

STRATEGICALLY REPRODUCIBLE BASES AND THE FACTORIZATION PROPERTY

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ABSTRACT

We introduce the concept of strategically reproducible bases in Banach spaces and show that operators which have large diagonal with respect to strategically reproducible bases are factors of the identity. We give several examples of classical Banach spaces in which the Haar system is strategically reproducible: multi-parameter Lebesgue spaces, mixed-norm Hardy spaces and most significantly the space L^1 . Moreover, we show the strategic reproducibility is inherited by unconditional sums.

1. Introduction

In this paper, we address the following question: Given a Banach space X with a basis $(e_i)_{i=1}^\infty$, let $T : X \rightarrow X$ be an operator, whose matrix representation has a diagonal whose elements are uniformly bounded away from 0. We say in that case that T has a large diagonal. Is it possible to factor the identity operator on X through T ?

The origin of this problem can be traced back to the work of Pełczyński [24], who proved that every infinite-dimensional subspace of ℓ^p , $1 \leq p < \infty$ and c_0 contains a further subspace which is complemented and isomorphic to the whole space.

Closely related is the concept of primarity of a Banach space. Recall that X is called **primary**, if for every bounded projection $P : X \rightarrow X$, either $P(X)$ or $(I - P)(X)$ is isomorphic to X . The connection between the primarity of a Banach space and the factorization problem is as follows: either P has large diagonal or $I - P$ has large diagonal on a “large” subsequence of the basis $(e_i)_{i=1}^\infty$ of the Banach space X . For example Enflo (according to [17]) proved primarity for $X = L^p$, $1 \leq p < \infty$, by showing that for every operator $T : L^p \rightarrow L^p$, the identity operator factors either through T or $I - T$; see also Alspach–Enflo–Odell [1]. Factorization and primarity theorems were obtained by Capon [4] for the mixed norm spaces $L^p(L^q)$, $1 < p, q < \infty$, and by the third named author [19] for H^1 and BMO.

Separately, Andrew [2] showed that for $1 < p < \infty$, every operator $T : L^p \rightarrow L^p$ which has large diagonal with respect to the Haar system is a factor of the identity operator on L^p . More recently in [13] it was proved that for $1 \leq p, q < \infty$, every operator $T : H^p(H^q) \rightarrow H^p(H^q)$ which has large diagonal is a factor of the identity operator on $H^p(H^q)$.

In this paper we introduce a new approach to the factorization problem by devising an infinite two-person game and isolate a property of a basis called strategic reproducibility, which implies the factorization of the identity through operators with large diagonal. We say in that case, the basis $(e_i)_{i=1}^{\infty}$ has the factorization property. By using this method, we obtain simplified proofs of existing results, and obtain the following new factorization theorems for L^1 and related spaces.

THEOREM: *The normalized Haar system of $L^1[0, 1]$ has the factorization property. Moreover, the normalized bi-parameter Haar system of $L^1([0, 1]^2)$ and the tensor product of the ℓ^p unit vector basis with the Haar system have the factorization property.*

The paper is organized as follows. Section 2 covers basic concepts relevant to this work. In Section 3, we define three notions of strategic reproducibility and show that those imply the factorization property. In Section 4 we review basic properties of multi-parameter Lebesgue- and Hardy spaces. In Section 5 we establish that the Haar system is strategically reproducible in several classical Banach spaces such as reflexive, multi-parameter Lebesgue spaces, H^1 and two-parameter Hardy spaces $HP(H^q)$, $1 \leq p, q < \infty$. In Section 6 we show that the Haar system is strategically reproducible in L_0^1 . In Section 7 we show that unconditional sums of spaces with strategically reproducible bases have themselves that property. Finally, we discuss open problems in Section 8.

2. A brief discussion of basic concepts

We discuss several closely related concepts for operators on Banach spaces.

Definition 2.1: Let X be a Banach space and $T : X \rightarrow X$ be a bounded linear operator.

- (i) We say that $T(X)$ **contains a copy** of X if there is a (necessarily closed) subspace Y of $T(X)$ that is isomorphic to X .
- (ii) We say that T **preserves a copy** of X (or **fixes a copy** of X) if there exists a subspace Y of X that is isomorphic to X and T restricted on Y is an isomorphism.
- (iii) We say that the identity operator I on X **factors through** T if there are bounded linear operators $R, S : X \rightarrow X$ with $I = STR$.

We also consider a quantified version of (iii). For $K > 0$ we say that the identity **K -factors through T** if there are bounded linear operators $R, S : X \rightarrow X$ with

$$\|R\| \cdot \|S\| \leq K \quad \text{and} \quad I = STR$$

and we say that the identity **almost K -factors through T** if it $(K + \varepsilon)$ -factors through T for all $\varepsilon > 0$.

Remark 2.2: In general, for a given operator T , it is easy to see that (iii) \Rightarrow (ii) and (ii) \Rightarrow (i). The converse implications are in general false. To see that (i) $\not\Rightarrow$ (ii) take a quotient operator $T_0 : L^1 \rightarrow \ell_1$ and a quotient operator $T_1 : \ell_1 \rightarrow L^1$. Then if $T = T_1 \circ T_0 : L^1 \rightarrow L^1$, $T(L^1) = L^1$ however T does not preserve a copy of L^1 . There is an example demonstrating (iii) $\not\Rightarrow$ (ii) but it is slightly more involved. We first observe that if $I = STR$ and $Z = TR(X)$, then Z is isomorphic to X and complemented in X . Indeed, it follows that R is bounded below and T is bounded below on $R(X)$ hence TR is an isomorphic embedding. Furthermore, S restricted on $Z = TR(X)$ is an isomorphism onto X . Therefore, we can define the inverse map $S|_Z^{-1} : X \rightarrow Z$. One can check that $Px = S|_Z^{-1}(Sx)$ defines a bounded projection onto Z . This easy fact implies that if X is a minimal space that is not complementably minimal, then there exists an operator $T : X \rightarrow X$ that is an into isomorphism so that the identity does not factor through X . To see this, choose a subspace Y of X that is isomorphic to X and does not contain a further subspace isomorphic to X and complemented in X . If $T : X \rightarrow X$ is an into isomorphism, the image of which is Y , then the identity does not factor through T . Indeed, if $I = STR$ then $Z = TR(X)$ is isomorphic to X and complemented to X . This is not possible because Z is a subspace of Y . In conclusion, the fact that (iii) $\not\Rightarrow$ (ii) is reduced to the existence of a minimal and not complementably minimal space X . It is well known that the dual of Tsirelson space has this property, however to the best of our knowledge there is no recorded proof of this fact so we give a short description of it here. Assume that T^* is complementably minimal. We will show that this would imply that T is minimal, which was proved to be false in [7]*Corollary VI.b.6, page 58. Let X be an infinite-dimensional subspace of T . By [6]*Theorem 1 X is isomorphic to a quotient of T and hence X^* is isomorphic to a subspace of T^* . If T^* is complementably minimal, then X^* contains a complemented copy of T^* which yields that X contains a complemented copy of T . In particular, T is minimal and this cannot be the case.

