 1/m_{2j}, b^*) : \xi \in \Gamma_n \text{ with } \mathrm{weight}(\xi) = 1/m_{2j}, \\ &\mathrm{age}(\xi) < n_{2j}, b^* \in A_n\}, \end{aligned} \tag{22c}$$

$$\begin{aligned}\Delta_{n+1}^{\mathrm{Odd}_1} = \{(n+1, \xi, 1/m_{2j-1}, \eta) : \xi \in \Gamma_n \text{ with weight}(\xi) = 1/m_{2j-1} \text{ and}\\ \text{age}(\xi) < n_{2j-1}, \eta \in \Gamma_n \text{ with rank}(\xi) < \text{rank}(\eta) \text{ and}\\ \text{weight}(\eta) = 1/m_{4\sigma(\xi)}\}.\end{aligned} \tag{22d}$$

We define, for each $\gamma \in \Delta_{n+1}$, the corresponding linear functional c_γ^*. Note that each e_η^* for $\eta \in \Gamma_n$ and each b^* for $b^* \in A_n$ act as a linear functional on $\mathcal{Z}_X^n$ in the natural way. We set

$$\begin{aligned}
c_\gamma^* &= \frac{1}{m_{2j}} b^* \circ P_{(0,n]}^n, \text{age}(\gamma) = 1, \text{ and weight}(\gamma) = 1/m_{2j}, \text{ if } \gamma \in \Delta_{n+1}^{\mathrm{Even}_0},\\
c_\gamma^* &= \frac{1}{m_{2j-1}} e_\eta^* \circ P_{(0,n]}^n, \text{age}(\gamma) = 1, \text{ and weight}(\gamma) = 1/m_{2j-1},\\
&\quad \text{if } \gamma \in \Delta_{n+1}^{\mathrm{Odd}_0},\\
c_\gamma^* &= e_\xi^* + \frac{1}{m_{2j}} b^* \circ P_{(p,n]}^n, \text{age}(\gamma) = \text{age}(\xi) + 1, \text{ and weight}(\gamma) = 1/m_{2j},\\
&\quad \text{if } \gamma \in \Delta_{n+1}^{\mathrm{Even}_1} \text{ and } p = \text{rank}(\xi),\\
c_\gamma^* &= e_\xi^* + \frac{1}{m_{2j-1}} e_\eta^* \circ P_{(p,n]}^n, \text{age}(\gamma) = \text{age}(\xi) + 1, \text{ and weight}(\gamma) = 1/m_{2j-1},\\
&\quad \text{if } \gamma \in \Delta_{n+1}^{\mathrm{Odd}_1} \text{ and } p = \text{rank}(\xi).
\end{aligned}$$

Finally, we extend the definition of the function σ to the set Δ_{n+1}, so that it remains one-to-one injective on Γ_{n+1} and $\max_{\gamma \in \Delta_n} \sigma(\gamma) < \min_{\gamma \in \Delta_{n+1}} \sigma(\gamma)$.

REMARK 3.9. Comparing the definition of this section to the definition presented in [Z, Section 4], modulo perhaps certain convexity conditions, the key addition is that we allow the functionals b^* to be chosen from the set $\mathcal{G}_n^{\mathrm{utc}}$ as well.

3.3. Basic Properties of $\mathcal{Y}_X$

For every $n \in \mathbb{N}$ and $\gamma \in \Delta_{n+1}$, the functional c_γ^* is defined on $\mathcal{Z}_X^n$. We extend its domain to the whole space $\mathcal{Z}_X^\infty$ and hence also to the space $\mathcal{Y}_X$ by taking $c_\gamma^* \circ R_n$, and we denote this "new" functional by c_γ^* as well. We also set $c_\gamma^* = 0$ if $\gamma \in \Delta_1$. We then define, for each $\gamma \in \Gamma$, the functional $d_\gamma^* = e_\gamma^* - c_\gamma^*$, which is defined on $\mathcal{Z}_X^\infty$ and hence also on $\mathcal{Y}_X$. An important fact is that if $\gamma \in \Delta_n$, then $d_\gamma^* = e_\gamma^* \circ P_{\{n\}}$.

The following result provides what is called the evaluation analysis of a coordinate functional e_γ^*. It is also used in [Z] (Proposition 5.3); however, it appeared earlier in [AH, Proposition 4.5] where a proof may be found.

PROPOSITION 3.10. *Let* $n \in \mathbb{N}$ *and* $\gamma \in \Delta_{n+1}$ *with* $\text{weight}(\gamma) = 1/m_j$ *and* $\text{age}(\gamma) = a \le n_j$. *Then there exist* $0 = p_0 < p_1 < \cdots < p_a = n+1$, *elements* $\xi_1, \xi_2, \ldots, \xi_a$ *of weight* $1/m_j$ *with* $\xi_i \in \Delta_{p_i}$ *and* $\xi_a = \gamma$, *and functionals* $b_i^* \in$

A_{p_i-1} *(see paragraph before* (22a)*) so that*

$$e_\gamma^* = \sum_{i=1}^{a} d_{\xi_i}^* + \frac{1}{m_j}\sum_{i=1}^{a} b_i^* \circ P_{(p_{i-1},p_i)}. \tag{23}$$

The sequence $(p_i, \xi_i, b_i^*)_{i=1}^a$ *is called the evaluation analysis of* γ.

The following lemma allows us to conversely build functionals with a certain prescribed evaluation analysis, provided that certain mild conditions are satisfied. For a proof, see [AH, Proposition 4.7]

LEMMA 3.11. *Let* $j \in \mathbb{N}$, $1 \le a \le n_{2j}$, $0 = p_0 < p_1 < \cdots < p_a$ *with* $2j \le p_1$, *and* $b_i^* \in A_{p_i-1}$. *Then there exist* $\xi_i \in \Delta_{p_i+1}$ *for* $1 \le i \le a$ *and a* $\gamma \in \Gamma$ *with* $\text{weight}(\gamma) = 2j$ *and evaluation analysis* $(p_i, \xi_i, b_i^*)_{i=1}^a$.

4. The Calkin Algebra of $\mathcal{Y}_X$

In this section we will assume a result from the sequel to prove the desired description of the Calkin algebra of $\mathcal{Y}_X$, namely that it is isomorphic as a Banach algebra to $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$. For $n \in \mathbb{N}$, we define a bounded linear operator $I_n : \mathcal{Z}_X \to \mathcal{Z}_X$ as follows. If z is in $\mathcal{Y}_X$ and $z = (x_k, y_k)_{k=1}^\infty$ is its representation in $\mathcal{Z}_X^\infty$, then set $R_{\{n\},0}z = (\tilde{x}_y, \tilde{y}_k)_{k=1}^n$ with $\tilde{y}_k = 0$ for $1 \le k \le n$, $\tilde{x}_k = 0$, for $1 \le k < n-1$, and $\tilde{x}_n = x_n$. Note that $R_{\{n\},0} : \mathcal{Y}_X \to \mathcal{Z}_X^n$ is a bounded linear operator with norm at most A_0. We then set $I_n = i_n \circ R_{\{n\},0}$, that is, $I_n(z) = i_n((0,0),\ldots,(x_n,0))$. The map I_n is a projection of norm at most A_0C_0 and its image is $(A_0 \sup_i \|t_{n,i}^*\|)$-isomorphic to the space X_n. Furthermore, the map $A_n : \mathcal{Y}_X \to \mathcal{Y}_X$ defined by $A_n z = i_n((0,0),\ldots,(0,y_n))$ is a finite rank operator and hence compact. It is important to note that $I_n = P_{\{n\}} - A_n$, that is, I_n is a compact perturbation of $P_{\{n\}}$. The following cannot be proved yet and requires some work. We use it in this section to describe the Calkin algebra of $\mathcal{Y}_X$ and postpone its proof until much later.

THEOREM 4.1 (Theorem 7.3). *For every bounded linear operator* $T : \mathcal{Y}_X \to \mathcal{Y}_X$, *there exist a sequence of real numbers* $(a_k)_{k=0}^\infty$ *and a sequence of compact operators* $(K_n)_n$ *such that*

$$T = \lim_n \left(a_0 I + \sum_{k=1}^n a_k I_k + K_n \right),$$

where the limit is taken in the operator norm.

COROLLARY 4.2. *If we denote by* $[I]$ *and* $[I_n]$ *the equivalence class of* I *and* I_n *respectively in* $\mathcal{C}al(\mathcal{Y}_X)$, *then the linear span of* $\{[I]\} \cup \{[I_n] : n \in \mathbb{N}\}$ *is dense in the space* $\mathcal{C}al(\mathcal{Y}_X)$.

We now proceed to making several estimates as to how the norm on the linear span $\{[I]\} \cup \{[I_n] : n \in \mathbb{N}\}$ compares to the norm of the linear span $\{I\} \cup \{e_i^* \otimes e_i : i \in \mathbb{N}\}$ in $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$.

LEMMA 4.3. *Let $i_0 \leq i \in \mathbb{N}$ and $(b_k)_{k=1}^{i_0}$ be a sequence of scalars. If*

$$u = (x_k, y_k)_{k=1}^{i_0} \in \left(\left(\sum_{k=1}^{i_0} \oplus X_k \right)_{\text{utc}}^{X} \oplus \left(\sum_{k=1}^{i_0} \oplus \ell_\infty(\Delta_k) \right)_\infty \right)_\infty$$

with $x_k = b_k t_{k,i}$ for $1 \leq k \leq i_0$ and $z = i_{i_0}(u)$ (which by (19) is in $\mathcal{Y}_X$), then we have that

$$\left\| \sum_{k=1}^{i_0} b_k e_k \right\|_X \leq \|z\| \leq C_0 \max\left\{ \left\| \sum_{k=1}^{i_0} b_k e_k \right\|_X, \max_{1\leq k\leq i_0} \|y_k\|_\infty \right\}.$$

In particular, the space X is C_0-crudely finitely representable in $\mathcal{Y}_X$.

Proof. By (18) we have $\|u\| \leq \|z\| \leq C_0\|u\|$, hence it is sufficient to show that $\| \sum_{k=1}^{i_0} b_k e_k \|_X \leq \|u\| \leq \max\{\| \sum_{k=1}^{i_0} b_k e_k \|_X, \max_{1\leq k\leq i_0} \|y_k\|_\infty\}$. We define the vector $x = (\lambda_k t_{k,i})_{k=1}^{i_0} \in (\sum_{k=1}^{i_0} \oplus X_k)_{\text{utc}}^X$. Then (20) yields

$$\|u\| = \max\left\{ \max_{1\leq k\leq i_0} (1/A_0)|\lambda_k|, \sup_{x^*\in\mathcal{G}^{\text{utc}}} |x^*(x)|, \max_{1\leq k\leq i_0} \|y_k\|_\infty \right\}.$$

From the fact that the basis of X has a bimonotone constant A_0, we obtain that $\max_{1\leq k\leq i_0} |\lambda_k| \leq A_0\| \sum_{k=1}^{i_0} b_k e_k \|_X$, whereas from the definition of $\mathcal{G}^{\text{utc}}$ we obtain

$$\sup_{x^*\in\mathcal{G}^{\text{utc}}} |x^*(x)| = \sup\left\{ \left| \sum_{k=1}^{i_0} a_k b_k \right| : (a_k)_{k=1}^\infty \text{ is such that } \left\| w^*\text{-}\sum_{k=1}^\infty a_k e_k^* \right\|_X \leq 1 \right\}$$
$$= \left\| \sum_{k=1}^{i_0} b_k e_k \right\|_X.$$

The conclusion immediately follows from the equations. □

LEMMA 4.4. *Let $n \in \mathbb{N}$, $(a_k)_{k=0}^n$ be a sequence of scalars and $T : \mathcal{Y}_X \to \mathcal{Y}_X$ be the bounded linear operator $T = a_0 I + \sum_{k=1}^n a_k I_k$. Then, for every compact operator $K : \mathcal{Y}_X \to \mathcal{Y}_X$, we have*

$$\|T - K\| \geq \frac{1}{C_0} \left\| a_0 I_X + \sum_{k=1}^n a_k e_k^* \otimes e_k \right\|_{\mathcal{L}(X)}.$$

Proof. Let us for the moment fix $i_0 \in \mathbb{N}$ with $i_0 \geq n + 1$ and $(b_k)_{k=1}^{i_0}$ so that $\| \sum_{k=1}^{i_0} b_k e_k \|_X \leq 1$. For $i \geq i_0$, define the vector x_i exactly as in the statement of Lemma 4.3. We observe that the sequence $(x_i)_{i\geq i_0}$ is weakly null. Recall that for $1 \leq k \leq i_0$ the basis $(t_{k,i})_i$ of the space X_k is shrinking (although this is not explicitly stated, it follows easily from the proof of [AH, Proposition 5.2]). The natural image of X_k in the space $((\sum_{k=1}^{i_0} \oplus X_k)_X^{\text{utc}} \oplus (\sum_{k=1}^{i_0} \oplus \ell_\infty(\Delta_k))_\infty)_\infty$ is a $(A_0 \sup_i \|t_{k,i}^*\|)$-embedding (see Remark 2.3) and the map i_{i_0} is a C_0-embedding as well (see (18)). If we consider the natural image $(\tilde{t}_{k,i}^{i_0})_i$ of the sequence $(t_{k,i})_i$ in the aforementioned space, then the sequence $(i_{i_0}(\tilde{t}_{k,i}^{i_0}))_i$ is weakly null. As

