Practice problems for the third midterm exam

Math 251, Fall 2015

Sections 504 & 513

Problem 1. Find the gradient vector field of:

(i) the function of two variables

$$f(x,y) = 2xe^{x^2y} + 5y^4 + x^3\sin(e^{-y}),$$

(ii) the function of three variables

$$f(x,y,z) = \sin(xe^y) + 2x^2z^3 + \cos(y - 3z^2).$$

Problem 2. If C is the curve given by $r(t) = \langle t^2, t, \sin(t)\rangle$, $0 \leqslant t \leqslant \pi$:

(i) evaluate the line integral $\int_C f\,\mathrm{d}s$, where $f(x,y,z) = 8y - 2z\cos(y)$,
(ii) evaluate the line integral $\int_C g\,\mathrm{d}z$, where $g(x,y,z) = z\sin(y)$.

Problem 3. If C is the part of the unit circle in the first quadrant of the xy-plane, oriented counter-clockwise:

(i) evaluate the line integral $\int_C f\,\mathrm{d}s$, where $f(x,y) = xy$,
(ii) evaluate the line integral $\int_C g\,\mathrm{d}x$, where $g(x,y) = -xy^2$.

Problem 4. If C is the line segment from $(1,0)$ to $(-1,2)$:

(i) evaluate the line integral $\int_C f\,\mathrm{d}s$, where $f(x,y) = (y-1)\sin(x^2)$,
(ii) evaluate the line integral $\int_C g\,\mathrm{d}y$, where $g(x,y) = x\cos((1-y)^2)$.

Problem 5. Evaluate the line integral $\int_C F\cdot \mathrm{d}r$ for the three-dimensional vector field $F(x,y,z) = \langle y\cos(x), -\sin(z^2), \sqrt{x}\rangle$ and the curve C given by $r(t) = \langle t, \cos(t), \sqrt{t}\rangle$, $0 \leqslant t \leqslant 2\pi$.

Problem 6. Find the work done by the force field $F(x,y) = \langle y, -x/2\rangle$ in moving a particle from $(-1,1)$ to $(2,7)$, along the curve $y = 2x^2 - 1$.

Problem 7. Determine whether the two-dimensional vector field $F(x,y) = \langle x\ln(y), \frac{x^2}{2y}\rangle$ is conservative.

Problem 8. Determine which of the following two-dimensional vector fields are conservative. For those that are, find a potential function.

(i) $F(x,y) = \langle \sin(xy) + e^y, \cos(xy) - e^x\rangle$,
(ii) $F(x,y) = \langle ye^x + 2xy, e^x + x^2\rangle$.

Problem 9. Use the Fundamental Theorem of Calculus and Problem 8 to evaluate the line integral $\int_C F\cdot \mathrm{d}r$, where $F(x,y) = \langle ye^x + 2xy, e^x + x^2\rangle$ and C is the part of the unit circle from $(\sqrt{2}/2, \sqrt{2}/2)$ to $(\sqrt{2}/2, -\sqrt{2}/2)$ oriented counter-clockwise.

Problem 10. Use Green's Theorem to evaluate the line integral $\int_C F \cdot \mathrm{d}r$ in the following cases:

(i) $F(x, y) = \langle -y^3, x^3 \rangle$ and C is the positively oriented curve consisting of the left half of the unit disk and the line segment connecting $(0, -1)$ to $(0, 1)$.
(ii) $F(x, y) = \langle -y^2/2, xy \rangle$ and C is the positively oriented triangle whose vertices are $(0, 0)$, $(1, 0)$ and $(2, 2)$.

Problem 11. Use Green's Theorem to find the work done by the force field $F(x, y) = \langle e^x, 5x^2 \cos(y^5) \rangle$ in moving a particle from $(1, 1)$ to $(0, 1)$ parallel to the x-axis, to the origin along the y-axis and then back to $(1, 1)$ along the curve $y = \sqrt{x}$.

Problem 12. Find the curl and divergence of the following three-dimensional vector fields:

(i) $F(x, y, z) = \langle xy^2, z^3 + 2xz^2 - 7y^2z, 2xy^4z^2 \rangle$,
(ii) $F(x, y, z) = \langle z \sin(xe^y), y\sqrt{x^2 + z^2}, 3xe^{yz^4} \rangle$.

Problem 13. Determine which of the following three-dimensional vector fields are conservative. For those that are, find a potential function.

(i) $F(x, y, z) = \langle z \sin(xy) + z^2 e^y, \cos(xy) - e^z x, x^2\sqrt{y^2 + x^2} \rangle$,
(ii) $F(x, y, z) = \langle 2xy + z \cos(x), x^2 + e^z, \sin(x) + ye^z \rangle$.

Problem 14. Find a parametric representation of the following surfaces:

(i) the part of the plane $x + 2y + 3z = 6$ that lies in the first octant,
(ii) the part of the plane $2x - y + z = 4$ that lies inside the cylinder $y^2 + z^2 = 4$,
(iii) the part of surface $x^2+z^2 = y$ that lies inside the cylinder $x^2+z^2 = 1$.