The following definition of C -perturbable was introduced by Andrew in [2]. The concept of large diagonal, which was implicitly present in [2], was formally introduced in [13].

Definition 2.3: Let X be a Banach space with a normalized Schauder basis $(e_k)_k$.

- (i) Let $0 < C \leq 1$. The basis $(e_k)_k$ is called **C -perturbable** if whenever $T : X \rightarrow X$ is a bounded linear operator for which there exists $\delta > 0$ with $\|T(e_k) - e_k\| < C - \delta$ for all $k \in \mathbb{N}$, then $T(X)$ contains a copy of X .
- (ii) If an operator T on X satisfies $\inf_k |e_k^*(T(e_k))| > 0$, then we say that T has **large diagonal**.
- (iii) An operator T on X satisfying $e_m^*(T(e_k)) = 0$ whenever $k \neq m$ is called a **diagonal operator**.
- (iv) We say that the basis $(e_k)_k$ has the **factorization property** if whenever $T : X \rightarrow X$ is a bounded linear operator with $\inf_k |e_k^*(T(e_k))| > 0$ then the identity of X factors through T .

Remark 2.4: It is easy to see that a basis that has the factorizing property is also 1-perturbable. However, there are bases that are C -perturbable without the factorization property, as the following example shows.

The norm on the boundedly complete basis of James space $(e_i)_i$ is defined as follows:

$$(1) \quad \left\| \sum_{i=1}^n a_i e_i \right\| = \sup \left(\sum_{k=1}^m \left(\sum_{i \in E_k} a_i \right)^2 \right)^{1/2},$$

where the supremum is taken over $m \in \mathbb{N}$ and sequences of successive intervals $(E_k)_{k=1}^m$ of natural numbers. Let J denote the completion of the linear span of $(e_i)_i$ with this norm. Some well known important properties of J are the following:

- (i) The basis $(e_i)_i$ is spreading. In particular, for any sequence scalars $(a_i)_{i=1}^n$ and natural numbers $k_1 < \dots < k_n$ we have

$$(2) \quad \left\| \sum_{i=1}^n a_i e_i \right\| = \left\| \sum_{i=1}^n a_i e_{k_i} \right\|.$$

- (ii) The sequence $(e_i)_i$ is non-trivial weak Cauchy, i.e., there is $e^{**} \in J^{**} \setminus J$ so that $w^* - \lim_i e_i = e^{**}$. Additionally, $\text{dist}(e^{**}, J) = 1$.

- (iii) The space J is quasi-reflexive of order one. In particular, $J^{**} = \mathbb{R}e^{**} \oplus J$, i.e., J^{**} is spanned by e^{**} and the canonical embedding of J in J^{**} .

Note that by (iii), if the identity factors through an operator on J then that operator cannot be weakly compact.

PROPOSITION 2.5: *There exists a weakly compact operator $T : J \rightarrow J$ with*

$$e_i^*(T(e_i)) = 1,$$

for all $i \in \mathbb{N}$. In particular, (e_i) does not have the factorization property in J .

Proof. By (ii) the operator $S : J \rightarrow J$ given by $Se_i = e_{i+1}$, for all $i \in \mathbb{N}$, is a linear isometry. We define $T = I - S$, which has norm at most two. We will show that S is weakly compact by showing that for every bounded sequence $(x_i)_i$ the sequence $(Tx_i)_i$ has a weakly convergent subsequence. By the separability of J^* , we pass to a subsequence so that $(x_i)_i$ converges in the w^* -topology to some $x^{**} \in J^{**}$. By (iii), there is $x \in J$ and $c \in \mathbb{R}$ so that $x^{**} = x + ce^{**}$. We have

$$\begin{aligned} S^{**}(x^{**}) &= S(x) + cS^{**}(e^{**}) = S(x) + c(w^* - \lim_i S(e_i)) \\ &= S(x) + c(w^* - \lim_i e_{i+1}) = S(x) + ce^{**}. \end{aligned}$$

Thus $w^* - \lim_i Tx_i = w^* - \lim_i (x_i - S(x_i)) = x + ce^{**} - (S(x) + ce^{**}) = x - S(x)$. Because the w^* -limit is in J it has to be a weak limit. ■

We wish to show now that the boundedly complete basis of James space is perturbable. To achieve that we shall need the following well known fact. We describe a proof for completeness.

PROPOSITION 2.6: *Let $(x_i)_i$ be a non-trivial weak Cauchy sequence in J . Then, $(x_i)_i$ has a subsequence $(x_{j_i})_i$ that is equivalent to $(e_i)_i$ so that there exists a bounded linear projection $P : J \rightarrow [(x_{j_i})_i]$.*

Proof. Proposition 7.4 from [3] says the result holds, provided that the sequence $(e_i)_i$ is equivalent to its convex block sequences and not equivalent to the summing basis of c_0 . Both of these properties follow from (1). ■

PROPOSITION 2.7: *Let $T : J \rightarrow J$ be a bounded linear operator with the property $\sup_i \|Te_i - e_i\| < 1$. Then the identity factors through T . That is, the boundedly complete basis of J is perturbable.*

Proof. If $C = \liminf \|T(e_i) - e_i\| < 1$ then we have $\|T^{**}(e^{**}) - e^{**}\| \leq C$ and therefore, from (ii), $\text{dist}(T^{**}(e^{**}), J) \geq 1 - C > 0$. This means that $T^{**}(e^{**})$, which is the w^* -limit of $(T(e_i))_i$, is not in J . In other words, $(T(e_i))_i$ is non-trivial weak Cauchy. By Proposition 2.6 there is a subsequence $(T(e_{j_i}))_i$ of $(T(e_i))_i$ that is equivalent to $(e_i)_i$ and a bounded linear projection

$$P : J \rightarrow W = [(T(e_{j_i}))_i].$$

Let $A : J \rightarrow J$ be the map defined by $Ae_i = e_{j_i}$, which by (i) is bounded. Let $R : W \rightarrow J$ be the isomorphism given by $R(T(e_{j_i})) = e_i$ and set $B : J \rightarrow J$ with $B = R \circ P$. It is easy to see that $I = B \circ T \circ A$. ■

3. Strategical reproducibility, a condition implying the factorization property

In this section we formulate several versions of a property of bases we call strategical reproducibility and show that they imply the factorization property.

NOTATION AND CONVENTIONS. All our Banach spaces are assumed to be over the real numbers \mathbb{R} ; B_X denotes the unit ball, S_X denotes the unit sphere of a Banach space X , and c_{00} denotes the sequences in \mathbb{R} which eventually vanish.