$x_i = \sum_{k=1}^{i_0} b_k i_{i_0}(\tilde{t}_{k,i}^{i_0})$, we have that $(x_i)_i$ is weakly null. This means that $(Kx_i)_i$ converges to zero in norm, that is, $\liminf_i \|Tx_i - Kx_i\| = \liminf_i \|Tx_i\|$. We combine this with Lemma 4.3, according to which $\|x_i\| \le C_0$ for all $i \ge i_0$, to deduce

$$\|T - K\| \ge \frac{1}{C_0} \liminf_i \|Tx_i\|. \tag{24}$$

For $i \ge i_0$, the vector Tx_i has the form $i_{i_0}(x_k, y_k)_{k=1}^{i_0}$, where $x_k = (a_0 + a_k)b_k t_{k,i}$ for $1 \le k \le n$ and $x_k = a_0 b_k t_{k,i}$ for $n+1 \le k \le i_0$. By Lemma 4.3 we obtain

$$\begin{aligned}\|Tx_i\| &\ge \Big\|\sum_{k=1}^{n}(a_0 + a_k)b_k e_k + \sum_{k=n+1}^{i_0} a_0 b_k e_k\Big\|_X \\ &= \Big\|\Big(a_0 I_X + \sum_{k=1}^{n} a_k e_k^* \otimes e_k\Big)\Big(\sum_{k=1}^{i_0} b_k e_k\Big)\Big\|_X.\end{aligned}$$

Taking a supremum over all $i_0 \ge n+1$ and $(b_k)_{k=1}^{i_0}$ as in the first line of the proof yields the desired result. □

The following is proved using Proposition 3.10 in an identical manner as in the proof of [MPZ, Lemma 1.7].

LEMMA 4.5. *Let $n, q \in \mathbb{N}$ with $q > n$ and $\gamma \in \Delta_q$ with* weight$(\gamma) = 1/m_j$ *for some $j \in \mathbb{N}$. Consider the functional $g : \mathcal{Y}_X \to \mathbb{R}$ with $g = e_\gamma^* \circ P_{[1,n]}$. Then one of the following holds*:

(i) $g = 0$;
(ii) *There are $p_1 \in \mathbb{N}$ with $p_1 \le n$ and b^* in the unit ball of the space $((\sum_{k=1}^n \oplus X_k)_{\mathrm{utc}}^X \oplus (\sum_{k=1}^n \oplus \ell_\infty(\Delta_k))_\infty)_\infty^*$ so that $g = (1/m_j) b^* \circ P_{(p_1,n]}$*;
(iii) *There are $p_1 \in \mathbb{N}$ with $p_1 \le n$, $\gamma' \in \Delta_{p_1}$, and b^* as above so that $g = e_{\gamma'}^* + (1/m_j) b^* \circ P_{(p_1,n]}$.*

PROPOSITION 4.6. *Let $n \in \mathbb{N}$, $(a_k)_{k=0}^n$ be a sequence of scalars, and $T : \mathcal{Y}_X \to \mathcal{Y}_X$ be the bounded linear operator $T = a_0 I + \sum_{k=1}^n a_k I_k$. Then there exists a compact operator $K : \mathcal{Y}_X \to \mathcal{Y}_X$ such that*

$$\|T - K\| \le (2C_0^2 - C_0)\Big\|a_0 I_X + \sum_{k=1}^{n} a_k e_k^* \otimes e_k\Big\|_{\mathcal{L}(X)}.$$

Proof. Let us set

$$\Big\|a_0 I_X + \sum_{k=1}^{n} a_k e_k^* \otimes e_k\Big\|_{\mathcal{L}(X)} = M.$$

Set $a_k = 0$ for all $k > n$. This way, for all $x = \sum_{k=1}^\infty c_k e_k$ in X, we have $(a_0 I_X + \sum_{k=1}^n a_k e_k^* \otimes e_k)x = \sum_{k=1}^\infty (a_0 + a_k) c_k e_k$. Let $A : \mathcal{Y}_X \to \mathcal{Y}_X$ be defined as follows: if $x = (x_k, y_k)_{k=1}^\infty$, then $Ax = i_n((0, y_k)_{k=1}^n)$. This is a finite rank operator and hence it is compact. We define $K = A\sum_{k=1}^n a_k I_k$. To show that K

satisfies the conclusion, let $x = (x_k, y_k)_{k=1}^{\infty}$ be an element in the unit ball of $\mathcal{Y}_X$. Note that

$$\left(\sum_{k=1}^{n} a_k I_k - K\right)x = i_n(a_k x_k, 0)_{k=1}^{n} \quad \text{and} \quad a_0 I x = (a_0 x_k, a_0 y_k)_{k=1}^{\infty}. \tag{25}$$

This means that $(T - K)x = ((a_0 + a_k)x_k, w_k)_{k=1}^{\infty}$, where $w_k = a_0 y_k$, if $1 \leq k \leq n$ and w_k are some vectors in $\ell_\infty(\Delta_k)$ for $k > n$.

As $\|x\| \leq 1$ we conclude that, for every $x^* \in \mathcal{G}^{\mathrm{utc}}$, we have $|x^*(x)| \leq 1$. This means that, for all $i_0 \in \mathbb{N}$ and for all $(b_k)_{k=1}^{\infty}$ for which $\|w^*\text{-}\sum_{k=1}^{\infty} b_k e_k^*\|_{X^*} \leq 1$, we have $|\sum_{k=1}^{i_0} b_k t_{k,i_0}^*(x_k)| \leq 1$, that is, for all $i_0 \in \mathbb{N}$, we have

$$\left\|\sum_{k=1}^{i_0} t_{k,i_0}^*(x_k) e_k\right\|_X \leq 1. \tag{26}$$

This yields that, for any $(b_k)_{k=1}^{\infty}$ with $\|w^*\text{-}\sum_{k=1}^{\infty} b_k e_k^*\|_{X^*} \leq 1$ and every $i_0 \in \mathbb{N}$, we have

$$\begin{aligned}\left|\sum_{k=1}^{i_0} b_k(a_0 + a_k)t_{k,i_0}^*(x_k)\right| &\leq \left\|\sum_{k=1}^{i_0}(a_k + a_0)t_{k,i_0}^*(x_k)e_k\right\|_X \\ &\leq M\left\|\sum_{k=1}^{i_0} t_{k,i_0}^*(x_k)e_k\right\|_X \leq M \quad \text{by (26).}\end{aligned} \tag{27}$$

The above easily implies that, for all $x^* \in \mathcal{G}^{\mathrm{utc}}$, we have

$$|x^*((T - K)x)| \leq M.$$

Similarly, for all $k \in \mathbb{N}$ and $x_k^* \in (1/A_0)B_{X_k^*}$, we have $|x_k^*(x_k)| \leq 1$, that is,

$$\begin{aligned}|x_k^*((T - K)x)| &= |x_k^*((a_0 + a_k)x_k)| \leq |a_0 + a_k| = \|(a_0 + a_k)e_k\|_X \\ &= \left\|\left(a_0 I_X + \sum_{k=1}^{n} a_k e_k^* \otimes e_k\right)e_k\right\|_X \leq M.\end{aligned} \tag{28}$$

To obtain the desired conclusion, it remains to show that, for any $\gamma \in \Gamma$, we have

$$|e_\gamma^*((T - K)x)| \leq (2C_0^2 - C_0)M.$$

For $\gamma \in \Gamma$ with $\mathrm{rank}(\gamma) = q \leq n$, we have

$$|e_\gamma^*((T - K)x)| = |e_\gamma^*(a_0 y_q)| \leq |a_0| = \left\|\left(a_0 I_X + \sum_{k=1}^{n} a_k e_k^* \otimes e_k\right)e_{n+1}\right\|_X \leq M.$$

We now consider the case in which $\gamma \in \Gamma$ with $\mathrm{rank}(\gamma) = q > n$. Set $u = (a_k x_k, 0)_{k=1}^{n} \in \mathcal{Z}_X^n$. Arguments identical to that used in obtaining (27) and (28) yield that

$$\|u\| \leq \left\|\sum_{k=1}^{n} a_k e_k^* \otimes e_k\right\|_{\mathcal{L}(X)} \leq 2M.$$

We observe that if $m \le n$ then

$$\|P_{(m,n]}i_n(u)\| = \|i_n(u) - i_m R_m(u)\| \le (C_0 + C_0 A_0)\|u\| \le 4C_0 A_0 M.$$

Also observe that, for all γ' with $\operatorname{rank}(\gamma') \le n$, we have $e^*_\gamma(i_n(u)) = 0$. We combine the two facts with Lemma 4.5 to obtain

$$|e^*_\gamma(i_n(u))| \le \beta_0 4C_0 A_0 M.$$

Finally,

$$\begin{aligned}
|e^*_\gamma((T-K)x)| &= |e^*_\gamma(a_0 x) + e^*_\gamma(i_n(u))| \text{ (from (25))} \\
&\le |a_0| + 4\beta_0 C_0 A_0 M \le M + 4\beta_0 C_0 A_0 M = (1 + 4\beta_0 C_0 A_0)M \\
&\le C_0(1 + 4\beta_0 A_0)M = C_0\left(C_0 - \frac{2\beta_0 A_0}{1-\beta_0 A_0} + 4\beta_0 C_0 A_0\right)M \\
&\le C_0\left(C_0 + \frac{2\beta_0 A_0}{1-\beta_0 A_0}\right)M = C_0(C_0 + C_0 - 1)M \\
&= (2C_0^2 - C_0)M. \qquad \square
\end{aligned}$$

We are ready to prove the main theorem of this paper before proceeding with the proof of Theorem 4.1 (otherwise known as Theorem 7.3). Note that, as it was pointed out in Remark 3.4, we can modify our construction in such a way that C_0 is arbitrarily close to one.

THEOREM 4.7. *The space $\mathcal{C}al(\mathcal{Y}_X)$ is isomorphic as a Banach algebra to the space $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$. Furthermore, the isomorphism Φ witnessing this fact satisfies $\|\Phi^{-1}\|\|\Phi\| \le 2C_0^3 - C_0^2$.*

Proof. If we denote by $[I]$ and $[I_n]$ the equivalence class of I and I_n respectively in $\mathcal{C}al(\mathcal{Y}_X)$, then by Lemma 4.4 and Proposition 4.6 we have that, for any sequence of scalars $(a_k)_{k=0}^n$, we have

$$\begin{aligned}
\frac{1}{C_0}\left\|a_0 I_X + \sum_{k=1}^n a_k e^*_k \otimes e_k\right\|_{\mathcal{L}(X)} &\le \left\|a_0[I] + \sum_{k=1}^n a_k[I_k]\right\| \\
&\le (2C_0^2 - C_0)\left\|a_0 I_X + \sum_{k=1}^n a_k e^*_k \otimes e_k\right\|_{\mathcal{L}(X)}.
\end{aligned}$$

That is, the space $\overline{\langle\{[I]\} \cup \{[I_n] : n \in \mathbb{N}\}\rangle}$ is $C_0(2C_0^2 - C_0)$-isomorphic to $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$. By Corollary 4.2 the space $\mathcal{C}al(\mathcal{Y}_X)$ is $(2C_0^3 - C_0^2)$-isomorphic to $\mathbb{R}I \oplus \mathcal{K}_{\text{diag}}(X)$. $\square$

5. Rapidly Increasing Sequences

The notions of rapidly increasing sequences (RIS) and the basic inequality can by now be considered standard tools used in most HI, and related, constructions. The version of an RIS presented herein adds an extra condition that eliminates the influence of the $(\sum \oplus X_k)_X^{\text{utc}}$ part in the definition of the space $\mathcal{Y}_X$ and keeps only the Bourgain–Delbaen part. Thus, RIS sequences can be treated identically to those in [Z]. The "utc" condition used when defining $(\sum \oplus X_k)_X^{\mathrm{utc}}$ is designed so that sufficiently many RIS sequences with this extra condition can be found in the space. Recall that every element of $\mathcal{G}^{\mathrm{utc}}$ is naturally identified with a functional acting on $\mathcal{Z}_X^\infty$, and hence also on $\mathcal{Y}_X$.

