

Fall 2016, Math 409, Section 502**Seventh Assignment**

Due Monday, March 28 (at the beginning of class)

Exercise 1. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions so that for all $q \in \mathbb{Q}$ we have $f(q) = g(q)$. Prove that $f = g$, i.e. $f(x) = g(x)$ for all $x \in \mathbb{R}$. *2 pts.*

Exercise 2. Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ be an enumeration of the set of rational numbers (i.e. for each $q \in \mathbb{Q}$ there is exactly one $n \in \mathbb{N}$ with $q = q_n$). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by the rule

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{n} & \text{if } x \in \mathbb{Q} \text{ and } x = q_n. \end{cases}$$

Prove that f is continuous at x if and only if $x \in \mathbb{R} \setminus \mathbb{Q}$. *2 pts.*

Exercise 3. Let a, b be real numbers with $a < b$ and $f : [a, b] \rightarrow [a, b]$ be a continuous function. Prove that there exists $x \in [a, b]$ with $f(x) = x$. *2 pts.*

Exercise 4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and assume that $f[\mathbb{R}] \subseteq \mathbb{R} \setminus \mathbb{Q}$. Prove that f is constant, i.e. there is some $c \in \mathbb{R}$ with $f(x) = c$ for all $x \in \mathbb{R}$. *2 pts.*

Exercise 5. Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be the function defined by $f(x) = \sqrt{x}$, for all $x \in [0, +\infty)$. Prove that f is uniformly continuous. *2 pts.*