For a Banach space X we denote by $\text{cof}(X)$ the set of cofinite-dimensional subspaces of X , while $\text{cof}_{w^*}(X^*)$ denotes the set of cofinite dimensional w^* -closed subspaces of X^* .

If $\bar{e} = (e_i)$ is a basis of a Banach space X , for $x = \sum_{i=1}^\infty x_i e_i \in X$ we call the set

$$\{i \in \mathbb{N} : x_i \neq 0\}$$

the **support of x with respect to (\bar{e})** and denote it by $\text{supp}_{\bar{e}}(x)$. If there is no confusion possible, we also may write $\text{supp}(x)$ instead of $\text{supp}_{\bar{e}}(x)$.

We recall that a basis (e_n) of a Banach space X is **shrinking** if the coordinate functionals (e_n^*) are a basis of X^* , and **unconditional** if for some constant $c \geq 1$ and all finite sequence of scalars $(a_i)_{i=1}^n$, and all $\sigma = (\sigma_i)_{i=1}^n \in \{\pm 1\}$,

$$\left\| \sum_{i=1}^n \sigma_i a_i e_i \right\| \leq c \left\| \sum_{i=1}^n a_i e_i \right\|.$$

Let $(x_i)_i$ and $(y_i)_i$ be Schauder basic sequences in (possibly different) Banach spaces and $C \geq 1$. We say that $(x_i)_i$ and $(y_i)_i$ are **C -equivalent** if there are $A, B > 0$ with $A \cdot B \leq C$ so that for any $(a_i)_i \in c_{00}$ of scalars we have

$$\frac{1}{A} \left\| \sum_{i=1}^{\infty} a_i y_i \right\| \leq \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \leq B \left\| \sum_{i=1}^{\infty} a_i y_i \right\|.$$

We say that $(x_i)_i$ and $(y_i)_i$ are **impartially C -equivalent** if for any finite choice of scalars $(a_i)_i \in c_{00}$ we have

$$\frac{1}{\sqrt{C}} \left\| \sum_{i=1}^{\infty} a_i y_i \right\| \leq \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \leq \sqrt{C} \left\| \sum_{i=1}^{\infty} a_i y_i \right\|.$$

Note that if two sequences are C -equivalent, then by scaling one of them we can always make them impartially C -equivalent.

We now formally define the concept of strategical reproducibility depending on properties of the basis of a Banach space. The most general form will be given in Definition 3.4. Nevertheless, under additional assumptions on the basis, this notion simplifies considerably. The proof that the different definitions of strategical reproducibility are equivalent under their respective assumptions on the basis will be given later.

If we demand that our basis is unconditional and shrinking, strategical reproducibility can be defined as follows.

Definition 3.1: Assume that X is a Banach space with a basis (e_n) which is unconditional and shrinking. Let $(e_n^*) \subset X^*$ be the corresponding coordinate functionals. We say that (e_n) is **strategically reproducible** if the following condition is satisfied for some $C \geq 1$:

$$\begin{aligned} (3) \quad & \forall n_1 \in \mathbb{N} \exists b_1 \in \text{span}(e_n : n \geq n_1) \exists b_1^* \in \text{span}(e_n^* : n \geq n_1), \\ & \forall n_2 \in \mathbb{N} \exists b_2 \in \text{span}(e_n : n \geq n_2) \exists b_2^* \in \text{span}(e_n^* : n \geq n_2), \\ & \forall n_3 \in \mathbb{N} \exists b_3 \in \text{span}(e_n : n \geq n_3) \exists b_3^* \in \text{span}(e_n^* : n \geq n_3), \\ & \quad \vdots \end{aligned}$$

so that

$$(3a) \quad \begin{aligned} & (b_k) \text{ is impartially } C\text{-equivalent to } (e_k), \quad \text{and} \\ & (b_k^*) \text{ is impartially } C\text{-equivalent to } (e_k^*), \end{aligned}$$

$$(3b) \quad b_k^*(b_l) = \delta_{k,l} \text{ for all } k, l \in \mathbb{N}.$$

Remark 3.2: Condition (3) in Definition 3.1 can be interpreted that one player in a two-person game has a winning strategy:

We fix $C \geq 1$. Player (I) chooses $n_1 \in \mathbb{N}$, then player (II) chooses

$$b_1 \in \text{span}(e_n : n \geq n_1) \quad \text{and} \quad b_1^* \in \text{span}(e_n^* : n \geq n_1).$$

They repeat the moves infinitely many times, obtaining for every $k \in \mathbb{N}$ numbers n_k , and vectors b_k and b_k^* . Player (II) wins if he was able to choose the sequences $(b_n) \subset X$ and $(b_n^*) \subset X^*$ so that (3a), (3b) are satisfied. Thus, the basis (e_i) is strategically reproducible if and only if for some $C \geq 1$ player (II) has a winning strategy.

In general it is not true that two-player games of infinite length are determined, i.e., that one of the players has a winning strategy. Nevertheless, for $C \geq 1$ it is easy to see that the set of all sequences (b_k, b_k^*) in $(X \times X^*)^{\mathbb{N}}$ which satisfy 3(a) and 3(b) is Borel measurable (it is actually closed) with respect to the product topology of the discrete topology on $X \times X^*$, and thus it follows from the main result in [15] that this game is determined. More on these Infinite Asymptotic Games can be found in [22].

Now we relax the condition on our basis (e_i) and only require it be unconditional. In that case we define strategical reproducible as follows.

Definition 3.3: Let X be a Banach space with an unconditional basis $(e_i)_i$ and fix positive constants $C \geq 1$.

Consider the following two-player game between player (I) and player (II). For $k \in \mathbb{N}$, turn k is played out in three steps.

Step 1: Player (I) chooses $\eta_k > 0$, $W_k \in \text{cof}(X)$, and $G_k \in \text{cof}_{w^*}(X^*)$.

Step 2: Player (II) chooses a finite subset E_k of \mathbb{N} and sequences of non-negative real numbers $(\lambda_i^k)_{i \in E_k}$, $(\mu_i^k)_{i \in E_k}$ satisfying

$$\sum_{i \in E_k} \lambda_i^{(k)} \mu_i^{(k)} = 1.$$

Step 3: Player (I) chooses $(\varepsilon_i^{(k)})_{i \in E_k}$ in $\{-1, 1\}^{E_k}$.