NOTATION 5.1. If $z \in \mathcal{Y}_X$ has finite BD-support, that is, $\max \mathrm{range}_{\mathrm{BD}}(z) = n$, then there is $u \in \mathcal{Z}_X^n$ with $z = i_n(u)$. If $u = (x_k, y_k)_{k=1}^n$ each vector x_k is in the space X_k, which has a Schauder basis $(t_{k,i})_{i=1}^\infty$, and hence we may define the set $\mathrm{supp}_k(z) = \mathrm{supp}_{(t_{k,i})}(x_k)$ (see also (12)). If $\mathrm{supp}_k(z)$ is finite for $1 \le k \le n$, then we say that z has coordinate-wise finite supports, and we may define the quantity

$$\max \underset{\mathrm{cw}}{\mathrm{supp}}(z) = \max\{\max \underset{k}{\mathrm{supp}}(z) : 1 \le k \le n\}.$$

DEFINITION 5.2. *Let $C > 0$ and $(j_n)_n$ be a strictly increasing sequence of natural numbers. A block sequence (which may be either finite or infinite) $(z_n)_n$ of elements with coordinate-wise finite supports is called a C-rapidly increasing sequence (or a C-RIS) if*

(i) $\|z_n\| \le C$ *for all* n,
(ii) $j_{n+1} > \max \mathrm{range}_{\mathrm{BD}}(z_n)$,
(iii) $|e_\gamma^*(z_n)| \le C/m_i$ *for all* $\gamma \in \Gamma$ *with* $\mathrm{weight}(\gamma) = 1/m_i$ *and* $i < j_n$, *and*
(iv) *for all* $z^* \in \mathcal{G}^{\mathrm{utc}}$, *there exists at most one* n *for which* $z^*(z_n) \neq 0$.

If we need to be specific about the sequence $(j_k)_k$ in the definition, we shall say that $(z_k)_k$ is a $(C, (j_k)_k)$-RIS.

LEMMA 5.3. *Let $(z_n)_n$ be a sequence in $\mathcal{Y}_X$ that satisfies item* (iv) *of Definition 5.2. Then, for every $z^* \in \mathcal{G}^{\mathrm{utc}}$ and $s \in \mathbb{N}$, there exists at most one $n \in \mathbb{N}$ so that $z^*(P_{(s,\infty)} z_n) \neq 0$.*

Proof. Let $s \in \mathbb{N}$ and $z^* = \sum_{k=1}^{i_0} a_k t_{k,i_0}^*$ with (a_k) such that $\|w^* \text{-} \sum_{k=1}^\infty a_k e_k^*\|_X \le 1$. If $s > i_0$, it follows that $P_{(s,\infty)}^* z^* = 0$, and there is nothing to prove. Otherwise, $s \le i_0$ and it follows that $P_{(s,\infty)}^* z^* = \sum_{k=s}^{i_0} a_k t_{k,i_0}^*$. Recall that the basis $(e_i)_i$ of the space X has a bimonotone constant A_0, which means that $\|\sum_{k=s}^{i_0} (a_k/A_0) e_k^*\|_X \le 1$, that is, $\tilde{z}^* = (1/A_0) P_{(s,\infty)}^* z^*$ is in $\mathcal{G}^{\mathrm{utc}}$. This easily implies the desired conclusion. □

5.1. The Basic Inequality

Let us denote by $(t_k^*)_k$ the unit vector basis of $c_{00}(\mathbb{N})$. Given a sequence of natural numbers $(l_j)_j$ and a sequence of positive real numbers $(\theta_j)_j$, we define $W[(\mathcal{A}_{l_j}, \theta_j)_j]$ to be the smallest subset W of $c_{00}(\mathbb{N})$ with the following properties:

(1) $\pm t_k^* \in W$ for all $k \in \mathbb{N}$,

(2) for all $j \in \mathbb{N}$, $n \le l_j$, and successive vectors $f_1, f_2, \dots, f_n$ in W, the vector

$$f = \theta_j \sum_{k=1}^{n} f_k \tag{29}$$

is in W as well.

The elements $\pm t_k^*$, $k \in \mathbb{N}$ will be referred to as type 0 elements of $W[(\mathcal{A}_{l_j}, \theta_j)_j]$. If an element of $W[(\mathcal{A}_{l_j}, \theta_j)_j]$ is as in (29), then we say that it is of type 1 and it has weight θ_j. We also use the notation $(t_k)_k$ for the unit vector basis of $c_{00}(\mathbb{N})$, and for f in $W[(\mathcal{A}_{l_j}, \theta_j)_j]$ and $x \in c_{00}(\mathbb{N})$ we define $f(x)$ to be the usual inner product $\langle f, x\rangle$ on $c_{00}(\mathbb{N})$. The following can be found in [AH, Proposition 2.5].

PROPOSITION 5.4. *If $j_0 \in \mathbb{N}$ and $f \in W[(\mathcal{A}_{4n_j}, 1/m_j)_j]$ is an element of weight $1/m_h$, then*

$$\left| f\left(\frac{1}{n_{j_0}} \sum_{l=1}^{n_{j_0}} t_l\right)\right| \le \begin{cases} 2/(m_h m_{j_0}), & \text{if } h < j_0, \\ 1/m_h, & \text{if } h \ge j_0. \end{cases} \tag{30}$$

If additionally $f \in W[(\mathcal{A}_{4n_j}, 1/m_j)_{j \neq j_0}]$, then

$$\left| f\left(\frac{1}{n_{j_0}} \sum_{l=1}^{n_{j_0}} t_l\right)\right| \le \begin{cases} 2/(m_h (m_{j_0})^2), & \text{if } h < j_0, \\ 1/m_h, & \text{if } h > j_0. \end{cases} \tag{31}$$

The proof of the following is practically identical to the proof of [AH, Proposition 5.4]. This is because on RIS functionals from $\mathcal{G}^{\text{utc}}$ act in a c_0 way, and hence they do not contribute to the norm of linear combinations of an RIS. We describe the proof for completeness.

PROPOSITION 5.5 (Basic Inequality). *Let $(z_k)_{k\in I}$ be a C-RIS, where I is an interval of $\mathbb{N}$, let $(\lambda_k)_{k\in I}$ be real numbers, let s be a natural number, and let γ be an element of Γ. Then there exist $k_0 \in I$ and $g \in W[(\mathcal{A}_{3n_j}, 1/m_j)_j]$ such that*

(1) *either $g = 0$, or* $\text{weight}(g) = \text{weight}(\gamma)$ *and* $\text{supp}(g) \subset \{k \in I : k > k_0\}$ *and*

(2)

$$\left| e_\gamma^*\left(P_{(s,\infty)} \sum_{k\in I} \lambda_k z_k\right)\right| \le (3A_0C_0C)|\lambda_{k_0}| + (4A_0C_0C) g\left(\sum_{k\in I} |\lambda_k| e_k\right).$$

Moreover, if j_0 is such that

$$\left| e_\xi^*\left(\sum_{k\in J} \lambda_k z_k\right)\right| \le C \max_{k\in J} |\lambda_k|$$

for all subintervals J of I and all $\xi \in \Gamma$ of weight $1/m_{j_0}$, then we may choose g to be in $W[(\mathcal{A}_{3n_j}, 1/m_j)_{j\neq j_0}]$.

Proof. This proof is along the lines of the proof of [AH, Proposition 5.4]. The difference is that here we also have to use property (iv) of Definition 5.2. Moreover, some constants are different as, according to Remark 3.7(iii), $\|P_n\| \le A_0 C_0$ for all $n \in \mathbb{N}$, whereas in [AH] the same quantity is bounded by two. We describe the points where this difference takes place and refer the reader to [AH] for the rest of the details. We shall only consider the statement without the additional assumption, as the modification required does not differ from that presented in [AH]. The proof proceeds by induction on the rank of γ. The case in which $\mathrm{rank}(\gamma) = 1$ is fairly easy. Next, consider an element γ of rank greater than one, of age a, and of weight $1/m_h$. Let $(j_k)_k$ be the sequence for which $(z_k)_k$ is a $(C, (j_k)_k)$-RIS, and assume that there is $l \in I$ for which $j_l \leq h < j_{l+1}$, as the other cases are simpler. Arguing identically as in [AH, Proposition 5.4], we obtain that, for some $k_0 \leq l$,

$$e_\gamma^*\left(P_{(s,\infty)}\sum_{\substack{k\in I\\ k\leq l}}\lambda_k z_k\right) \leq (3A_0C_0C)|\lambda_{k_0}|. \tag{32}$$

Set $I' = \{k \in I : k > l\}$ and let

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + (1/m_h)\sum_{r=1}^{a} b_r^* \circ P_{(p_{r-1},p_r)}$$

be the evaluation analysis of γ. Set

$$I_0' = \{k \in I' : \underset{\mathrm{BD}}{\mathrm{range}}(z_k) \text{ contains } p_r \text{ for some } r\} \text{ and for } 1 \leq r \leq a \text{ set}$$
$$I_r' = \{k \in I' \setminus I_0' : \underset{\mathrm{BD}}{\mathrm{range}}(z_k) \cap (p_{r-1}, p_r) \neq \emptyset\}.$$

Note that $\# I_0' \leq a$. Arguing identically as in [AH], we obtain

$$\begin{aligned} e_\gamma^*\left(P_{(s,\infty)}\sum_{k\in I'}\lambda_k z_k\right) &\leq (4A_0C_0C)(1/m_h^{-1})\sum_{k\in I_0'}|\lambda_k| \\ &\quad + (1/m_h)\left|\sum_{r=1}^{a} b_r^* \circ P_{(s\vee p_{r-1},\infty)}\sum_{k\in I_r'}\lambda_k z_k\right|. \end{aligned}$$

Recall that each $b_r^* \in A_{p_r-1} = K_{p_r-1} \cup \mathcal{G}_{p_r-1}^{\mathrm{utc}} \cup B_{p_r-1}$. For r such that $b_r \in B_{p_r-1}$, we have that b_r^* is a convex combination of functionals $\pm e_\eta^*$, $\eta \in \Gamma_{p_r-1}$. Hence, there exists η_r with $\mathrm{rank}(\eta) < p_r$ so that

$$\left|\sum_{r=1}^{a} b_r^* \circ P_{(s\vee p_{r-1},\infty)}\sum_{k\in I_r'}\lambda_k z_k\right| \leq \left|\sum_{r=1}^{a} e_{\eta_r}^* \circ P_{(s\vee p_{r-1},\infty)}\sum_{k\in I_r'}\lambda_k z_k\right|. \tag{33}$$

Applying the inductive assumption to η_r, $(z_k)_{k\in I_r'}$ and $s \vee p_{r-1}$, we obtain $k_r \in I_r'$ and $g_r \in W[(\mathcal{A}_{3n_j}, 1/m_j)_j]$ with $\mathrm{supp}(g_r) \subset \{k \in I_r' : k > k_r\}$ so that

$$\begin{aligned} &\left|e_{\eta_r}^* \circ P_{(s\vee p_{r-1},\infty)}\sum_{k\in I_r'}\lambda_k z_k\right| \\ &\qquad \leq (3A_0C_0C)|\lambda_{k_r}| + (4A_0C_0C)g_r\left(\sum_{k\in I_r'}|\lambda_k|e_k\right). \end{aligned} \tag{34}$$

For r such that $b_r^* \in K_{p_r-1}$, we have that $\mathrm{supp}_{\mathrm{BD}}(b_r^*)$ is a singleton, whereas if $r \in \mathcal{G}_{p_r-1}^{\mathrm{utc}}$ by Definition 5.2(iv) and Lemma 5.3, there is at most one k_r in I_r' for which $b_r^* \circ P_{(s\vee p_{r-1},\infty)} z_{k_r} \neq 0$. In either case, there is $k_r \in I_r'$ such that

$$\left| b_r^* \circ P_{(s\vee p_{r-1},\infty)} \sum_{k\in I_r'} \lambda_k z_k \right| \le |b_r^* \circ P_{(s\vee p_{r-1},\infty)}(\lambda_{k_r} z_{k_r})| \le (2A_0C_0C)|\lambda_{k_r}|. \quad (35)$$

We may, for each such r, set $g_r = 0$. We combine (33) and (34) with (35) to obtain that for all r there are $k_r \in I_r'$ and $g_r \in W[(\mathcal{A}_{3n_j}, 1/m_j)_j]$ $\mathrm{supp}(g_r) \subset \{k \in I_r' : k > k_r\}$ such that

$$\left| b_r^* \circ P_{(s\vee p_{r-1},\infty)} \sum_{k\in I_r'} \lambda_k z_k \right| \le (3A_0C_0C)|\lambda_{k_r}| + (4A_0C_0C) g_r\left(\sum_{k\in I_r'} |\lambda_k| e_k\right). \quad (36)$$

If we now define $g = (1/m_h)(\sum_{k\in I_0'} t_k^* + \sum_{r=1}^{a}(t_{k_r}^* + g_r))$, then we conclude that $g \in W[(\mathcal{A}_{3n_j}, 1/m_j)_j]$ with $\mathrm{supp}(g) \subset \{k \in I_r' : k > k_0\}$. As in [AH], we combine (32) with (36) to obtain

$$e_\gamma^*\left(\sum_{k\in I} \lambda_k z_k\right) \le (3A_0C_0C)|\lambda_{k_0}| + (4A_0C_0C) g\left(\sum_{k\in I} |\lambda_k| e_k\right).$$ □

A combination of (30) with Proposition 5.5 directly yields the following estimate.

