Fall 2016, Math 409, Section 502 **Eleventh Assignment**

Due Friday, April 29 (at the beginning of class)

Exercise 1. *3 pts.*

(i) Let x, y be real numbers with $0 \leqslant x \leqslant y$. Show that

$$x \leqslant \sqrt{\frac{x^2+y^2+xy}{3}} \leqslant y.$$

(ii) Let a, b be non-negative real numbers with $a < b$. Use Riemann sums to prove that $\int_a^b x^2 \mathrm{d}x = \frac{1}{3}(b^3 - a^3)$.

Exercise 2. Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ be a bounded function. *7 pts.*

(i) If $Q_1 = \{x_0, x_1, \ldots, x_n\}$ and $Q_2 = \{y_0, y_1, \ldots, y_m\}$ are partitions of the interval $[a, b]$ and for each $i = 1, \ldots, n$ we define the set $E_i = \{1 \leqslant j \leqslant m : [y_{j-1}, y_j] \subseteq [x_{i-1}, x_i]\}$, prove

$$\left|(x_i - x_{i-1}) - \sum_{j \in E_i} (y_j - y_{j-1})\right| \leqslant 2\|Q_2\|.$$

(ii) If $Q_1 = \{x_0, x_1, \ldots, x_n\}$, $Q_2 = \{y_0, y_1, \ldots, y_m\}$ are partitions of $[a, b]$ and we define $M = \sup\{|f(x)| : x \in [a, b]\}$, prove the following inequalities:

$$U(f, Q_2) \leqslant U(f, Q_1) + 3nM\|Q_2\| \text{ and,}$$
$$L(f, Q_2) \geqslant L(f, Q_1) - 3nM\|Q_2\|.$$

(iii) Prove that the following statements are equivalent.

(a) The function f is Riemann integrable.

(b) For every $\varepsilon > 0$ there exists $\delta > 0$, so that for every partition P of $[a, b]$ with $\|P\| < \delta$ we have $U(f, P) - L(f, P) < \varepsilon$.

(c) For every $\varepsilon > 0$ there exists $\delta > 0$, so that for every partition P of $[a, b]$ with $\|P\| < \delta$ and all samples t_j, s_j over P we have $|\mathcal{S}(f, P, t_j) - \mathcal{S}(f, P, s_j)| < \varepsilon$.

(iv) If f is assumed to be Riemann integrable, prove that if $(P_n)_n$ is a sequence of partitions of $[a, b]$, with $\lim_n \|P_n\| = 0$, and t_j^n are samples over P_n for all n, then $\lim_n \mathcal{S}(f, P_n, t_j^n) = \int_a^b f(x)\mathrm{d}x$.

(v) Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function. Prove that $\lim_n \frac{1}{n} \sum_{k=1}^{n} f(k/n) = \int_0^1 f(x)\mathrm{d}x$.

Comment: Statement (iii) of Exercise 2 is called the Darboux criterion of Riemann integrability.