We say that player (II) has a winning strategy in the game $\text{Rep}_{(X, (e_i))}(C)$ if he can force the following properties on the result:

For all $n \in \mathbb{N}$ we set $x_k = \sum_{i \in E_k} \varepsilon_i^{(k)} \lambda_i^{(k)} e_i$ and $x_k^* = \sum_{i \in E_k} \varepsilon_i^{(k)} \mu_i^{(k)} e_i^*$ and demand:

- (i) the sequences $(x_k)_k$ and $(e_k)_k$ are impartially C -equivalent,

- (ii) the sequences $(x_k^*)_k$ and $(e_k^*)_k$ are impartially C -equivalent,
- (iii) for all $n \in \mathbb{N}$ we have $\text{dist}(x_k, W_k) < \eta_k$, and
- (iv) for all $n \in \mathbb{N}$ we have $\text{dist}(x_k^*, G_k) < \eta_k$.

We say that $(e_i)_i$ is **C -strategically reproducible** in X if for every $\eta > 0$ player (II) has a winning strategy in the game $\text{Rep}_{(X, (e_i))}(C + \eta)$.

Finally we will not even require the basis (e_j) to be unconditional and define strategical reproducible as follows:

Definition 3.4: Let X be a Banach space with a normalized Schauder basis $(e_i)_i$ and fix positive constants $C \geq 1$, and $\eta > 0$.

Consider the following two-player game between player (I) and player (II):

Before the first turn player (I) is allowed to choose a partition of $\mathbb{N} = N_1 \cup N_2$. For $k \in \mathbb{N}$, turn k is played out in three steps.

Step 1: Player (I) chooses $\eta_k > 0$, $W_k \in \text{cof}(X)$, and $G_k \in \text{cof}_{w^*}(X^*)$.

Step 2: Player (II) chooses $i_k \in \{1, 2\}$, a finite subset E_k of N_{i_k} and sequences of non-negative real numbers $(\lambda_i^k)_{i \in E_k}$, $(\mu_i^k)_{i \in E_k}$ satisfying

$$1 - \eta < \sum_{i \in E_k} \lambda_i^{(k)} \mu_i^{(k)} < 1 + \eta.$$

Step 3: Player (I) chooses $(\varepsilon_i^{(k)})_{i \in E_k}$ in $\{-1, 1\}^{E_k}$.

We say that player (II) has a winning strategy in the game $\text{Rep}_{(X, (e_i))}(C, \eta)$ if he can force the following properties on the result:

For all $n \in \mathbb{N}$ we set $x_k = \sum_{i \in E_k} \varepsilon_i^{(k)} \lambda_i^{(k)} e_i$ and $x_k^* = \sum_{i \in E_k} \varepsilon_i^{(k)} \mu_i^{(k)} e_i^*$ and demand:

- (i) the sequences $(x_k)_k$ and $(e_k)_k$ are impartially $(C + \eta)$ -equivalent,
- (ii) the sequences $(x_k^*)_k$ and $(e_k^*)_k$ are impartially $(C + \eta)$ -equivalent,
- (iii) for all $n \in \mathbb{N}$ we have $\text{dist}(x_k, W_k) < \eta_k$, and
- (iv) for all $n \in \mathbb{N}$ we have $\text{dist}(x_k^*, G_k) < \eta_k$.

We say that $(e_i)_i$ is **C -strategically reproducible in X** if for every $\eta > 0$ player II has a winning strategy in the game $\text{Rep}_{(X, (e_i))}(C, \eta)$.

Remark 3.5: We first want to observe that if (e_i) is a normalized shrinking and unconditional basis, then being strategically reproducible in the sense of Definition 3.1 is equivalent with being C -strategically reproducible for some $C \geq 1$ in the sense of Definition 3.4.

Indeed, assume (e_i) is 1-unconditional and shrinking and assume that for some $\tilde{C} \geq 1$, (3) of Definition 3.1 holds. We will show that (e_i) is $3\tilde{C}$ -strategically reproducible in the sense of Definition 3.4.

Let $1/3 > \eta > 0$ be given and assume player (I) has at the beginning of the game chosen a partition (N_1, N_2) of \mathbb{N} . At the k -th step player (I) chooses $\eta_k > 0$ and spaces $W_k \in \text{cof}(X)$ and $G_k \in \text{cof}_{w^*}(X)$. Since (e_k) is shrinking, player (II) can “approximate W_k by a tail space” as follows: there is $n_k^{(1)} \in \mathbb{N}$ so that for all $x \in B_X \cap [e_i : i \geq n_k^{(1)}]$ it follows that $\text{dist}(x, W_k) < \eta_k/2\tilde{C}$. Secondly, since G_k is a w^* -closed and cofinite-dimensional subspace of X^* , and thus the annihilator of a finite subset of X , we find $n_k^{(2)} \in \mathbb{N}$ so that for all $x^* \in B_{X^*} \cap [e_i^* : i \geq n_k^{(2)}]$ it follows that $\text{dist}(x^*, W_k) < \eta_k/2\tilde{C}$. Finally we let $n_k^{(3)} = 1 + \max(\bigcup_{j=1}^{k-1} \text{supp}(x_j) \cup \text{supp}(x_j^*))$. Let $n_k = \max(n_k^{(1)}, n_k^{(2)}, n_k^{(3)})$ and let player (II) follow his winning strategy, assuming player (I) has chosen $n_k \in \mathbb{N}$ in his k -th move of the game described in Definition 3.1, and let $b_k \in [e_i : i \geq n_k]$ and $b_k^* \in [e_i^* : i \geq n_k]$ be chosen according to that strategy, which in particular implies that $\|b_k\|, \|b_k^*\| \leq \sqrt{\tilde{C}}$. We write b_k and b_k^* as

$$b_k = \sum_{j=1}^{\infty} \tilde{\lambda}_j^{(k)} e_j \quad \text{and} \quad b_k^* = \sum_{j=1}^{\infty} \tilde{\mu}_j^{(k)} e_j.$$

By reducing the supports, if necessary we can assume, by using Proposition 3.8, that $\tilde{E}_k = \text{supp}(b_k) = \text{supp}(b_k^*)$ and, since $b_k^*(b_k) = \sum_{j=1}^{\infty} \tilde{\mu}_j^{(k)} \tilde{\lambda}_j^{(k)} = 1$, we can choose $i_k \in \{1, 2\}$, so that

$$\rho_k = \sum_{j \in N_{i_k}} \tilde{\mu}_j^{(k)} \tilde{\lambda}_j^{(k)} \geq \frac{1}{2}.$$

Then we let

$$E_k = \tilde{E}_k \cap N_{i_k}, \quad \mu_j^{(k)} = \tilde{\mu}_j^{(k)} / \sqrt{\rho_k} \quad \text{and} \quad \lambda_j^{(k)} = \tilde{\lambda}_j^{(k)} / \sqrt{\rho_k}$$

for $j \in E_k$. After player (I) has chosen $(\varepsilon_j^{(k)})_{j \in E_k}$ we also put

$$x_k = \sum_{j \in E_k} \varepsilon_j^{(k)} \mu_j^{(k)} e_j \in \sqrt{2\tilde{C}} B_X \cap [e_j : j \geq n_k]$$

and

$$x_k^* = \sum_{j \in E_k} \varepsilon_j^{(k)} \lambda_j^{(k)} e_j^* \in \sqrt{2\tilde{C}} B_{X^*} \cap [e_i^* : i \geq n_k].$$