COROLLARY 5.6. *Let* $(z_k)_{k=1}^{n_j}$ *be a* C*-RIS in* $\mathcal{Y}_X$. *Then*

$$\left\| \frac{1}{n_j} \sum_{k=1}^{n_j} z_k \right\| \le A_0C_0C\left(\frac{3}{n_j} + \frac{4}{m_j}\right).$$

Recall that the sequence $(m_j, n_j)_j$ is in fact of the form $(\tilde{m}_j, \tilde{n}_j)_{j\in L_0}$, that is, it is a subsequence of the sequence $(\tilde{m}_j, \tilde{n}_j)_j$ from page 126.

COROLLARY 5.7. *Let* $j_0 \in \mathbb{N} \setminus L_0$ *and* $(z_k)_{k=1}^{\tilde{n}_{j_0}}$ *be a* C*-RIS in* $\mathcal{Y}_X$. *Then*

$$\left\| \frac{\tilde{m}_{j_0}}{\tilde{n}_{j_0}} \sum_{k=1}^{\tilde{n}_{j_0}} z_k \right\| \le \frac{11A_0C_0C}{\tilde{m}_{j_0}}.$$

Proof. By Proposition 5.5 there is $g \in W[(\mathcal{A}_{3n_j}, 1/m_j)_j]$ with

$$\left\| \frac{\tilde{m}_{j_0}}{\tilde{n}_{j_0}} \sum_{k=1}^{\tilde{n}_{j_0}} z_k \right\| \le 3A_0C_0C\frac{\tilde{m}_{j_0}}{\tilde{n}_{j_0}} + 3A_0C_0C g\left(\frac{\tilde{m}_{j_0}}{\tilde{n}_{j_0}} \sum_{k=1}^{\tilde{n}_{j_0}} e_k\right).$$

Note that $g \in W[(\mathcal{A}_{4\tilde{n}_j}, 1/\tilde{m}_j)_{j\neq j_0}]$ and, although (31) is not formulated in terms of this set, it applies for it as well. This yields $g((\tilde{m}_{j_0}/\tilde{n}_{j_0})\sum_{k=1}^{\tilde{n}_{j_0}} e_k) \le 2/\tilde{m}_{j_0}^2$ from which the desired estimate follows. □

5.2. *Rapidly Increasing Sequences and Bounded Linear Operators*

In this subsection we prove that whether a bounded linear operator on $\mathcal{Y}_X$ is horizontally compact (see Definition 6.1) is witnessed on RIS.

NOTATION 5.8. An important conclusion of (19) is that if $z \in \mathcal{Y}_X$ and $\mathrm{range}_{\mathrm{BD}}(z) = (m, n]$, then there is $u \in \mathcal{Z}_X^n$ with $z = i_n(u)$, so that if $u = (x_k, y_k)_{k=1}^n$, we have $x_k = 0$ and $y_k = 0$ for $1 \le k \le m$. We define

$$\operatorname*{supp}_{\mathrm{loc}}^{\Gamma}(z) = \{i \in \mathbb{N} : \text{ there is } \gamma \in \Gamma_n \text{ with } \mathrm{weight}(\gamma) = 1/m_i \text{ and } e_\gamma^*(z) \neq 0\}.$$

This set is called the Γ-local support of z.

REMARK 5.9. For every $z \in \mathcal{Y}_X$ with finite BD-support and $\varepsilon > 0$, there is $\tilde{u}$ with coordinate-wise finite supports, $\mathrm{supp}_{\mathrm{BD}}(u) = \mathrm{supp}_{\mathrm{BD}}(\tilde{u})$, and $\|u - \tilde{u}\| < \varepsilon$. Indeed, if $u = i_n(x_k, y_k)_{k=1}^n$, take for each $1 \le k \le n$ a finitely supported vector $\tilde{x}_k$ in X_k with $\|x_k - \tilde{x}_k\| \le \varepsilon/(nC_0 \sup_i \|t_{k,i}^*\|)$ (see Remark 2.3) and set $\tilde{u} = i_n(\tilde{x}_k, y_k)_{k=1}^n$. By (17) $\|i_n\| \le C_0$, which yields the desired estimate.

LEMMA 5.10. *Let $(z_n)_n$ be a block sequence with each z_n having coordinate-wise finite supports. If, for all $n \in \mathbb{N}$, we have $\max \mathrm{supp}_{\mathrm{cw}}(z_n) < \min \mathrm{supp}_{\mathrm{BD}}(z_{n+1})$, then $(z_n)_n$ satisfies item* (iv) *of Definition 5.2.*

Proof. If $z^* \in \mathcal{G}^{\mathrm{utc}}$, then $z^* = \sum_{k=1}^{i_0} a_k t_{k,i_0}^*$. If, for some $n \in \mathbb{N}$, we have $z^*(z_n) \neq 0$, then $i_0 \ge \min \mathrm{supp}_{\mathrm{BD}}(z_n)$ and $i_0 \le \max \mathrm{supp}_{\mathrm{cw}}(z_n)$. That is,

$$\min \operatorname*{supp}_{\mathrm{BD}}(z_n) \le i_0 \le \max \operatorname*{supp}_{\mathrm{cw}}(z_n),$$

which can be satisfied for at most one $n \in \mathbb{N}$. $\square$

REMARK 5.11. Lemma 5.10 easily implies that if $(z_k)_k$ and $(w_k)_k$ are both C-RIS, then $(z_k + w_k)_k$ has a subsequence that is $2C$-RIS.

The following is essentially the same as [AH, Lemma 5.8] but for completeness we describe a proof.

LEMMA 5.12. *Let z be a vector for which the set $\mathrm{supp}_{\mathrm{BD}}(z)$ is finite. For every $\gamma \in \Gamma$ with $\mathrm{weight}(\gamma) = 1/m_i$ and $i \notin \mathrm{supp}_{\mathrm{loc}}^{\Gamma}(z)$, we have $|e_\gamma^*(z)| \le 2C_0\|z\|/m_i$ (see (16)).*

Proof. We show this is by induction on $\mathrm{rank}(\gamma)$. If $\mathrm{rank}(\gamma) \le q_0 = \max \mathrm{supp}_{\mathrm{BD}}(z)$ with $\mathrm{weight}(\gamma) = m_i$ and $i \notin \mathrm{supp}_{\mathrm{loc}}^{\Gamma}(z)$, then $e_\gamma^*(z) = 0$. Assume that the result holds for all γ with $\mathrm{weight}(\gamma) = 1/m_i$ and $i \notin \mathrm{supp}_{\mathrm{loc}}^{\Gamma}(z)$ so that $\mathrm{rank}(\gamma) \le n$ for some $n \ge q_0$, and let $\gamma \in \Gamma$ with $\mathrm{weight}(\gamma) = 1/m_i$, $i \notin \mathrm{supp}_{\mathrm{loc}}^{\Gamma}(z)$, and $\mathrm{rank}(\gamma) = n + 1$. Then $e_\gamma^* = d_\gamma^* + c_\gamma^*$ (see p. 127, Section 3.3) and $d_\gamma^*(z) = 0$, that is, $e_\gamma^*(z) = c_\gamma^*$. Either $c_\gamma^* = e_\xi^* + (1/m_i^*)b^* \circ P_{(p,n]}$, where $\mathrm{weight}(\xi) = 1/m_i$ and $\mathrm{rank}(\xi) = p$ and $b^* \in A_n$ or $c_\gamma^* = (1/m_i^*)b^* \circ P_{(0,n]}$, where $b^* \in A_n$. In the second case it trivially follows that $|c_\gamma^*(z)| \le (1/m_i)\|P_{(0,n]}\|\|z\|$. In the first case, if $p \le n$, then $e^*_\xi(z) = 0$ and similarly $|c^*_\gamma(z)| \le (1/m_i)\|P_{(0,n]}\|\|z\|$. Otherwise, if $p > n$, then $P_{(p,n]}(z) = 0$, and the result follows from the inductive assumption. □

We shall say that a block sequence $(z_n)_n$ has bounded local weights if there exists i_0, so that, for all $n \in \mathbb{N}$ and all $\gamma \in \mathrm{supp}^\Gamma_{\mathrm{loc}}(z_n)$, we have $\mathrm{weight}(\gamma)^{-1} \le m_{i_0}$. We shall say that a block sequence $(z_n)_n$ has rapidly decreasing local weights if there exists a sequence of natural numbers $(j_n)_n$ such that $\lim_n j_n = \infty$ and, for all $n \in \mathbb{N}$ and $\gamma \in \mathrm{supp}^\Gamma_{\mathrm{loc}}(z_n)$, we have $\mathrm{weight}(\gamma) \le 1/m_{j_n}$.

The next result is a combination of Lemma 5.10 and [AH, Proposition 5.10].

PROPOSITION 5.13. *Let $(z_n)_n$ be a bounded block sequence in $\mathcal{Y}_X$ so that each z_n has finite coordinate-wise supports. If $(z_n)_n$ has bounded local weights or $(z_n)_n$ has rapidly decreasing local weights, then $(z_k)_k$ has a subsequence that is an RIS.*

Proof. We first pass to a subsequence so that the assumption of Lemma 5.10 is satisfied. We refer to this sequence as $(z_n)_n$ as well. If $(z_n)_n$ has bounded local weights witnessed by m_{i_0}, set $C = \max\{m_{i_0}, 2C_0\}\sup_n \|z_n\|$. Then, for all $\gamma \in \Gamma$ with $\mathrm{weight}(\gamma) = 1/m_i$ and $i \le i_0$ and for all $n \in \mathbb{N}$, we have $|e^*_\gamma(z_n)| \le \|z_n\| \le (m_{i_0}/m_i)\|z_n\| \le C/m_i$. On the other hand, for $\gamma \in \Gamma$ with $\mathrm{weight}(\gamma) = 1/m_i$ and $i > i_0$ by Lemma 5.12, we obtain $|e^*_\gamma(z_n)| \le 2C_0\|z\|/m_i \le C/m_i$. If we define $j_n = \min \mathrm{range}_{\mathrm{BD}}(z_n)$, then all assumptions of Definition 5.2 are satisfied. If, on the other hand, the sequence $(z_n)_n$ has rapidly increasing weights witnessed by $(j_n)_n$, pass to common subsequences of $(z_n)_n$ and $(j_n)_n$, again denoted by $(z_n)_n$ and $(j_n)_n$, so that $j_{n+1} > \max \mathrm{range}_{\mathrm{BD}}(z_n)$ for all $n \in \mathbb{N}$. If $C = 2C_0 \sup_n \|z_n\|$, then Lemma 5.12 yields that $(z_n)_n$ is a $(C, (j_n)_n)$-RIS. □

The following is almost identical to the proof of [AH, Proposition 5.11]. We describe the proof for completeness.

PROPOSITION 5.14. *Let Y be a Banach space and $T : \mathcal{Y}_X \to Y$ be a bounded linear operator. If for all RIS $(z_n)_n$ in $\mathcal{Y}_X$ we have $\lim_n \|Tz_n\| = 0$, then for every bounded block sequence $(z_n)_n$ we have $\lim_n \|Tz_n\| = 0$.*

Proof. We will start with an arbitrary block sequence $(z_n)_n$ and show that it has a subsequence that satisfies the conclusion. By Remark 5.9 we may perturb the sequence so that all of its elements have finite coordinate-wise supports. For each $n \in \mathbb{N}$, there is $q_n \in \mathbb{N}$ and $(x_{n,k}, y_{n,k})_{k=1}^{q_n}$ so that $z_n = i_{q_n}(x_{n,k}, y_{n,k})_{k=1}^{q_n}$. Define, for each $n, N \in \mathbb{N}$ and $k \le q_n$, the following vectors in $\ell_\infty(\Delta_k)$:

$$v^N_{n,k}(\gamma) = \begin{cases} y_k(\gamma), & \text{if } \mathrm{weight}(\gamma) \ge 1/m_N, \\ 0, & \text{otherwise} \end{cases} \quad \text{and}$$

$$w^N_{n,k}(\gamma) = \begin{cases} y_k(\gamma), & \text{if } \mathrm{weight}(\gamma) < 1/m_N, \\ 0, & \text{otherwise,} \end{cases}$$

and, for all $n, N \in \mathbb{N}$, define the vectors

$$\chi^N_n = i_{q_n}(x_{n,k}, v^N_{n,k})_{k=1}^{q_n} \quad \text{and} \quad \psi^N_n = i_{q_n}(0, w^N_{n,k})_{k=1}^{q_n}.$$

For each $n, N \in \mathbb{N}$, we have $z_n = \chi_n^N + \psi_n^N$ and $\|\chi_n^N\| \leq \|i_{q_n}\| \|(x_k, v_k^N)_{k=1}^{q_n}\| \leq C_0 \|R_{q_n} z_n\| \leq C_0 A_0 \|z_n\|$. For fixed $N \in \mathbb{N}$, the sequence $(\chi_n^N)_n$ has finite coordinate-wise supports and bounded weights, that is, by Proposition 5.13 any of its subsequences has a further subsequence that is an RIS. By assumption, $\lim_n \|T\chi_n^N\| = 0$. We may therefore choose an increasing sequence of indices $(n_N)_N$ so that $\lim_N \|T\chi_{n_N}^N\| = 0$. Consider now the sequence $(\psi_{n_N}^N)_N$ which is bounded, it has finite coordinate-wise supports, and it has rapidly decreasing local weights. By Proposition 5.13 any of its subsequences has a further subsequence that is an RIS and, by assumption, $\lim_N \|T\psi_{n_N}^N\| = 0$. As $z_{n_N} = \chi_{n_N}^N + \psi_{n_N}^N$, we have $\lim_N \|T z_{n_N}\| = 0$. □

Corollary 5.15. *The Schauder decomposition $(Z_n)_n$ of $\mathcal{Y}_X$ is shrinking.*

Proof. If it were not shrinking, then there would be a functional $x^* \in \mathcal{Y}_X^*$ and a normalized block sequence $(x_k)_k$ with $\liminf |x^*(x_k)| > 0$. By Proposition 5.14 there would exist a C-RIS $(z_k)_k$ and $\varepsilon > 0$ with $x^*(z_k) > \varepsilon$ for all $k \in \mathbb{N}$. By Corollary 5.6 we would have

$$\varepsilon < \left\| \frac{1}{n_j} \sum_{k=1}^{n_j} z_k \right\| \leq A_0 C_0 C \left(\frac{3}{n_j} + \frac{4}{m_j} \right),$$

which is absurd for j sufficiently large. □

6. The Scalar-Plus-Horizontally Compact Property

In this section we prove one of the most important features of this construction, namely that every operator on $\mathcal{Y}_X$ is a scalar multiple of the identity plus a horizontally compact operator. The following was introduced in [Z, Definition 7.1].

