Fall 2017, Math 304

Second Midterm

Sample

Last name:

First name:

UIN:

"An Aggie does not lie, cheat or steal or tolerate those who do."

This exam consists of **seven** problems, the total point value of which is 100 points.

The answer to each question must be **justified in detail**.

The time length of this exam is 50 minutes.

The use of electronic devices, such as cellphones, tables, laptops, and calculators is prohibited.

Good luck!

Prb. 1	Prb. 2	Prb. 3	Prb. 4	Prb. 5	Prb. 6	Prb. 7	Total
16	14	14	14	14	14	14	100

Problem 1. Let $C[0,1]$ denote the vector space of all continuous functions with domain $[0,1]$, consider the functions $f_1(x) = 4x^{1/2}$, $f_2(x) = 4x^{3/2}$, $f_3(x) = 4x^{5/2}$.

(i) What is the Wronskian of $f_1(x)$, $f_2(x)$, and $f_3(x)$? *4 pts.*

(ii) Are $f_1(x)$, $f_2(x)$, and $f_3(x)$ linearly independent? *4 pts.*

(iii) What is the dimension of the vector space $V = \text{Span}\{f_1, f_2, f_3\}$? *4 pts.*

(iv) Consider the linear operator $M : V \to V$ given by $M(f(x)) = xf'(x)$. Find the matrix representing M with respect to the basis $E = \{f_1(x), f_2(x), f_3(x)\}$. *4 pts.*

Problem 2.

Consider the basis E of $\mathbb{R}^3$ given by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

(i) Find the transition matrix from E to the usual basis $\{e_1, e_2, e_3\}$. *4 pt.*

(ii) Find the transition matrix from the usual basis to E. *4 pt.*

(iii) Consider the basis F of $\mathbb{R}^3$ given by the vectors

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad w_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the transition matrix form F to E. *4 pt.*

(iv) Let v be a vector with $[v]_F = [-4, 5, 8]^\top$. Find the coordinate vector of v with respect to the basis E and with respect to the standard basis. *2 pt.*

Problem 3. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ -1 & 2 & 1 & 0 \end{bmatrix}.$$

(i) Find the rank and nullity of A. *2 pt.*
(ii) Write a basis for the row space of A. *2 pt.*
(iii) Write a basis for the column space of A. *3 pt.*
(iv) Write a basis for the null space of $A^\top$. *3 pt.*
(v) Consider $S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$ as a subspace of $\mathbb{R}^3$. What are $\dim(S)$ and $\dim(S^\perp)$? *2 pt.*
(vi) Write a basis for S and a basis for $S^\perp$. *2 pt.*

Problem 4.

Consider the transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b+c \\ b+c \\ a+2b+2c \end{bmatrix}.$$

(i) Show that L is linear. *2 pt.*

(ii) Find the standard matrix representation A of L. *2 pt.*

(iii) Find the x_0 for which $\|L(x_0) - b\|$ is minimized, where $\mathbf{b} = [1, 1, 1]^\top$. *3 pt.*

(iii) Find the kernel of L. *3 pt.*

(iv) If $E = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ find the matrix B representing L with respect to E. *3 pt.*

(v) If $v = 2v_1 - v_2 + 3v_3$ find $L(v) = c_1v_1 + c_2v_2 + c_3v_3$. *1 pt.*

Problem 5.
Consider the functions $f_1(x) = x^{1/2}$, $f_2(x) = x^{3/2}$, $f_3(x) = x^{5/2}$ and let V denote the vector space $V = \text{Span}\{f_1(x), f_2(x), f_3(x)\}$. Consider the transformations $L_1 : \mathbb{R}^2 \to V$ and $L_2 : V \to \mathbb{R}^2$ given by

$$L_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ax^{1/2} + (a+b)x^{3/2} - (a+b)x^{5/2} \text{ and } L_2(f(x)) = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}.$$

(i) Show that L_1 is linear. *3 pt.*
(ii) Show that L_2 is linear. *3 pt.*
(iii) Find the 3×2 matrix A representing L_1 relative to the bases $E = \{e_1, e_2\}$ and $F = \{f_1(x), f_2(x), f_3(x)\}$. *4 pt.*
(iv) Find the 2×3 matrix B representing L_2 relative to F and E. *4 pt.*

Problem 6. Let A, B be $m \times n$ matrices. That satisfy the following:

(a) $R(A) \cap R(B) = \{0\}$, i.e. zero is the only common vector in $R(A)$ and $R(B)$.

(b) $N(A) \cap N(B) = \{0\}$, i.e. zero is the only common vector in $N(A)$ and $N(B)$.

(i) Show that if some $x \in \mathbb{R}^n$ satisfies $Ax = Bx$ then $Ax = Bx = 0$. *2 pt.*

(ii) Prove $N(A - B) \subset N(A) \cap N(B)$, i.e. that any $x \in N(A - B)$ is in $N(A)$ and in $N(B)$. *4 pt.*

(iii) Prove $N(A - B) = \{0\}$ i.e. that $N(A - B)$ contains only the zero vector. *4 pt.*

(iv) Show that rank(A − B) = n. *4 pt.*

Problem 7.
Let A by an $m \times n$ matrix with $\text{rank}(A) = n$.
(i) What is $\text{nullity}(A)$? *1 pt.*
(ii) Let x be a vector in $\mathbb{R}^n$ for which $(Ax)^\top(Ax) = 0$. Show that x must be the zero vector. *3 pt.*
(iii) Consider for each x, y in $\mathbb{R}^n$ the quantity $\langle x, y\rangle = (Ax)^\top(Ay)$. Show that it defines an inner product. *3 pt.*

Consider the vector space $C[0,1]$ equipped with the inner product given by $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. If $f(x) = \sqrt{3}x$, $g(x) = \sqrt{15}x^7$ find
(iv) The norms of f and g. *3 pt.*
(v) The cosine of the angle of f and g. *3 pt.*
(vi) The vector projection $p_g(f)$ of f onto g. *1 pt.*

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