From the choice of n_k , and the fact that $\|x_k\| \leq \sqrt{2\tilde{C}}$ and $\|x_k^*\| \leq \sqrt{2\tilde{C}}$, it follows that $\text{dist}(x_k, W_k) < \eta_k$ and $\text{dist}(x_k^*, G_k) < \eta_k$. From the 1-unconditionality of (e_j) it follows for $(\xi_k) \in c_{00}$ that

$$\left\| \sum_{k=1}^{\infty} \xi_k x_k \right\| \leq \sqrt{2} \left\| \sum_{k=1}^{\infty} \xi_k b_k \right\| \leq \sqrt{2} \sqrt{\tilde{C}} \leq \left\| \sum_{k=1}^{\infty} \xi_k e_k \right\|$$

and

$$\left\| \sum_{k=1}^{\infty} \xi_k x_k^* \right\| \leq \sqrt{2} \left\| \sum_{k=1}^{\infty} \xi_k b_k^* \right\| \leq \sqrt{2} \sqrt{\tilde{C}} \leq \left\| \sum_{k=1}^{\infty} \xi_k e_k^* \right\|.$$

Thus, by Proposition 3.8 below, (x_k) is impartially $2\tilde{C}$ -equivalent to (e_k) and (e_k^*) is impartially $2\tilde{C}$ -equivalent to (e_k^*) .

Conversely, it is easy to deduce that if (e_j) is unconditional and shrinking and strategically reproducible in the sense of Definition 3.4, then it is also strategically reproducible in the sense of Definition 3.1.

In a similar way we can show that for an unconditional and normalized basis (e_j) strategical reproducibility in the sense of Definitions 3.3 and 3.4 are equivalent.

Remark 3.6: The unit vector basis of ℓ_1 has the factorization property yet it is not strategically reproducible under any of the above definitions. It is possible to give a fourth notion of strategic reproducibility that covers ℓ_1 , is strictly less restrictive than Definition 3.4, and implies the factorization property. This formulation is rather technical, so we will not discuss it in the present paper.

We will now show that a basis which is strategically reproducible has the factorization property. We will first need the following two observations.

LEMMA 3.7: *Assume that X is a Banach space with a basis (e_n) , whose basis constant is $\lambda \geq 1$ and biorthogonal functionals (e_n^*) . Let (b_n) and (b_n^*) be block bases of (e_n) and (e_n^*) , respectively, so that $b_m^*(b_n) = \delta_{m,n}$, for $m, n \in \mathbb{N}$, and so that for some $C \geq 1$ it follows that*

$$(4) \quad \left\| \sum_{j=1}^{\infty} \xi_j b_j \right\|_X \leq \sqrt{C} \left\| \sum_{j=1}^{\infty} \xi_j e_j \right\|_X \quad \text{and} \quad \left\| \sum_{j=1}^{\infty} \xi_j b_j^* \right\|_{X^*} \leq \sqrt{C} \left\| \sum_{j=1}^{\infty} \xi_j e_j^* \right\|_{X^*},$$

for all $(\xi_j) \in c_{00}$.

Then $Y = \overline{\text{span}(b_j : j \in \mathbb{N})}$ is a complemented subspace of X and

$$P : X \rightarrow Y, \quad x \mapsto \sum_{n=1}^{\infty} b_n^*(x)b_n$$

is well defined and a bounded projection onto Y with $\|P\| \leq \lambda C$. Moreover, if (e_n) is shrinking, then $\|P\| \leq C$.

Proof. If (e_n) is shrinking then $\text{span}(e_n^* : n \in \mathbb{N})$ is norm dense in X^* and therefore we have in that case

$$\|x\| = \sup_{x^* \in \text{span}(e_n^*, n \in \mathbb{N})} x^*(x).$$

If (e_n) is a general basis whose basis constant is λ , we denote by P_n the projection

$$P_n : X \rightarrow X, \quad \sum_{j=1}^{\infty} x_j e_j \mapsto \sum_{j=1}^n x_j e_j,$$

and since $\|P_n\| = \|P_n^*\| \leq \lambda$ we deduce for $x \in X$

$$\begin{aligned} \|x\| &= \lim_{n \rightarrow \infty} \sup_{x^* \in B_{X^*}} x^*(P_n(x)) \leq \sup_{n \in \mathbb{N}, x^* \in B_{X^*}} P_n^*(x^*)(x) \\ (5) \quad &\leq \sup_{z^* \in \lambda B_{X^*} \cap \text{span}(e_j^* : j \in \mathbb{N})} z^*(x) \\ &= \lambda \sup_{z^* \in B_{X^*} \cap \text{span}(e_j^* : j \in \mathbb{N})} z^*(x). \end{aligned}$$

If $x \in \text{span}(e_j)$ then $P(x)$ is a finite linear combination of elements of (b_n) . We compute

$$\begin{aligned} \sup_{x \in B_X \cap \text{span}(e_j : j \in \mathbb{N})} \|P(x)\| &= \sup_{x \in B_X \cap \text{span}(e_j : j \in \mathbb{N})} \left\| \sum_{j=1}^{\infty} b_j^*(x)b_j \right\| \\ &\leq \sqrt{C} \sup_{x \in B_X \cap \text{span}(e_j : j \in \mathbb{N})} \left\| \sum_{j=1}^{\infty} b_j^*(x)e_j \right\|. \end{aligned}$$

Using (5) yields

$$\begin{aligned} \sup_{x \in B_X \cap \text{span}(e_j : j \in \mathbb{N})} \|P(x)\| &\leq \lambda \sqrt{C} \sup_{\substack{x \in B_X \cap \text{span}(e_j : j \in \mathbb{N}) \\ x^* \in B_{X^*} \cap \text{span}(e_j^* : j \in \mathbb{N})}} x^* \left(\sum_{j=1}^{\infty} b_j^*(x)e_j \right) \\ &= \lambda \sqrt{C} \sup_{x^* \in B_{X^*} \cap \text{span}(e_j^* : j \in \mathbb{N})} \left\| \sum_{j=1}^{\infty} x^*(e_j)b_j^* \right\|. \end{aligned}$$

By (4), we obtain therefore

$$\sup_{x \in B_X \cap \text{span}(e_j: j \in \mathbb{N})} \|P(x)\| \leq \lambda \sqrt{C} \sup_{(\xi_j) \in c_{00}, \|\sum \xi_j e_j^*\| \leq 1} \left\| \sum_{j=1}^{\infty} \xi_j b_j^* \right\| \leq \lambda C.$$