Definition 6.1. *Let X be a Banach space with a Schauder decomposition $(Y_n)_n$. An operator $T : X \to X$ is called horizontally compact (with respect to $(Y_n)_n$) if, for every bounded block sequence $(x_k)_k$, we have $\lim_k \|Tx_k\| = 0$.*

A standard argument yields that $T : X \to X$, where X has a Schauder decomposition with associated projections $(P_n)_n$, is horizontally compact precisely when $\lim_n \|TP_{(n,\infty)}\| = 0$ or, equivalently, $T = \lim_n TP_n$ in operator norm. If one furthermore assumes that the Schauder decomposition is shrinking, then T is horizontally compact if and only if its restriction on any block subspace is compact.

6.1. *Exact Pairs and Exact Sequences*

The definition of exact pairs and exact sequences is based on that from [AH, Definition 6.1]. Some modification is made to take into consideration the set $\mathcal{G}^{\mathrm{utc}}$. Exact sequences are a delicate tool necessary to prove properties about operators in $\mathcal{Y}_X$. Similar to [Z, Theorem 1.1(2)], one can use these tools to show that block sequences in $\mathcal{Y}_X$ span HI spaces. We do not prove this result as we do not use it.

DEFINITION 6.2. *Let* $C > 0$ *and* $\varepsilon \in \{0, 1\}$. *A pair* (z, γ) *where* $z \in \mathcal{Y}_X$ *and* $\gamma \in \Gamma$ *is said to be a* (C, j, M, ε)*-exact pair if the following are satisfied*:

(i) $|d_\xi^*(z)| \leq C/m_j$ *for all* $\xi \in \Gamma$;
(ii) $\mathrm{weight}(\gamma) = 1/m_j$;
(iii) $\|z\| \leq C$ *and* $e_\gamma^*(z) = \varepsilon$;
(iv) *for every element* $\xi \in \Gamma$ *with* $\mathrm{weight}(\xi) \neq m_j$, *we have*

$$|e_\xi^*(z)| \leq \begin{cases} C/m_i, & \text{if } i < j, \\ C/m_j, & \text{if } i > j; \end{cases}$$

(v) $\mathrm{supp}_{\mathrm{BD}}(z)$ *is finite and* $\max \mathrm{supp}_{\mathrm{cw}}(z) \leq M$.

The following is very similar to [AH, Lemma 6.2]. We include the proof for the sake of completeness.

LEMMA 6.3. *Let* $(z_k)_{k=1}^{n_{2j}}$ *be* C*-RIS and assume that there are natural numbers* $0 = q_0 < 2j \leq q_1 < \cdots < q_k$ *so that* $\mathrm{supp}_{\mathrm{BD}}(z_k) \subset (q_{k-1}, q_k)$ *and that there are* $b_k \in A_{q_{k-1}}$ *with* $b_k(x_k) = 0$ *for* $k = 1, \ldots, n_{2j}$. *Then there exist* $\zeta_k \in \Delta_{q_k}$ *such that if* $\gamma = \zeta_{n_{2j}}$, $M = \max_{1 \leq k \leq n_{2j}} \max \mathrm{supp}_{\mathrm{cw}}(z_k)$, *and*

$$z = \frac{m_{2j}}{n_{2j}} \sum_{k=1}^{n_{2j}} z_k,$$

then (z, γ) *is a* $(7C, 2j, M, 0)$*-exact pair. Furthermore, the evaluation analysis of* γ *is* $(q_k, b_k^*, \zeta_k)_{k=1}^{n_{2j}}$.

Proof. The existence of γ with the desired properties is a consequence of Lemma 3.11, whereas (23) yields $e_\gamma^*(z) = 0$. For every $\xi \in \Gamma$, the functional d_ξ^* acts on at most one z_k, so we have $|d_\xi^*(z)| \leq \|d_\xi^*\| C m_{2j}/n_{2j} \leq CC_0/m_{2j}$, so (i) holds. Corollary 5.6 yields $\|z\| \leq 5A_0C_0C$, so (iii) holds. Also, combining Proposition 5.5 with (30), we deduce that, for every element $\xi \in \Gamma$ with $\mathrm{weight}(\xi) \neq m_j$, we have

$$|e_\xi^*(z)| \leq \begin{cases} 7A_0C_0C/m_i, & \text{if } i < j, \\ 7A_0C_0C/m_j, & \text{if } i > j. \end{cases}$$

Additionally, (v) clearly holds from the choice of M. □

DEFINITION 6.4. *A sequence* $(z_k)_{k=1}^{n_{2j_0-1}}$ *is called a* $(C, 2j_0 - 1, \varepsilon)$*-dependent sequence if there exist natural numbers* $0 = p_0 < p_1 < \cdots < p_{n_{2j_0-1}}$ *and elements* η_k, ξ_k *in* Γ *for* $k = 1, \ldots, n_{2j_0-2}$ *so that*

(i) (z_1, η_1) *is a* $(C, 4j_1 - 2, p_1, \varepsilon)$*-exact pair and* (z_k, η_k) *is a* $(C, 4j_k, p_k, \varepsilon)$*-exact pair for* $k = 2, \ldots, n_{2j_0-1}$;
(ii) *the element* $\gamma = \xi_{n_{2j_0-1}}$ *has* $\mathrm{weight}(\gamma) = 1/m_{2j_0-1}$ *and evaluation analysis* $(p_k, e_{\eta_k}^*, \xi_k)_{k=1}^{n_{2j_0-1}}$; *and*
(iii) $\mathrm{range}_{\mathrm{BD}}(z_k) \subset (p_{k-1}, p_k)$ *for* $k = 1, \ldots, n_{2j_0-1}$.

REMARK 6.5. If $(z_k)_{k=1}^{n_{2j_0-1}}$ is a $(C, 2j_0 - 1, \varepsilon)$-dependent sequence, then $e_\gamma^*(z_k) = \varepsilon/m_{2j_0-1}$ for $1 \le k \le n_{2j_0-1}$ (where $\gamma = \xi_{n_{2j_0-1}}$). In addition, by the definition of functionals of odd weight for the associated components, we can deduce that $p_{k-1} < \mathrm{rank}(\eta_k) < p_k$, $\mathrm{rank}(\xi_k) = p_k$, and for $2 \le k \le n_{2j_0-1}$ $\mathrm{weight}(\xi_k) = 1/m_{2j_0-1}$, $\mathrm{weight}(\eta_1) = 1/m_{4j_1-2} < 1/n_{2j_0-1}^2$, and for $2 \le k \le n_{2j_0-1}$ $\mathrm{weight}(\eta_k) = 1/m_{4j_k} = 1/m_{4\sigma(\xi_{k-1})}$.

The following is an easy consequence of the definition of an exact pair, the growth condition of the function σ, as well as Lemma 5.10. A short proof can be found in [AH, Lemma 6.4].

LEMMA 6.6. *A $(C, 2j_0 - 1, \varepsilon)$-dependent sequence $(z_k)_{k=1}^{n_{2j_0-1}}$ is a C-RIS.*

The following requires some calculations; however, its proof is entirely identical to [AH, Lemma 6.5] so we omit it.

LEMMA 6.7. *Let $(z_k)_{k=1}^{n_{2j_0-1}}$ be a $(C, 2j_0-1, 0)$-dependent sequence, J be a subinterval of $\{1, \ldots, n_{2j_0-1}\}$, and $\zeta \in \Gamma$ with* $\mathrm{weight}(\zeta) = 1/m_{2j_0-1}$. *Then we have*

$$\left| e_\zeta^*\left(\sum_{k\in J} z_k\right)\right| \le 3C.$$

The following is only a minor modification of [AH, Proposition 6.6]. We include a proof for the sake of completeness.

PROPOSITION 6.8. *Let $(z_k)_{k=1}^{n_{2j_0-1}}$ be a $(C, 2j_0 - 1, 0)$-dependent sequence. Then we have*

$$\left\| \frac{1}{n_{2j_0-1}} \sum_{k=1}^{n_{2j_0-1}} z_k \right\| \le 33 A_0 C_0 C \frac{1}{m_{2j_0-1}^2}.$$

Proof. Set $z = 1/n_{2j_0-1} \sum_{k=1}^{n_{2j_0-1}} z_k$. We will use (20). By condition (iv) of Definition 5.2 every $x^* \in \mathcal{G}^{\mathrm{utc}}$ acts on at most one z_k, so we have $|x^*(z)| \le C/n_{2j_0-1}$. The same holds for $x^* \in \bigcup_k (1/A_0) B_{X_k^*}$. For $\gamma \in \Gamma$ with $\mathrm{weight}(\gamma) = 1/m_{2j_0-1}$, by Lemma 6.7, we have $|e_\gamma^*(z)| \le 3C/n_{2j_0-1}$. Fix $\gamma \in \Gamma$ with $\mathrm{weight}(\gamma) \ne 1/m_{2j_0-1}$. We will use the full statement of Proposition 5.2. By Lemmas 6.6 and 6.7 it follows that the sequence $(z_k)_{k=1}^{n_{2j_0-1}}$ satisfies the additional assumption of 5.2 for $3C$ and $2j_0 - 1$. This means that there exists $g \in W[(\mathcal{A}_{3n_j}, 1/m_j)_{j \ne 2j_0-1}]$ with

$$|e_\gamma^*(z)| \le \frac{9A_0C_0C}{n_{2j_0-1}} + 12A_0C_0C g\left(\frac{1}{n_{2j_0-1}} \sum_{k=1}^{n_{2j_0-1}} e_k\right).$$

We conclude the desired estimate by applying (31). ☐

6.2. *Scalar-Plus-Horizontally Compact*

As the technical tool of dependent sequences has been discussed, we may now proceed to proving that every operator on $\mathcal{Y}_X$ is a scalar operator plus a horizontally compact operator.

PROPOSITION 6.9. *Let $T : \mathcal{Y}_X \to \mathcal{Y}_X$ be a bounded linear operator. Then, for every N in $\mathbb{N}$ and every bounded block sequence $(z_k)_k$, we have* $\lim_k \|P_N T z_k\| = 0$.

