Fall 2016, Math 409, Section 502 **Second Assignment**
Due Friday, February 5 (at the beginning of class)

Exercise 1. Let A be a non-empty subset of $\mathbb{R}$. Assume that s is a real number satisfying the following:

(i) s is an upper bound for A and
(ii) for every $\varepsilon > 0$ there exists $a \in A$ with $s - \varepsilon < a \leqslant s$.

Show that $s = \sup A$. *1 pt.*

Exercise 2 (Approximation property for infima)**.** Let A be a non-empty subset of $\mathbb{R}$ so that $s = \inf A$ exists. Show that for all $\varepsilon > 0$ there exists $a \in A$ with $s \leqslant a < s + \varepsilon$. *1 pt.*

Exercise 3 (Completeness property for infima)**.** Let A be a non-empty subset of $\mathbb{R}$ that is bounded below. Show that $\inf A$ exists. *1 pt.*

Exercise 4. If a, b are real numbers with $a < b$ and $A = (a, b)$, show that $\sup A = b$. *1 pt.*

Exercise 5. Let A and B be non-empty subsets of $\mathbb{R}$ with the property $a \leqslant b$ for all $a \in A$ and $b \in B$. Show that $\sup A$ and $\inf B$ exist and in particular $\sup A \leqslant \inf B$. *2 pts.*

Exercise 6.

(i) Show that $2^n < n!$, for all $n \in \mathbb{N}$ with $n \geqslant 4$.
(ii) Show that $n2^n < n!$, for all $n \in \mathbb{N}$ with $n \geqslant 6$. *1 pt.*

Exercise 7. Let $a \in \mathbb{R}$ with $a \geqslant 0$. The goal is to show that there exists a non-negative real number r, with $r^2 = a$. We call r the square root of a and denote it by $r = \sqrt{a}$ or $r = a^{1/2}$. Define

$$A = \{x \in \mathbb{R} : 0 \leqslant x \text{ and } x^2 \leqslant a\} \text{ and } B = \{x \in \mathbb{R} : 0 \leqslant x \text{ and } a \leqslant x^2\}.$$

Without assuming the existence of $\sqrt{a}$ prove:

(i) A is non-empty and $\max\{1, a\}$ is an upper bound for A,
(ii) B is non-empty and $\min\{1, a\}$ is a lower bound for B,
(iii) if $s = \sup A$, then $s^2 \leqslant a$ (Hint: use the approximation property for suprema),
(iv) if $t = \inf B$, then $a \leqslant t^2$ (Hint: use the approximation property for infima),
(v) $s = t$ and hence $s^2 = a$. *3 pts.*