In the case that (e_j) is shrinking we can replace in the first inequality λ by 1, and therefore obtain that $\|P(x)\| \leq C$ for $x \in B_X$. ■

PROPOSITION 3.8: Assume that X is a Banach space with a basis (e_n) and biorthogonal functionals (e_n^*) . Let (b_n) and (b_n^*) be block bases of (e_n) and (e_n^*) , respectively, so that $b_m^*(b_n) = \delta_{m,n}$, for $m, n \in \mathbb{N}$, and so that for some $C \geq 1$ it follows that

$$(6) \quad \left\| \sum_{j=1}^{\infty} \xi_j b_j \right\|_X \leq \sqrt{C} \left\| \sum_{j=1}^{\infty} \xi_j e_j \right\|_X \quad \text{and} \quad \left\| \sum_{j=1}^{\infty} \xi_j b_j^* \right\|_{X^*} \leq \sqrt{C} \left\| \sum_{j=1}^{\infty} \xi_j e_j^* \right\|_{X^*},$$

for all $(\xi_j) \in c_{00}$.

Then (b_n) is λC -impartially equivalent to (e_n) and (b_n^*) is C -impartially equivalent to (e_n) . If $(e_n)_n$ is shrinking then (b_n) is C -impartially equivalent to (e_n) .

Proof. For a sequence $(\xi_j) \in c_{00}$ we compute

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} \xi_j b_j \right\| &\geq \sup_{(\eta_j) \in c_{00}, \|\sum_{j=1}^{\infty} \eta_j b_j^*\| \leq 1} \sum_{j=1}^{\infty} \eta_j b_j^* \left(\sum_{j=1}^{\infty} \xi_j b_j \right) \\ &= \sup_{(\eta_j) \in c_{00}, \|\sum_{j=1}^{\infty} \eta_j b_j^*\| \leq 1} \sum_{j=1}^{\infty} \xi_j \cdot \eta_j. \end{aligned}$$

By (6) and then (5) we obtain

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} \xi_j b_j \right\| &\geq \sup_{(\eta_j) \in c_{00}, \|\sum_{j=1}^{\infty} \eta_j e_j^*\| \leq 1/\sqrt{C}} \sum_{j=1}^{\infty} \xi_j \cdot \eta_j \\ &= \frac{1}{\sqrt{C}} \sup_{(\eta_j) \in c_{00}, \|\sum_{j=1}^{\infty} \eta_j e_j^*\| \leq 1} \sum_{j=1}^{\infty} \xi_j \cdot \eta_j \geq \frac{1}{\lambda \sqrt{C}} \left\| \sum_{j=1}^{\infty} \xi_j e_j \right\|. \end{aligned}$$

Similarly we show that

$$\left\| \sum_{j=1}^{\infty} \xi_j b_j^* \right\| \geq \frac{1}{\sqrt{C}} \left\| \sum_{j=1}^{\infty} \xi_j e_j^* \right\|. \quad \blacksquare$$

In order to deduce that strategical reproducibility implies the factorization property, we will also need a condition on diagonal operators which is automatically satisfied in the case that the given basis is unconditional.

Definition 3.9: Let X be a Banach space with a normalized Schauder basis $(e_n)_n$. We say that the basis $(e_n)_n$ has the **uniform diagonal factorization property** if for every $\delta > 0$ there exists $K(\delta) \geq 1$ so that for every bounded diagonal operator $T : X \rightarrow X$ with $\inf_n |e_n^*(T(e_n))| \geq \delta$ the identity almost $K(\delta)$ -factors through T . If we wish to be more specific we shall say that $(e_n)_n$ has the **$K(\delta)$ -diagonal factorization property**.

Note that if (e_n) is unconditional then it has the uniform diagonal factorization property. The following definition quantifies the uniform factorization property.

Definition 3.10: Let X be a Banach space with a normalized Schauder basis $(e_n)_n$. We say that the basis $(e_n)_n$ has the **uniform factorization property** if for every $\delta > 0$ there exists $K(\delta) \geq 1$ so that for every bounded linear operator $T : X \rightarrow X$ with $\inf_n |e_n^*(T(e_n))| \geq \delta$ the identity almost $K(\delta)$ -factors through T . If we wish to be more specific we shall say that $(e_n)_n$ has the **$K(\delta)$ -factorization property**.

Remark 3.11: Note that in Definitions 3.9 and 3.10, $K(\delta) \geq 1/\delta$. This can be witnessed by taking $T = \delta I$. Also, whenever the function $K : (0, \infty) \rightarrow \mathbb{R}$ is well defined it is also continuous. In fact, for $0 < \varepsilon < \delta$ a simple scaling argument yields

$$(7) \quad K(\delta) \leq K(\delta - \varepsilon) \leq \frac{\delta}{\delta - \varepsilon} K(\delta).$$

To see this, use the following trick: if T has a diagonal whose elements are absolutely bounded below by $\delta - \varepsilon$, then the elements of the diagonal of $\delta/(\delta - \varepsilon)T$ are absolutely bounded below by δ . Inspecting (7) we also deduce $K(\delta) \leq K(1)/\delta$ and therefore

$$\frac{1}{\delta} \leq K(\delta) \leq \frac{K(1)}{\delta},$$

for all $\delta > 0$.

THEOREM 3.12: *Let X be a Banach space with a Schauder basis $(e_n)_n$ that has a basis constant λ . Assume also that*

- (i) *the basis $(e_i)_i$ has the $K(\delta)$ -diagonal factorization property and*
- (ii) *the basis $(e_i)_i$ is C -strategically reproducible in X .*

Then $(e_i)_i$ has the $\lambda C^2 K(\delta)$ -factorization property.

Remark 3.13: It is worth pointing out that in Definition 3.10 the norm of T does not appear and the factorization constant of the identity through T depends only on the diagonal of T . This means that having the uniform factorization property is formally stronger than having the factorization property. Theorem 3.12 yields the stronger property. It is unclear whether these two properties are actually distinct.

We postpone the proof of Theorem 3.12 to first present a necessary lemma.

LEMMA 3.14: *Let X be a Banach space with a normalized Schauder basis $(e_n)_n$ with a basis constant λ , let $T : X \rightarrow X$ be a bounded linear operator, let $(x_n)_n, (x_n^*)_n$ be sequences in X and X^* respectively and let $\eta > 0, C \geq 1$. Assume that the following are satisfied:*

- (i) $(x_n)_n$ and $(e_n)_n$ are impartially C -equivalent,
- (ii) $(x_n^*)_n$ and $(e_n^*)_n$ are impartially C -equivalent,
- (iii) there exists a block sequence $(\tilde{x}_n^*)_n$ of $(e_n^*)_n$ so that $\sum_n \|x_n^* - \tilde{x}_n^*\| < \infty$,
- (iv) $\sum_n \sum_{m \neq n} |x_n^*(T(x_m))| < \eta$.