Proof. By Corollary 5.15 every bounded block sequence is weakly null. Also, $\sum_{k=1}^{N} \oplus Z_k \simeq (\sum_{k=1}^{N} \oplus X_k) \oplus \ell_\infty(\Gamma_N)$. Therefore, it is sufficient to check that, for every k_0, every bounded block sequence $(z_k)_k$, and every bounded linear operator $S : \mathcal{Y}_X \to X_{k_0}$, we have $\lim_k \|S z_k\| = 0$. By Proposition 5.14 it is sufficient to check this for a C-RIS $(z_k)_k$. If the conclusion were false, then, for some $\delta > 0$, $(S z_k)_k$ would be equivalent to a bounded block sequence $(w_k)_k$ in X_{k_0} with $\|w_k\| > \delta$ for all $k \in \mathbb{N}$, that is, there would exist a constant M such that, for all scalars $(a_k)_k$, we would have

$$\Big\|\sum_k a_k w_k\Big\| \le M \Big\|\sum_k a_k z_k\Big\|.$$

By [AH, Proposition 4.8] for any $\ell_{2j} \in L_{k_0} = \{\ell_1 < \ell_2 < \cdots\}$ (see page 126), we would have

$$\Big\|\frac{\tilde{m}_{\ell_{2j}}}{\tilde{n}_{\ell_{2j}}} \sum_{k=1}^{\tilde{n}_{\ell_{2j}}} w_k\Big\| \ge \delta,$$

whereas by Corollary 5.7 we would have

$$\Big\|\frac{\tilde{m}_{\ell_{2j}}}{\tilde{n}_{\ell_{2j}}} \sum_{k=1}^{\tilde{n}_{\ell_{2j}}} z_k\Big\| \le \frac{11 A_0 C_0 C}{\tilde{m}_{\ell_{2j}}}.$$

This would mean $\tilde{m}_{\ell_{2j}} \le 11 A_0 C_0 C M/\delta$ for arbitrary j, which is absurd. □

LEMMA 6.10. *For every $z \in \mathcal{Y}_X$, $N \in \mathbb{N}$, and $\varepsilon > 0$, there exists $\gamma \in \Gamma$ with* $\operatorname{rank}(\gamma) \ge N$ *such that*

$$|e_\gamma^*(z)| \ge \frac{1-\varepsilon}{m_2}\|z\|.$$

Proof. Approximate z by a vector with finite BD-support $\tilde{z}$ so that $\|z - \tilde{z}\| \le \varepsilon/(2m_2)$. By Remark 3.8 we may choose a natural number M so that $(1-\varepsilon/2)\|\tilde{z}\| \le \max\{|b^*(w)| : b^* \in A_M\}$ (for the definition of A_M, see page 126, paragraph before (22a)). Fix b_0^* achieving this maximum. Then, for every $n \ge \max\{M, N-1, \max \operatorname{supp}_{\mathrm{BD}}(\tilde{z})\}$ and $b^* \in A_M \subset A_n$, we have that the triple $\gamma = (n+1, 1/m_2, b_0^*)$ is in $\Delta_{n+1}^{\mathrm{Even}_0}$. It follows that $|e_\gamma^*(\tilde{z})| = |c_\gamma^*(\tilde{z})| = (1/m_2)|b_0^*(\tilde{z})| \ge (1-\varepsilon/2)/m_2\|\tilde{z}\|$. Hence, $|e_\gamma^*(z)| \ge ((1-\varepsilon)/m_2)\|z\|$. □

Lemma 6.11 is an alternative approach to what is presented in [Z, Section 7], where an element x is approached by another element with the property that the action of every functional in $\bigcup_n A_n$ yields a rational number. A modification of the approach from [Z] would work here as well; however, the factor of $1/m_2$ in item (iii) in what follows would not be avoided. The reason is that the construction presented here is designed to yield a Calkin algebra that is $(1+\varepsilon)$-isomorphic to $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$.

LEMMA 6.11. *Let x be in $\mathcal{Y}_X$, $T : \mathcal{Y}_X \to \mathcal{Y}_X$ be a bounded linear operator, and $\delta = \mathrm{dist}(Tx, \mathbb{R}x)$. Then, for every $\varepsilon > 0$ and $N \in \mathbb{N}$, there exist $b^* \in \bigcup_n B_n$ (see page 126), $\gamma_0 \in \Gamma$, and $0 < \theta < \varepsilon$ so that if $\tilde{x} = x - \theta d_{\gamma_0}$:*

(i) $\mathrm{rank}(\gamma_0) \geq N$,

(ii) $b^*(\tilde{x}) = 0$, *and*

(iii) $b^*(T\tilde{x}) \geq \delta/(5m_2)$.

Proof. Use the Hahn–Banach theorem to find a functional $x^* \in \mathcal{Y}_X^*$ with $\|x^*\| = 1$, $x^*(Tx) = \delta$, and $x^*(x) = 0$. Lemma 6.10 and a separation theorem imply that $\overline{\mathrm{conv}}^{w^*}\{\pm m_2 e_\gamma^* : \gamma \in \Gamma\}$ contains the unit ball of $\mathcal{Y}_X^*$. This means that there are a finite set E and $f_0 = \sum_{\gamma \in E} c_\gamma e_\gamma^*$ with $f_0(Tx) = x^*(Tx) = \delta$ and $f_0(x) = x^*(x) = 0$ such that $\sum_{\gamma \in E} |c_\gamma| \leq 2m_2$. Choose collections of rational numbers $(c_\gamma^n)_{\gamma \in E}$ with $\lim_n c_\gamma^n = c_\gamma$ for all $\gamma \in E$.

Fix $\gamma_0 \in \Gamma$ with $\mathrm{rank}(\gamma_0) > \max\{N, \max_{\gamma \in E} \mathrm{rank}(\gamma)\}$ and $\mathrm{dist}(d_{\gamma_0}, \langle\{x, Tx\}\rangle) = \eta > 0$. Arguing as before, find another finite subset F of Γ and $g_0 = \sum_{\gamma \in F} a_\gamma e_\gamma^*$ with $g_0(d_{\gamma_0}) = \eta$, $g_0(x) = g_0(Tx) = 0$ so that $\sum_{\gamma \in E} |a_\gamma| \leq 2m_2$. Choose collections of rational numbers $(a_\gamma^n)_{\gamma \in F}$ with $\lim_n a_\gamma^n = a_\gamma$ for all $\gamma \in F$.

Define, for each $n \in \mathbb{N}$,

$$f_n = \frac{1}{2}\left(\frac{1}{\sum_{\gamma \in E} |c_\gamma^n|} \sum_{\gamma \in E} c_\gamma^n e_\gamma^* + \frac{1}{\sum_{\gamma \in F} |a_\gamma^n|} \sum_{\gamma \in F} a_\gamma^n e_\gamma^*\right) \quad \text{and}$$

$$x_n = x - \frac{f_n(x)}{f_n(d_{\gamma_0})} d_{\gamma_0}.$$

Observe that for each $n \in \mathbb{N}$ we have $f_n \in \bigcup_m B_m$, that $f_n(x_n) = 0$, and that $\lim_n f_n(Tx) = \delta/(2\sum_{\gamma \in E} |c_\gamma|) \geq \delta/(4m_2)$. Furthermore, observe that $f_n(d_{\gamma_0}) = \sum_{\gamma \in F} a_\gamma^n e_\gamma^*(d_{\gamma_0})/(2\sum_{\gamma \in F} |a_\gamma^n|)$, which yields that $\lim_n f_n(d_{\gamma_0}) = \eta/(2\sum_{\gamma \in F} |a_\gamma|) \geq \eta/(4m_2)$. The proof is concluded by setting $b^* = f_n$ and $\theta = f_n(x)/f_n(d_{\gamma_0})$ for n sufficiently large. $\square$

REMARK 6.12. Let $(z_k)_k$ be a $(C, (j_k)_k)$-RIS, $(\gamma_k)_k$ be a sequence in Γ such that $\mathrm{rank}(\gamma_k)_k$ increases to infinity, and $(\theta_k)_k$ be real numbers with $0 < \theta_k < C/m_{j_k}$ for all $k \in \mathbb{N}$. Then the sequence $(\tilde{z}_k)_k = (z_k - \theta_k d_{\gamma_k})_k$ has a subsequence that is a $2C$-RIS. Indeed, conditions (i), (ii), and (iii) from Definition 5.2 are straightforwardly satisfied by a suitable subsequence. Condition (iv) is trivial once it is observed that, for all $\gamma \in \Gamma$ and all $x^* \in \mathcal{G}^{\mathrm{utc}}$, we have $x^*(d_\gamma) = 0$.

PROPOSITION 6.13. *Let $T : \mathcal{Y}_X \to \mathcal{Y}_X$ be a bounded linear operator. Then, for every infinite RIS $(z_k)_k$, we have $\lim_k \mathrm{dist}(Tz_k, \mathbb{R}z_k) = 0$.*

Proof. Assume that there is a $(C,(j_k)_k)$-RIS $(z_k)_k$ and $\delta > 0$ such that, for all $k \in \mathbb{N}$, we have $\operatorname{dist}(Tz_k, \mathbb{R}z_k) > \delta$. We will show that this would mean that T is unbounded.

We shall first prove the following claim. For every j and $N \in \mathbb{N}$, there exists an $(14C, 2j, M, 0)$-exact pair (z,γ) such that

$$e_\gamma^*(Tz) \geq \delta/(6m_2) \quad \text{and} \quad \min_{\mathrm{BD}} \operatorname{supp}(z) > N.$$

For every $k \in \mathbb{N}$, we apply Lemma 6.11 to find a sequence $(\tilde{z}_k)_k = (z_k - \theta_k d_{\gamma_k})_k$, with $\operatorname{rank}(\gamma_k)$ increasing to infinity and $0 < \theta_k < C/m_{j_k}$, and $(b_k^*)_k$ in A_{N_k} for some N_k, so that $b_k^*(\tilde{z}_k) = 0$ and $b_k^*(T\tilde{z}_k) \geq \delta/(5m_2)$. By Remark 6.12 we may assume that $(\tilde{z}_k)_k$ is $2C$-RIS. Define $p_0 = 0$ and $p_k = \max\{N_k, \max \operatorname{range}_{\mathrm{BD}}(\tilde{z}_k) + 1\}$. By passing, if it is necessary, to a subsequence we may assume $\operatorname{range}_{\mathrm{BD}}(z_k) \subset (p_{k-1}, p_k)$ for all $k \in \mathbb{N}$. Utilizing the weakly null property of $(T\tilde{z}_k)_k$, we may assume that $\sum_k \|T\tilde{z}_k - P_{(p_{k-1},p_k)}\tilde{z}_k\| < \eta$, where η is a positive number to be determined later. We may assume that $\min \operatorname{supp}_{\mathrm{BD}}(\tilde{z}_1) > N$ and that $q_1 > 2j$. By Lemma 6.3, if $z = (m_{2j}/n_{2j}) \sum_{k=1}^{n_{2j}} \tilde{z}_k$ and there are $(\zeta_k)_{k=1}^{n_{2j}}$ with $\operatorname{rank}(\zeta_k) = p_k$ and $\gamma \in \Gamma$ with $e_\gamma^* = 1/m_{2j} \sum_{k=1}^{n_{2j}} b_k^* \circ P_{(p_{k-1},p_k)} + \sum_{k=1}^{n_{2j}} d_{\zeta_k}^*$ so that (z,γ) is a $(14C, 2j, M, 0)$-exact pair, we calculate

$$\begin{aligned}
e_\gamma^*(Tz) &= \frac{1}{n_{2j}} \sum_{k=1}^{n_{2j}} b_k^* \circ P_{(p_{k-1},p_k)}(T\tilde{z}_k) + \sum_{k=1}^{n_{2j}} d_{\zeta_k}^*(Tz) \\
&= \frac{1}{n_{2j}} \sum_{k=1}^{n_{2j}} b_k^*(T\tilde{z}_k) - \frac{1}{n_{2j}} \sum_{k=1}^{n_{2j}} b_k^*(T\tilde{z}_k - P_{(p_{k-1},p_k)}T\tilde{z}_k) \\
&\quad + \sum_{k=1}^{n_{2j}} e_{\zeta_k}^* \circ P_{\{q_k\}}(Tz) \\
&\geq \frac{\delta}{5m_2} - \frac{1}{n_{2j}} \sum_{k=1}^{n_{2j}} \|T\tilde{z}_k - P_{(p_{k-1},p_k)}T\tilde{z}_k\| - \sum_{k=1}^{n_{2j}} \|P_{\{p_k\}}Tz\| \\
&\geq \frac{\delta}{5m_2} - \frac{\eta}{n_{2j}} - \frac{m_{2j}}{n_{2j}} \sum_{k=1}^{n_{2j}} \sum_{l=1}^{n_{2j}} \|P_{\{p_k\}}T\tilde{z}_m\| \\
&\geq \frac{\delta}{5m_2} - \frac{\eta}{n_{2j}} - (2A_0C_0)\frac{m_{2j}}{n_{2_j}} \sum_{k=1}^{n_{2j}} \sum_{l=1}^{n_{2j}} \|T\tilde{z}_m - P_{(p_{k-1},p_k)}T\tilde{z}_m\| \\
&\geq \frac{\delta}{5m_2} - \frac{\eta}{n_{2j}} - (2A_0C_0)m_{2j}\eta \geq \frac{\delta}{6m_2}
\end{aligned}$$

for $\eta > 0$ sufficiently small.

