Fall 2016, Math 409, Section 502 **Fifth Assignment**

Due Friday, March 4 (at the beginning of class) *(15 pts. total)*

Exercise 1. Let $(x_n)_n$ be an increasing sequence that is **not** bounded above. Show that $\lim_n x_n = +\infty$ (i.e. $(x_n)_n$ **diverges** to $+\infty$.) *1 pt.*

Exercise 2. Let A be a non-empty subset of $\mathbb{R}$ that is bounded above. Assume that $\sup A \notin A$. Prove that there exists a strictly increasing sequence $(x_n)_n$ of A so that $\lim_n x_n = \sup A$. *2 pts.*

Exercise 3. Let $(x_n)_n$ be a real sequence and assume that $x_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$. *2 pts.*

(i) If $(x_n)_n$ is Cauchy, show that it is eventually constant (i.e. there exists $n_0 \in \mathbb{N}$ so that for all $n \geqslant n_0$ we have $x_n = x_{n_0}$).

(ii) If $(x_n)_n$ converges to some $a \in \mathbb{R}$, then $a \in \mathbb{Z}$.

Exercise 4. Let $(x_n)_n$ be a real sequence. Assume that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for all $m \geqslant n \geqslant n_0$ we have $\left|\sum_{k=n}^{m} x_k\right| < \varepsilon$. Show that there exists $S \in \mathbb{R}$ with $\lim_n \left(\sum_{k=1}^{n} x_k\right) = S$. *2 pts.*

Exercise 5. Let $(x_n)_n$ be a real sequence and assume that there exists $a \in \mathbb{R}$ with $0 < a < 1$, so that $|x_n - x_{n+1}| \leqslant a^n$ for all $n \in \mathbb{N}$. Show that $(x_n)_n$ converges to some real number x. *3 pts.*

Exercise 6. Calculate $\liminf_n x_n$ and $\limsup_n x_n$ in the cases below (fully explain your answer). *2 pts.*

(i) $x_n = \sin\left(n\frac{\pi}{2}\right)$, for all $n \in \mathbb{N}$.

(ii) $x_n = 2^{(-1)^n n}$ for all $n \in \mathbb{N}$.

Exercise 7. *3 pts.*

(i) Let $(x_n)_n$ be a real sequence and r be a real number. Assume that $r < \liminf_n x_n$. Show that there exists $n_0 \in \mathbb{N}$ with $r < x_n$ for all $n \geqslant n_0$.

(ii) Let $(x_n)_n$, $(y_n)_n$ be real sequences so that $y_n \neq 0$ for all $n \in \mathbb{N}$ and $0 < \liminf_n y_n < +\infty$. Show that

$$\limsup_n \left(\frac{x_n}{y_n}\right) \leqslant \frac{\limsup_n x_n}{\liminf_n y_n}.$$