Then the diagonal operator $D : X \rightarrow X$ given by $D(e_n) = x_n^*(T(x_n))e_n$ is bounded and there exist bounded linear operators $B, A : X \rightarrow X$ with $\|A\|\|B\| \leq \lambda C^2$ so that $\|D - BTA\| < 2\lambda\eta$.

If we additionally assume that $K \geq 1$ is such that the identity K -factors through D and $\eta < 1/(2\lambda K)$, then the identity $(\frac{KC^2}{1-2\lambda K\eta})$ -factors through T .

Proof. The maps $A : X \rightarrow X, S : [x_n : n \in \mathbb{N}] \rightarrow X$ with $A(e_n) = x_n, S(x_n) = e_n$ are well defined and satisfy $\|A\|\|S\| \leq C$.

From Lemma 3.7 it follows that the map $R : X \rightarrow [x_n : n \in \mathbb{N}]$ given by

$$R(x) = \sum_{n=1}^{\infty} x_n^*(x)x_n$$

is well defined and $\|R\| \leq \lambda C$.

Define $B = S \circ R : X \rightarrow X$. Then $\|B \circ A\| \leq \lambda C^2$. It also follows that for each $m \in \mathbb{N}$ we have $BTA(e_m) = \sum_{n=1}^{\infty} x_n^*(T(x_m))e_n$. By (iv) we deduce that for each $m \in \mathbb{N}$ we have

$$\|BTA(e_m) - x_m^*(T(x_m))e_m\| \leq \sum_{n \neq m} |x_n^*(T(x_m))|.$$

Combining this with the triangle inequality we obtain that the desired diagonal map D is bounded and $\|D - BTA\| < 2\lambda\eta$.

For the additional part, assume that $\hat{B} : X \rightarrow X$ and $\hat{A} : X \rightarrow X$ are such that $\|\hat{B}\|\|\hat{A}\| \leq K$ and $I = \hat{B}\hat{D}\hat{A}$. It follows that

$$\|I - \hat{B}BTA\hat{A}\| = \|\hat{B}(D - BTA)\hat{A}\| < 2\lambda K\eta < 1.$$

Hence, the map $Q = \hat{B}BTA\hat{A}$ is invertible with $\|Q^{-1}\| \leq 1/(1 - 2\lambda K\eta)$. In conclusion, if we set

$$\tilde{B} = Q^{-1}\hat{B}B, \quad \tilde{A} = A\hat{A},$$

then $\tilde{B}T\tilde{A} = I$ and $\|\tilde{B}\|\|\tilde{A}\| \leq \lambda KC^2/(1 - 2\lambda K\eta)$. ■

Proof of Theorem 3.12. Let $\delta > 0$, $T : X \rightarrow X$ be a bounded linear operator and $\inf_n |e_n^*(T(e_n))| \geq \delta$. Let us fix $\eta > 0$ to be determined later.

We will now describe a strategy for player (I) in a game $\text{Rep}_{(X, (e_i))}(C, \eta)$, and assume player (II) answers by following his winning strategy.

At the beginning, player (I) chooses

$$N_1 = \{n \in \mathbb{N} : e_n^*(T(e_n)) \geq \delta\} \quad \text{and} \quad N_2 = \{n \in \mathbb{N} : e_n^*(T(e_n)) \leq -\delta\}.$$

In the first step of the n 'th turn he chooses

$$\eta_n < \eta(\|T\|n2^n\sqrt{C + \eta})^{-1}$$

and if $l_n = \max_{1 \leq k < n}(E_k)$ he chooses $G_n = A_n^\perp$ and $W_n = (B_n)_\perp$, where

$$\begin{aligned} A_n &= \{x_1, T(x_1), \dots, x_{n-1}, T(x_{n-1}), e_1, T(e_1), \dots, e_{l_n}, T(e_{l_n})\}, \\ B_n &= \{x_1^*, T^*(x_1), \dots, x_n^*, T^*(x_n^*), e_1^*, T^*(e_1^*), \dots, e_{l_n}^*, T^*(e_{l_n}^*)\}. \end{aligned}$$

Player (II), following a winning strategy, chooses $i_n = 1$ or $i_n = 2$, picks $E_n \subset N_{i_n}$, and non-negative scalars $(\lambda_i^{(n)})_{i \in E_n}$, $(\mu_i^{(n)})_{i \in E_n}$ with

$$1 - \eta < \sum_{i \in E_n} \lambda_i^{(n)} \mu_i^{(n)} < 1 + \eta.$$

Then player (I) picks signs $(\varepsilon_i^{(n)})_{i \in E_n}$ so that if $x_n^* = \sum_{i \in E_n} \mu_i^{(n)} \varepsilon_i^{(n)} e_i^*$ then

$$(8) \quad \left| x_n^* \left(T \left(\sum_{i \in E_n} \varepsilon_i^{(n)} \lambda_i^{(n)} e_i \right) \right) \right| > (1 - \eta)\delta.$$

That this is possible follows using the following probabilistic argument:

Let $r = (r_j)_{j \in E_n}$ be a Rademacher sequence, i.e., $r_j, j \in E_n$, are independent random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$, with $\mathbb{P}(r_j = 1) = \mathbb{P}(r_j = -1) = \frac{1}{2}$.

$$\begin{aligned} \mathbb{E} \left(\left(\sum_{i \in E_n} r_i \mu_i^{(n)} e_i^* \right) \left(T \left(\sum_{j \in E_n} r_j \lambda_j^{(n)} e_j \right) \right) \right) &= \mathbb{E} \left(\sum_{i, j \in E_n} r_i r_j \mu_i^{(n)} \lambda_j^{(n)} e_j^*(T(e_i)) \right) \\ &= \sum_{i \in E_n} \mu_i^{(n)} \lambda_i^{(n)} e_i^*(T(e_i)) \\ &> \delta(1 - \eta). \end{aligned}$$

It follows therefore that we can choose $(\varepsilon_j^{(n)})_{j \in E_n}$ appropriately to satisfy (8).