We now use the claim to construct, for any given $j_0 \in \mathbb{N}$, a $(14C, 2j_0 - 1, 0)$-dependent sequence with associated components $0 = p_1 < \cdots < p_{n_{2j_0}-1}$,

$(\eta_k)_{k=1}^{n_{2j_0-1}}$, and $(\xi_k)_{k=1}^{n_{2j_0-1}}$ such that

$$e^*_{\xi_k}(Tz_k) > \delta/(6m_2) \quad \text{and} \quad \|Tz_k - P_{(p_{k-1},p_k)}Tz_k\| < \varepsilon/n_{2j_0-1},$$

where $0 < \varepsilon < \delta/(84m_2m_{2j_0-1})$ for $1 \leq k \leq n_{2j_0-1}$. We start by choosing a $(14C, 4j_1 - 2, M, 0)$-exact pair (z_1, η_1), for j_1 with $m_{4i_1-2} > n^2_{2j_0-1}$, satisfying the conclusion of the claim, and we also choose p_1 sufficiently large so that $\max\{M, \operatorname{rank}(\eta_1), \max\operatorname{supp}_{BD}(z_1)\} < p_1$ as well as $\|Tz_1 - P_{(0,p_1)}Tz_1\| < \varepsilon/n_{2j_0-1}$. Clearly, (z_1, η_1) is also a $(14C, 4j_1 - 2, p_1, 0)$-exact pair. Set $\xi_1 = (p_1, 1/m_{2j_0-1}, \eta_1)$, which is in $\Delta^{\mathrm{Odd}_0}_{p_1}$. If we have chosen (z_k, η_k), ξ_k, p_k for $1 \leq k \leq a < n_{2j_0-1}$, set $j_{a+1} = \sigma(\xi_k)$ and apply the claim to find a sequence of $(14C, 2j_{a+1}, M_n, 0)$-exact pairs $(z^{a+1}_n, \eta^{a+1}_n)_{n\in\mathbb{N}}$ with $p_a < \min\operatorname{supp}_{\mathrm{BD}}(z^{a+1}_n) \to \infty$. By the weak null property of $(z^{a+1}_n)_n$, for n sufficiently large, we have $\|Tz^{a+1}_n - P_{(0,p_a)}Tz^{a+1}_n\| < \varepsilon/(2n_{2j_0-1})$. Set $(z_{a+1}, \eta_{a+1}) = (z^{a+1}_n, \eta^{a+1}_n)$ and choose p_{a+1} sufficiently large so that $\max\{M_n, p_a, \operatorname{rank}(\eta_{a+1}), \max\operatorname{supp}_{BD}(z_{a+1})\} < p_{a+1}$, and $\|P_{(p_{a+1},\infty)}Tz_1\| < \varepsilon/(2n_{2j_0-1})$. Set $\xi_{a+1} = (p_{a+1}, \xi_a, 1/m_{2j_0-1}, \eta_{a+1})$, which is in $\Delta^{\mathrm{Odd}_1}_{p_{a+1}}$.

Having chosen the dependent sequence, set $z = (m_{2j_0-1}/n_{2j_0-2})\sum_{k=1}^{n_{2j_0-1}} z_k$ and $\gamma = \xi_{n_{2j_0-1}}$. By the analysis of γ, we obtain $e^*_\gamma = (1/m_{2j_0-1})\sum_{k=1}^{n_{j_0-1}} e^*_{\eta_k} \circ P_{(p_{k-1},p_k)} + \sum_{k=1}^{n_{2j_0-1}} d^*_{\xi_k}$. By Proposition 6.8 we have $\|z\| \leq 462A_0C_0C/m_{2j_0-1}$. The same calculations as the ones above yield that $e^*_\gamma(Tz) \geq \delta/(6m_2) - \varepsilon/n_{2j_0-1} - m_{2j_0-1}\varepsilon \geq \delta/ \geq \delta/(7m_2)$ by the choice of ε. This yields $\|T\| \geq \frac{\delta}{3{,}234m_2A_0C_0C}m_{2j_0-1}$. Choosing m_{2j_0-1} large enough yields a contradiction. □

PROPOSITION 6.14. *Let $T : \mathcal{Y}_X \to \mathcal{Y}_X$ be a bounded linear operator. Then there exists a scalar λ such that $T - \lambda I$ is horizontally compact.*

Proof. Let $(z_k)_k$ be an arbitrary C-RIS that is bounded from below, say by some $\varepsilon > 0$. By Proposition 6.13 we may find a scalar λ and pass to a subsequence so that $\lim_k \|Tz_k - \lambda z_k\| = 0$. We will show that, for any bounded block sequence $(w_k)_k$, we have $\lim_k \|Tw_k - \lambda w_k\| = 0$. By Proposition 5.14 it is sufficient to verify this only for a C'-RIS $(w_k)_k$. If $\lim_k \|w_k\| = 0$, then this is true. Otherwise we may assume $\|w_k\| > \tilde{\varepsilon}$ for all $k \in \mathbb{N}$ and for some $\tilde{\varepsilon} > 0$. By passing to subsequences of (z_k) and $(w_k)_k$, we may assume that range(z_k) and range(w_m) are disjoint for all k and m in $\mathbb{N}$. By Remark 5.11 the sequence $(z_k - w_k)_k$ has a $(C + C')$-RIS subsequence. Apply Lemma 6.13 and pass to further subsequences to find scalars λ' and μ such that $\lim_k \|Tw_k - \lambda' w_k\| = 0$ and $\lim_k \|T(z_k - w_k) - \mu(z_k - w_k)\| = 0$. We can consequently deduce that $\lim_k \|\lambda z_k - \lambda' w_k - \mu(z_k - w_k)\| = 0$. By the fact that the ranges of z_k and w_k are disjoint, we obtain

$$\frac{1}{2A_0C_0}\max\{|\lambda - \mu|\varepsilon, |\lambda' - \mu|\tilde{\varepsilon}\} \leq \lim_k \|(\lambda - \mu)z_k - (\lambda' - \mu)w_k\| = 0,$$

which yields $\lambda = \lambda'$, as desired. □

7. Diagonal Plus Compact Approximations

In this section we finally prove that every bounded linear operator on the space $\mathcal{Y}_X$ can be approximated by a sequence of operators of the form diagonal (with respect to the Schauder decomposition $(Z_k)_k$) plus compact. This is the main theorem used to prove the desired property of the Calkin algebra of $\mathcal{Y}_X$.

PROPOSITION 7.1. *Let $(z_i)_i$ be a block sequence in $\mathcal{Y}_X$ (with respect to the Schauder decomposition $(Z_k)_k$). Then $(z_i)_i$ has a subsequence $(z_{i_k})_k$ with the property that, for every $j \in \mathbb{N}$, $1 \leq a \leq n_{2j}$, and every further block vectors $(w_i)_{i=1}^a$ of $(z_{i_k})_k$, there exists $\gamma \in \Gamma$ with*

$$\Big\|\sum_{i=1}^a w_i\Big\| \geq e_\gamma^*\Big(\sum_{i=1}^a w_i\Big) \geq \frac{1}{3m_{2j}}\sum_{i=1}^a \|w_i\|.$$

Proof. Let us denote by E_i the support of each vector z_i with respect to $(Z_k)_k$. We fix $0 < \varepsilon \leq 1/3$. By Remark 3.8 we may choose, for each $i_0 \in \mathbb{N}$, an index N_{i_0} such that, for every w in the linear span of $(z_i)_{i \leq i_0}$, it satisfies

$$(1-\varepsilon)\|w\| \leq \max\{b^*(w) : b^* \in A_{N_{i_0}}\}$$

(for the definition of $A_{N_{i_0}}$, see page 126, paragraph before (22a)). As the sets $(A_n)_n$ are increasing, we may choose the sequence $(N_i)_i$ to be strictly increasing. Pass to a subsequence $(i_k)_k$ so that $N_{i_k} + 1 < \min(E_{i_{k+1}})$ as well as $2k \leq N_{i_k}$ for all $k \in \mathbb{N}$.

Let now $j \in \mathbb{N}$, $1 \leq a \leq n_{2j}$, and $(w_d)_{d=1}^a$ be block vectors of $(z_{i_k})_k$. We may assume that $a = n_{2j}$. For each d, let i_{k_d} be the maximum of the support of the vector w_d with respect to $(z_{i_k})_k$. Then there is $\tilde{b}_d^* \in A_{N_{i_{k_d}}}$ with $\tilde{b}_d^*(w_d) \geq (1-\varepsilon)\|w_d\|$. We set $b_1^* = 0$, $p_1 = N_{i_{k_j}}$, $b_2^* = \tilde{b}_{j+1}^*$, $p_2 = N_{i_{k_{j+1}}}, \dots, b_{n_{2j}-j+1}^* = \tilde{b}_{n_{2j}}^*$, $p_{n_{2j}-j+1} = N_{i_{k_{n_{2j}-j+1}}}$. Then there are $\xi_l \in \Delta_{p_l+1}$ for $1 \leq l \leq n_{2j} - j + 1$ and $\gamma \in \Gamma$ with evaluation analysis $(p_l, \xi_l, b_l^*)_{l=1}^{n_{2j}-j+1}$. It follows that

$$\Big\|\sum_{d=1}^{n_{2j}} w_d\Big\| \geq e_\gamma^*\Big(\sum_{d=1}^{n_{2j}} w_d\Big) = \frac{1}{m_{2j}}\sum_{d=j+1}^{n_{2j}} \tilde{b}_d^*(w_d) \geq (1-\varepsilon)\frac{1}{m_{2j}}\sum_{d=j+1}^{n_{2j}}\|w_d\|.$$

Similarly, for $1 \leq d \leq j$, there is $\gamma_d \in \Gamma$ with $\|\sum_{d=1}^{n_{2j}} w_d\| \geq e_{\gamma_d}^*(\sum_{i=1}^a w_i) \geq (1-\varepsilon)/m_2\|w_d\|$. Finally,

$$\begin{aligned}\frac{1}{m_{2j}}\sum_{i=1}^a \|w_i\| &\leq \frac{j}{m_{2j}}\max_{1\leq d\leq j}\|w_d\| + \frac{1}{m_{2j}}\sum_{d=j+1}^{n_{2j}}\|w_d\| \\ &\leq \frac{1}{m_2}\max_{1\leq d\leq j}\|w_d\| + \frac{1}{m_{2j}}\sum_{d=j+1}^{n_{2j}}\|w_d\| \leq \frac{2}{1-\varepsilon}\Big\|\sum_{d=1}^{n_{2j}} w_d\Big\|. \qquad \square\end{aligned}$$

PROPOSITION 7.2. *Let $T : \mathcal{Y}_X \to \mathcal{Y}_X$ be a bounded linear operator. Then, for every $k \in \mathbb{N}$, the operator $TP_{\{k\}} - P_{\{k\}}TP_{\{k\}}$ is compact.*

Proof. We first observe that $P_{(0,k)} T P_{\{k\}}$ is compact. Recall $Z_k \simeq X_k \oplus \ell_\infty(\Delta_k)$ and $\sum_{i=1}^{k-1} \oplus Z_k \simeq (\sum_{i=1}^{k-1} \oplus X_i) \oplus \ell_\infty(\Gamma_{k-1})$. We may identify $P_{(0,k)} T P_{\{k\}}$ with an operator on these spaces. As it is mentioned below (21), every operator from X_k to X_i with $k \neq i$ is compact. This yields that $P_{(0,k)} T P_{\{k\}}$ is compact.

We now show that $P_{(k,\infty)} T P_{\{k\}}$ is compact as well and the proof will be complete. This requires a bit more work. Omitting the finite dimensional component $\ell_\infty(\Delta_k)$, it is sufficient to show that every bounded linear operator $S : X_k \to \sum_{i=k+1}^{\infty} \oplus Z_i$ is compact. We will show that, for a C-RIS $(x_n)_n$ in X_k, $\lim_n \|Sx_n\| = 0$. If this is true, then [AH, Proposition 5.11] implies that S is indeed compact. We start by observing that by [AH, Corollary 5.5], for any scalars $(a_n)_n$, we have $\|\sum_{n=1}^{\infty} a_n x_n\| \leq 10C \left|\left|\left| \sum_{k=1}^{\infty} a_n e_n \right|\right|\right|$, where $|||\cdot|||$ is the norm induced by $W[(\mathcal{A}_{3n_j}, 1/\tilde{m}_j)_{j \in L_k}]$. The argument used in the proof of Corollary 5.7 yields that, for any $j \in \mathbb{N}$, we have

$$\left\| \frac{m_{2j}}{n_{2j}} \sum_{n=1}^{n_{2j}} x_n \right\| \leq \frac{20C}{m_{2j}}. \tag{37}$$

Towards a contradiction assume that $\liminf_n \|Sx_n\| > 0$. Arguing as in the first part of this proof, for every $N > k$, the operator $P_{(k,N)} S$ is compact. Recall that bounded block sequences in X_k are weakly null (see [AH, Proposition 5.12]). This means that, for all $N > k$, $\lim_n \|P_{(k,N)} S x_n\| = 0$. We conclude that $(Sx_n)_n$ has a subsequence that is equivalent to a block sequence in $\mathcal{Y}_X$. By Proposition 7.1 and passing to a further subsequence, there exists $\theta > 0$ such that, for all $j \in \mathbb{N}$, we have $\|\sum_{n=1}^{n_{2j}} x_n\| \geq \theta n_{2j}/m_{2j}$. Combining this with (37), we obtain $m_{2j} \leq 20C/\theta$ for all $j \in \mathbb{N}$, which is absurd. $\square$

THEOREM 7.3. *For every bounded linear operator $T : \mathcal{Y}_X \to \mathcal{Y}_X$, there exist a sequence of real numbers $(a_k)_{k=0}^{\infty}$ and a sequence of compact operators $(K_n)_n$ such that*

$$T = \lim_n \left(a_0 I + \sum_{k=1}^{n} a_k I_k + K_n \right),$$

where the limit is taken in the operator norm.

