Spring 2018, Math 304 | **Second Midterm**

Sample

Last name:

First name:

UIN:

"An Aggie does not lie, cheat or steal or tolerate those who do."

This exam consists of **ten** problems, the total point value of which is 100 points.

The answer to each question must be **justified in detail**, unless otherwise specified.

The time length of this exam is 1 hour and 15 minutes.

The use of electronic devices, such as cellphones, tables, laptops, and calculators is prohibited.

Good luck!

Prb. 1	Prb. 2	Prb. 3	Prb. 4	Prb. 5	Prb. 6	Prb. 7	Prb. 8	Prb. 9	Prb. 10	Total
10	10	10	10	10	10	10	10	10	10	100

Problem 1. Let P_3 denote the vector space of polynomials of degree at most two and consider $q_1(x) = 1$, $q_2(x) = x + 1$, $q_3(x) = x^2 - 1$.

(i) Write the Wronskian of $q_1(x)$, $q_2(x)$, and $q_3(x)$. *3 pts.*

(ii) Are $q_1(x)$, $q_2(x)$, and $q_3(x)$ linearly independent? *3 pts.*

(iii) Is $F = \{q_1(x), q_2(x), q_3(x)\}$ a basis of P_3? *3 pts.*

(iv) If $p_1(x) = 1, p_2(x) = x, p_3(x) = x^2$, it is given that $E = \{p_1(x), p_2(x), p_3(x)\}$ is a basis of P_2. Write the coordinate vectors $[q_1(x)]_E$, $[q_2(x)]_E$ and $[q_3(x)]_E$. *1 pts.*

Problem 2. Let $C[0,1]$ denote the vector space of all continuous functions with domain $[0,1]$, consider the functions $f_1(x) = 4x^{1/2}$, $f_2(x) = 4x^{3/2}$, $f_3(x) = 4x^{5/2}$.

(i) What is the Wronskian of $f_1(x)$, $f_2(x)$, and $f_3(x)$? *4 pts.*

(ii) Are $f_1(x)$, $f_2(x)$, and $f_3(x)$ linearly independent? *4 pts.*

(iii) What is the dimension of the vector space $V = \text{Span}\{f_1, f_2, f_3\}$? *2 pts.*

Problem 3.

(i) Consider the following subsets of $\mathbb{R}^3$:

$$S_1 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - c = b \right\}, \quad \text{and} \quad S_2 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a^2 + b^2 + c^2 = 1 \right\}$$

Which of the above are subspaces and which are not? *5 pt.*

(ii) Consider the following subsets of $C[0,1]$:

$$X_1 = \{f : f(0) = f(1)\} \quad \text{and} \quad X_2 = \{f : f(0) = 1\}$$

Which of the above are subspaces and which are not? *5 pt.*

Problem 4.

(i) Do the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a spanning set of $\mathbb{R}^3$? *4 pt.*

(ii) Do the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ form a spanning set of $\mathbb{R}^3$? *4 pt.*

(iii) Do the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a spanning set of $\mathbb{R}^3$? *2 pt.*

Problem 5.
Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & -2 & 1 & 1 \\ -2 & 4 & 4 & 1 & 3 \\ 1 & -2 & -2 & 2 & 1 \end{bmatrix}.$$

(i) Find the null space of A. *8 pt.*
(ii) Solve the equation $Ax = \mathbf{b}$ if x_0 is a known solution in the cases: *2 pt.*

$$(1)\ \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},\ x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad (2)\ \mathbf{b} = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix},\ x_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Problem 6.
Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 2 & 6 & 2 \\ 2 & 7 & 3 \end{bmatrix}.$$

(i) Find the null space of A. *5 pt.*

(ii) Are the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ linearly independent? *3 pt.*

(iii) Are the vectors $v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 4 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ linearly independent? *2 pt.*

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Problem 7.
Consider the basis E of $\mathbb{R}^3$ given by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

(i) Find the transition matrix from E to the usual basis $\{e_1, e_2, e_3\}$. *3 pt.*
(ii) Find the transition matrix from the usual basis to E. *3 pt.*
(iii) Consider the basis F of $\mathbb{R}^3$ given by the vectors

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad w_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the transition matrix form F to E. *2 pt.*
(iv) Let v be a vector with $[v]_F = [-4, 5, 8]^\top$. Find the coordinate vector of v with respect to the basis E and with respect to the standard basis. *2 pt.*

Problem 8. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ -1 & 2 & 1 & 0 \end{bmatrix}.$$

(i) Find the rank and nullity of A. *2 pt.*
(ii) Write a basis for the row space of A. *2 pt.*
(iii) Write a basis for the column space of A. *3 pt.*
(iv) Write a basis for the null space of A. *3 pt.*
(v) Consider $S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$ as a subspace of $\mathbb{R}^3$. What is $\dim(S)$? *1 pt.*
(vi) Write a basis for S. *1 pt.*

Problem 9. (i) Let A be a non-singular $n \times n$ matrix and V be an $n \times m$ matrix with $N(B) = \{0\}$, i.e., the null space of B consists only of the zero vector. Show that $N(AB) = \{0\}$, i.e., show that the null space of AB consists only of the zero vector. *5 pt.*

(ii) Let A be a non-singular $n \times n$ matrix and let $x_1, \dots, x_m$ be linearly independent vectors in $\mathbb{R}^n$. Show that the vectors $y_1 = Ax_1, \dots, y_m = Ax_m$ are also linearly independent. *5 pt.*

Problem 10. Let A, B be $m \times n$ matrices. That satisfy the following:

(a) $\mathrm{col}(A) \cap \mathrm{col}(B) = \{0\}$, i.e. zero is the only common vector in the column space of A and the column space of B.

(b) $N(A) \cap N(B) = \{0\}$, i.e. zero is the only common vector in the null space of A and the null space of B.

(i) Show that for any $x \in \mathbb{R}^n$ the vector Ax is in $\mathrm{col}(A)$ and Bx is in $\mathrm{col}(B)$. *2 pt.*

(ii) Show that if some $x \in \mathbb{R}^n$ satisfies $Ax = Bx$ then $Ax = Bx = 0$. *2 pt.*

(iii) Prove $N(A - B) \subset N(A) \cap N(B)$, i.e. that any $x \in N(A - B)$ is in $N(A)$ and in $N(B)$. *2 pt.*

(iv) Prove $N(A-B) = \{0\}$ i.e. that $N(A-B)$ contains only the zero vector. *2 pt.*

(v) Show that rank(A − B) $= n$. *2 pt.*