Fall 2016, Math 409, Section 502 **Fourth Assignment**

Due Friday, February 19 (at the beginning of class)

Exercise 1. Let $(x_n)_n$ be a real sequence and $a \in \mathbb{R}$. Show that $\lim_n x_n = a$ if and only if $\lim_n |x_n - a| = 0$. *2 pts.*

Exercise 2. *3 pts.*

Using the definition of convergence only, show the following:

(i) $\lim\limits_n \dfrac{n-1}{n} = 1$,

(ii) $\lim\limits_n \dfrac{3n + \frac{2}{n}}{5n} = \dfrac{3}{5}$,

Show that each of the following sequences diverges, to either $-\infty$ or $+\infty$:

(iii) $(x_n)_n$ with $x_n = n^2 - n$ for all $n \in \mathbb{N}$,

(iv) $(x_n)_n$ with $x_n = \dfrac{n - n^2}{n}$ for all $n \in \mathbb{N}$.

Exercise 3. Let $(x_n)_n$, $(y_n)_n$ be real sequences and $a \in \mathbb{R}$. Assume that $\lim_n x_n = 0$ and that there exists $n_0 \in \mathbb{N}$ so that for all $n \geqslant n_0$ we have $|y_n - a| \leqslant |x_n|$. Using the definition of convergence only, show that $\lim_n y_n = a$. *2 pts.*

Exercise 4. Let $(x_n)_n$ be a real sequence and $a \in \mathbb{R}$. *3 pts.*

(i) If the sequence $(x_n)_n$ **does not** converge to a, prove that there exists an $\varepsilon > 0$ and a subsequence $(x_{n_k})_k$ of $(x_n)_n$, so that $|x_{n_k} - a| \geqslant \varepsilon$ for all $k \in \mathbb{N}$.

(ii) Assume that for every subsequence $(x_{n_k})_k$ of $(x_n)_n$, there exists a further subsequence $(x_{n_{k_m}})_m$ of $(x_{n_k})_k$ that converges to a. Prove that the entire sequence $(x_n)_n$ converges to a.