Proof. Use Proposition 6.14 to write $T = a_0 I + S$ with S horizontally compact. Recall that, for each $k \in \mathbb{N}$, $I_k : \mathcal{Y}_X \to \mathcal{Y}_X$ is a projection whose image is isomorphic to the space X_k, a space that has the scalar-plus-compact property. This means that, for each $k \in \mathbb{N}$, the operator $I_k S I_k = a_k I_k + \tilde{C}_k$, where a_k is a scalar and $\tilde{C}_k$ is compact. Since I_k is a finite rank perturbation of $P_{\{k\}}$, we conclude that $P_{\{k\}} S P_{\{k\}} = a_k I_k + C_k$ with C_k compact. Furthermore, by Proposition 7.2, for every $n \in \mathbb{N}$, the operator $\tilde{K}_n = SP_{(0,n]} - \sum_{k=1}^{n} P_{\{k\}} S P_{\{k\}}$ is compact. Summarizing, if we define the compact operator $K_n = \sum_{k=1}^{n} C_k + \tilde{K}_n$, then $SP_{(0,n]} = \sum_{k=1}^{n} a_k I_k + K_n$. As S is horizontally compact, $SP_{(0,n]}$ converges to S in operator norm. In conclusion, $\lim_n \|T - (a_0 I + \sum_{k=1}^{n} a_k I_k + K_n)\| = 0$. $\square$

Remark 7.4. Theorem 7.3 easily implies that strictly singular operators on $\mathcal{Y}_X$ are always compact.

8. Remarks and Problems

We conclude this paper with a section containing general remarks based on our results as well as several related open problems.

Remark 8.1. In [MPZ] for every countable compactum K, a Banach space X_K is presented, the Calkin algebra of which is isomorphic as a Banach algebra to $C(K)$. There exist K_1 and K_2 such that $C(K_1)$ are isomorphic as Banach spaces but not as Banach algebras. Such an example is provided by $K_1 = \omega$ and $K_2 = \omega \cdot 2$. Hence, it is possible for Calkin algebras to be isomorphic to one another as Banach spaces but not as Banach algebras. There is also an additional manner of achieving this. In [Ta] a $\mathscr{L}_\infty$-space $\mathfrak{X}_\infty$ is presented, the Calkin algebra of which is isometric, as a Banach algebra, to the convolution algebra of $\ell_1(\mathbb{N}_0)$ (where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$). Note that this commutative Banach algebra has continuum many maximal ideals (each ideal corresponds to a number $\lambda \in [-1, 1]$ by being the kernel of the functional $\phi_\lambda \in \ell_\infty(\mathbb{N}_0) = \ell_1(\mathbb{N}_0^*)$ with $\phi_\lambda = (1, \lambda, \lambda^2, \lambda^3, \dots)$). Here, if X is the space c endowed with the monotone summing basis, then as it follows by Example 1.5, the Calkin algebra of $\mathcal{Y}_X$ is isomorphic to ℓ_1. However, by Corollary 1.4 it has countably many maximal ideals. Therefore it is different, as a Banach algebra, to the convolution algebra of $\ell_1(\mathbb{N}_0)$.

Remark 8.2. All preexisting examples of Banach spaces, the Calkin algebras of which were explicitly described, were $\mathscr{L}_\infty$-spaces (see [Ta; MPZ; KL]). We point out that, as it was observed in [Sk], the Calkin algebra of the space X from [AM] is the unitization of a Banach algebra with trivial multiplication. This space is not a $\mathscr{L}_\infty$-space; however, the norm structure of its Calkin algebra is not explicitly described. In the present paper we use tools from the theory of $\mathscr{L}_\infty$-spaces, but the space $\mathcal{Y}_X$ is not necessarily a $\mathscr{L}_\infty$-space. For example, it can be shown using the set $\mathcal{G}^{\text{utc}}$ that X^* is crudely finitely representable in $\mathcal{Y}_X^*$. This implies that when X^* does not embed in an L_1 space, $\mathcal{Y}_X^*$ is not a $\mathscr{L}_1$-space. Such a space is, for example, $X = J^* = \mathcal{J}_*(\ell_2)$, the dual of which is isomorphic to J, which has trivial cotype. In fact, with some work, something stronger can be shown: if $\mathcal{Y}_X$ is a $\mathscr{L}_\infty$-space, then so is X.

Remark 8.3. The space $\mathcal{Y}_X$ has the bounded approximation property. To see this, recall that $(Z_n)_n$ is a Schauder decomposition of $\mathcal{Y}_X$ and the spaces $\sum_{1=k}^n \oplus Z_k$ are uniformly isomorphic to $((\sum_{i=1}^n \oplus X_k)_{\text{utc}}^X \oplus \ell_\infty(\Gamma)_n)_\infty = (\mathcal{X}_n \oplus \ell_\infty(\Gamma_n))_\infty$. Recall that each X_k has a 2-Schauder basis $(t_{k,i})_i$. Using the definition of $\mathcal{G}^{\text{utc}}$, one can see that the finite dimensional subspaces F_m spanned by $((t_{k,i})_{i=1}^m)_{k=1}^m$ are increasing, uniformly complemented in $\mathcal{X}_n$, and their union is dense $\mathcal{X}_n$.

Remark 8.4. When a space X has a finite dimensional Calkin algebra or one that is isomorphic to ℓ_1, then $\mathcal{K}(X)$ is complemented in $\mathcal{L}(X)$. In the examples from

[MPZ] that have infinite dimensional $C(K)$ spaces as Calkin algebras, the space $\mathcal{K}(X)$ is not complemented in $\mathcal{L}(X)$. Regarding the space $\mathcal{Y}_X$, this is not always clear. If $\mathcal{K}_{\mathrm{diag}}(X)$ is isomorphic to ℓ_1, then $\mathcal{K}(\mathcal{Y}_X)$ is complemented (this happens, e.g., when X is c_0 endowed with the summing basis). On the other hand, if $\mathcal{K}_{\mathrm{diag}}(X)$ contains c_0, then the argument used in [MPZ] goes through and $\mathcal{K}(\mathcal{Y}_X)$ is not complemented in $\mathcal{L}(\mathcal{Y}_X)$.

REMARK 8.5. The construction of spaces with quasi-reflexive Calkin algebras is a step towards trying to find a space with a reflexive and infinite dimensional Calkin algebra. One way for this to be possible would be to find a space X with $\mathcal{L}(X)$ reflexive. As it was pointed out to us by Chávez–Domínguez, [B, Corollary 2] implies that such an X must be finite dimensional so that this route would be a dead end.

PROBLEM 1. Does there exist a Banach space the Calkin algebra of which is reflexive and infinite dimensional?

REMARK 8.6. Given a Banach space X with a basis, we have used the Argyros–Haydon scheme for defining spaces with the scalar-plus-compact property to obtain a Calkin algebra that is isomorphic as a Banach algebra to the space $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(X)$. It is conceivable that one may use the Gowers–Maurey scheme from [G] for constructing a space with an unconditional basis with the "diagonal plus strictly singular" property to construct a space with the property that algebra $\mathcal{L}(X)/\mathcal{SS}(X)$ is isomorphic as a Banach algebra to the whole space $\mathcal{L}_{\mathrm{diag}}(X)$. If one would like to have a space with Calkin algebra $\mathcal{L}_{\mathrm{diag}}(X)$, then a new scheme would be necessary, one that is used to a define Banach space with an unconditional basis with the diagonal-plus-compact property.

PROBLEM 2. Does there exist a Banach space with an unconditional basis such that every bounded linear operator on that space is the sum of a diagonal operator with a compact operator?

PROBLEM 3. Let X be a Banach space with a Schauder basis $(e_i)_i$. Does there exist a Banach space Y, the Calkin algebra of which is isomorphic as a Banach algebra to $\mathcal{L}_{\mathrm{diag}}(X)$?

REMARK 8.7. Recall that, for all spaces X with an unconditional basis, $\mathcal{L}_{\mathrm{diag}}(X)$ is isomorphic as a Banach algebra to ℓ_∞ with point-wise multiplication. As it was explained earlier, a positive answer to Problem 2 could perhaps yield a positive answer to Problem 3 and hence also a positive solution to the following.

PROBLEM 4. Does there exist a Banach space the Calkin algebra of which is isomorphic as a Banach algebra to $C(K)$ for an uncountable compact topological space K?

REMARK 8.8. In a personal communication with the first author, Kania asked whether the unitization of Schreier space from [Schr] endowed with coordinate-wise multiplication with respect to its standard unconditional basis is a Calkin algebra. Our paper does not provide an answer to this. More generally, we observe that our method does not work directly to show that the unitization of an arbitrary Banach space with an unconditional basis is a Calkin algebra. If X is a Banach space with an unconditional basis $(x_i)_i$ and there exists a Banach space Y with a basis such that the unitization of X is isomorphic as a Banach algebra to $\mathbb{R}I \oplus \mathcal{K}_{\mathrm{diag}}(Y)$, then $(x_i)_i$ is equivalent to the unit vector basis of c_0. Indeed, assume that $T : \mathbb{R}e \oplus X \to \mathbb{R}I \oplus \mathcal{K}(Y)$ is an algebra isomorphism. Then, if $(e_i)_i$ is the basis if Y, write for all $k \in \mathbb{N}$ $Tx_k = \mathrm{SOT} - \sum_i a_i^k e_i^* \otimes e_i$. Since $x_k^n = x_k$ for all $n \in \mathbb{N}$, we obtain that in fact there is a subset F_k of $\mathbb{N}$ such that $Tx_k = \mathrm{SOT} - \sum_{i\in F_k} e_i^* \otimes e_i$, and since $x_i x_j = 0$ for $i \neq j$, the sets F_k must be pairwise disjoint. Clearly, $Te = \mathrm{SOT} - \sum_i e_i^* \otimes e_i$. The fact that T is onto implies that each F_k is a singleton $F_k = \{\phi(k)\}$ and that $\bigcup_k F_k = \mathbb{N}$. If we reorder the basis $(x_k)_k$ as $(x_{\phi^{-1}(k)})_k$, then for all $n \in \mathbb{N}\|\sum_{k=1}^n x_{\phi^{-1}(k)}\| \leq \|T^{-1}\|\|\sum_{k=1}^n e_k^* \otimes e_k\| \leq C\|T^{-1}\|$, where C is the monotone constant of $(e_i)_i$. Unconditionality yields that $(x_i)_i$ is equivalent to the unit vector basis of c_0.

PROBLEM 5. Find a Banach space X with an unconditional basis $(x_i)_i$ that is not equivalent to the unit vector basis of c_0, so that there exist a Banach space the Calkin algebra of which is isomorphic as a Banach algebra to the unitization of X.

REMARK 8.9. The unitization of James space $\mathbb{R}e_\omega \oplus J$ may be viewed as the space of all scalar sequences of bounded quadratic variation and similarly, for every Banach space X with a subsymmetric sequence, the space $\mathbb{R}e_\omega \oplus J(X)$ is the space of all scalar sequences of bounded X-variation. One may consider, for any such X and any linearly ordered set I, the space $V_X(I)$ of all functions $f : I \to \mathbb{R}$ of bounded X-variation. The norm on such a space is submultiplicative. The spaces $V_X[0, 1]$ were introduced in [AMP]. By very carefully combining the method of the present paper with the method from [MPZ], it is conceivable that one may obtain for every countable well-ordered set I a Banach space the Calkin algebra of which is isomorphic as a Banach algebra to $V_X(I)$.

PROBLEM 6. Let X be a Banach space with a subsymmetric basis. For what ordered sets I does there exist a Banach space the Calkin algebra of which is isomorphic as a Banach algebra to $V_X(I)$?

REMARK 8.10. In [AMS] the notion of a convex block homogeneous sequence is introduced, that is, a sequence that is equivalent to its convex block sequences. Known examples of such sequences are constructed using a space X with a subsymmetric basis $(x_i)_i$. The bases $(e_i)_i$ of $J(X)$ and $(v_i)_i$ of $\mathcal{J}_*(X)$ are both convex block homogenous. In either case, the difference basis is submultiplicative.

Problem 7. Let X be a Banach space with a convex block homogeneous basis $(e_i)_i$ and set $d_1 = e_1$ and for $i \in \mathbb{N}$ $d_{i+1} = e_{i+1} - e_i$. Is X endowed with $(d_i)_i$ submultiplicative?

Regardless of the discussion, there is little reason to believe that the answer to this problem should be positive. In fact, by [AMS, Theorem II], a positive answer would imply that, for every conditional spreading sequence $(e_i)_i$, the difference basis $(d_i)_i$ is submultiplicative.

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