Problem 15. Find a parametric representation of the following surfaces:

(i) the surface of the entire unit sphere,
(ii) the part of the surface of the unit sphere that lies in the first octant,
(iii) the part of the surface of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $2x + y - z = 4$,
(iv) the part of the surface of the cylinder $x^2 + y^2 = 9$ that lies in the first octant, below the plane $z = 5$.

Problem 16. Find an equation for the plane tangent to the surface given by $r(u, v) = \langle u, v, \cos(u) \sin(v) \rangle$ at the point $(\pi/4, \pi/4, 1/2)$.

Problem 17. Find the area of the surfaces from Problem 14.

Problem 18. Find the area of the surfaces (iii) and (iv) from Problem 15.

Problem 19. Find the area of the surface given by $r(u, v) = \langle u, v, \sin(u) \rangle$, where $0 \leqslant u \leqslant \pi/4$ and $0 \leqslant v \leqslant \sin(u) \cos(u)$.

Problem 20. Evaluate the surface integral $\iint_S f \, \mathrm{d}S$ in the following cases:

(i) $f(x, y, z) = x^2y$ and S is the part of the plane $x + 2y + 3z = 6$ that lies in the first octant,
(ii) $f(x, y, z) = y^2 + z^2$ and S is the part of the plane $2x - y + z = 4$ that lies inside the cylinder $y^2 + z^2 = 4$,
(iii) $f(x, y, z) = x^2/y$ and S is the part of surface $x^2 + z^2 = y$ that lies inside the cylinder $x^2+z^2 = 1$ but outside the cylinder $x^2+z^2 = 1/4$.

Problem 21. Evaluate the mass and find the center of mass of an object in the shape of the surface given by $r(u, v) = \langle u, v, \sin(u)\rangle$, where $0 \leqslant u \leqslant 2\pi$ and $-\pi \leqslant v \leqslant \pi$, if its density function is $\rho(x, y, z) = \sqrt{2 - z^2}$.

Problem 22. Evaluate the surface integral $\iint_S f \, \mathrm{d}S$ in the following cases:
(i) $f(x, y, z) = x$ and S is the part of the surface of the cylinder $y^2+z^2 = 1$ that lies between the planes $x = 0$ and $2x + y - z = 4$,
(ii) $f(x, y, z) = x^3 + xy^2$ and S is the part of the surface of the cylinder $x^2 + y^2 = 9$ that lies in the first octant, below the plane $z = 5$.

Problem 23. Evaluate the flux of $F(x, y, z) = \langle y, -x, \sqrt{1 - z}\rangle$ across the boundary of the solid region bounded by the surface $x^2 + y^2 = 1 - z$ and the xy-plane, oriented upwards.

Problem 24. A fluid has density $3/4$ and its velocity field is given by $V = \langle y^2 + z^2, xy, xz\rangle$. Find the flow of the fluid outwards the boundary of the solid region bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = 0$ and $2x + y - z = 4$.

Problem 25. Use Stoke's Theorem to evaluate $\int_C F \cdot \mathrm{d}r$, where $F(x, y, z) = \langle -y^2, x, z^2\rangle$ and C is the intersection of the plane $y+z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented clockwise as viewed from above.

Problem 26. Use Stoke's Theorem to evaluate $\iint_S \mathrm{curl} F \cdot \mathrm{d}S$, if $F(x, y, z) = \langle yz, xz, xy\rangle$ and S is the part of the surface $x^2 + y^2 + z^2 = 10$ inside the cylinder $x^2 + y^2 = 1$, above the xy-plane, oriented upwards.

Problem 27. Use Stoke's Theorem to evaluate $\int_C F \cdot \mathrm{d}r$, where $F(x, y, z) = \langle z^2, y^2, x\rangle$ and C is the boundary of the part of the surface of the unit sphere that lies in the first octant, oriented counter-clockwise when viewed from above.

Problem 28. Use Stoke's Theorem to evaluate $\iint_S \mathrm{curl} F \cdot \mathrm{d}S$, if $F(x, y, z) = \langle x^2y, z^2, yz\rangle$ and S is the box without a lid, whose vertices are $(\pm 1, \pm 1, \pm 1)$, oriented outwards.

Problem 29. Use the Divergence Theorem to evaluate the flux of F across the positively oriented closed surface S:
(i) $F(x, y, z) = \langle x^2y, xy^2, x^3y^9\rangle$ and S is the surface of the rectangle $[0, 1] \times [-1, 1] \times [-1, 0]$.
(ii) $F(x, y, z) = \langle x, xy, zy\rangle$ and S is the surface of the solid region bounded by the planes $x + y + z = 1$, $x = 0$, $y = 0$ and $z = 0$.

Problem 30. Use the Divergence Theorem to evaluate the flux of the vector field $F(x, y, z) = \langle x^3, y^3, x^3 + y^3 + z^3 \rangle$ across the lower part of the unit sphere, oriented outwards from the center.