After the game is completed, put

$$x_n = \sum_{i \in E_n} \varepsilon_i^{(n)} \lambda_i^{(n)} e_i \quad \text{and} \quad x_n^* = \sum_{i \in E_n} \varepsilon_i^{(n)} \mu_i^{(n)} e_i^*.$$

Conditions (i) to (iv) of Definition 3.4 are satisfied. Then (8) can be rewritten as

$$(9) \quad |x_n^*(T(x_n))| \geq (1 - \eta)\delta.$$

Furthermore, observe that for any $k < n$ we have

$$\begin{aligned} |x_k^*(T(x_n))| &= |T^*(x_k^*)(x_n)| \leq \|T^*(x_k^*)\| \cdot \text{dist}(x_n, W_n) \\ &\leq \|T\| \sqrt{C + \eta} \cdot \text{dist}(x_n, W_n), \\ |x_n^*(T(x_k))| &\leq \|T(x_k)\| \cdot \text{dist}(x_n^*, G_n) \\ &\leq \|T\| \sqrt{C + \eta} \cdot \text{dist}(x_n^*, G_n). \end{aligned}$$

We conclude that $\sum_n \sum_{m \neq n} |x_n^*(T(x_m))| < \eta$. A similar argument yields that $(x_n^*)_n$ is summably close to a block sequence of $(e_i^*)_i$. By Lemma 3.14, the diagonal operator $D : X \rightarrow X$ given by $De_n = x_n^*(T(x_n))$ is bounded. By assumption, for any $\xi > 0$, the identity $(K(\delta - \eta) + \xi)$ -factors through D and, if η is sufficiently small, then by the second part of Lemma 3.14 the identity $(\frac{\lambda K(\delta - \eta) + \xi}{1 - 2\lambda(K(\delta - \eta) + \xi)\eta})$ -factors through T . Recall that by (7) the function $K : (0, \infty) \rightarrow \mathbb{R}$ is continuous. As we could have picked η and ξ arbitrarily close to zero we deduce that the identity almost $\lambda K(\delta)C^2$ -factors through T . ■

4. A basic overview: multi-parameter Lebesgue and Hardy spaces

Here we give a preparation for the following sections in which we exhibit examples of strategically reproducible bases.

4.1. THE MULTI-PARAMETER HAAR SYSTEM. We denote by \mathcal{D} the collection of all dyadic intervals in $[0, 1)$, namely $(\mathbb{N}_0 = \mathbb{N} \cup \{0\})$

$$\mathcal{D} = \left\{ \left[\frac{i-1}{2^j}, \frac{i}{2^j} \right) : j \in \mathbb{N}_0, 1 \leq i \leq 2^j \right\}.$$

For each $n \in \mathbb{N}_0$ we define $\mathcal{D}_n = \{I \in \mathcal{D} : |I| = 2^{-n}\}$ and $\mathcal{D}^n = \bigcup_{k=0}^n \mathcal{D}_k$. We define the bijective function $\mathcal{O} : \mathcal{D} \rightarrow \mathbb{N}$ by

$$\left[\frac{i-1}{2^j}, \frac{i}{2^j} \right) \mapsto 2^j + i - 1.$$

The function \mathcal{O} defines a linear order on \mathcal{D} . Recall that Haar system $(h_I)_{I \in \mathcal{D}}$ is defined as follows: if $I = [(i-1)/2^j, i/2^j)$ then set $I^+ = [(i-1)/2^j, (2i-1)/2^{j+1})$, $I^- = [(2i-1)/2^{j+1}, i/2^j)$, and

$$h_I = \chi_{I^+} - \chi_{I^-}.$$

The d -parameter dyadic rectangles \mathcal{R}_d are given by

$$\mathcal{R}_d = \{I_1 \times \dots \times I_d : I_1, \dots, I_d \in \mathcal{D}\},$$

and the d -parameter tensor product Haar system $(h_{\bar{I}})_{\bar{I} \in \mathcal{R}_d}$ is given by

$$h_{\bar{I}}(t_1, t_2, \dots, t_d) = h_{I_1}(t_1) \cdot h_{I_2}(t_2) \cdots h_{I_d}(t_d), \quad t_1, t_2, \dots, t_d \in [0, 1).$$

4.2. A LINEAR ORDER ON \mathcal{R}_2 . First, we define the bijective function $\mathcal{O}_{\mathbb{N}_0^2} : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ by

$$\mathcal{O}_{\mathbb{N}_0^2}(m, n) = \begin{cases} n^2 + m, & \text{if } m < n, \\ m^2 + m + n, & \text{if } m \geq n. \end{cases}$$

To see that $\mathcal{O}_{\mathbb{N}_0^2}$ is bijective consider that for each $k \in \mathbb{N}$:

- $\mathcal{O}_{\mathbb{N}_0^2}(0, 0) = 0$,
- $m \mapsto \mathcal{O}_{\mathbb{N}_0^2}(m, k)$ maps $\{0, \dots, k-1\}$ bijectively onto $\{k^2, \dots, k^2 + k - 1\}$ and preserves the natural order on \mathbb{N}_0 ,
- $\mathcal{O}_{\mathbb{N}_0^2}(k, 0) = \mathcal{O}_{\mathbb{N}_0^2}(k-1, k) + 1$,
- $n \mapsto \mathcal{O}_{\mathbb{N}_0^2}(k, n)$ maps $\{0, \dots, k\}$ bijectively onto $\{k^2 + k, \dots, k^2 + 2k\}$ and preserves the natural order on \mathbb{N}_0 ,
- $\mathcal{O}_{\mathbb{N}_0^2}(0, k+1) = \mathcal{O}_{\mathbb{N}_0^2}(k, k) + 1$.

0		4		24										
		3		23										
		6		8	28	32								
		5		7	27	31								
1	2	3		22										
		6		8	26	30								
9		10	11	12	14	16	18	20	36	40	44	48		
13		15	17	19	35	39	43	47	25	29	33	37	41	45

Figure 1. The first 49 rectangles and their indices $\mathcal{O}_{\triangleleft}$.

Now, let $<_{\ell}$ denote the lexicographic order on \mathbb{R}^3 . For two dyadic rectangles $I_k \times J_k \in \mathcal{R}_2$ with $|I_k| = 2^{-m_k}$, $|J_k| = 2^{-n_k}$, $k = 0, 1$, we define $I_0 \times J_0 \triangleleft I_1 \times J_1$ if and only if

$$(\mathcal{O}_{\mathbb{N}_0^2}(m_0, n_0), \inf I_0, \inf J_0) <_{\ell} (\mathcal{O}_{\mathbb{N}_0^2}(m_1, n_1), \inf I_1, \inf J_1).$$

Associated to the linear ordering \triangleleft is the bijective index function $\mathcal{O}_{\triangleleft} : \mathcal{R}_2 \rightarrow \mathbb{N}_0$ defined by

$$\mathcal{O}_{\triangleleft}(R_0) < \mathcal{O}_{\triangleleft}(R_1) \Leftrightarrow R_0 \triangleleft R_1, \quad R_0, R_1 \in \mathcal{R}_2.$$

See Figure 1 for a picture of $\mathcal{O}_{\triangleleft}$.

4.3. MULTI-PARAMETER LEBESGUE SPACES. Given $1 \leq p \leq \infty$, we define L_0^p as the closed subspace of $L^p[0, 1]$ given by

$$L_0^p = \left\{ f \in L^p : \int_0^1 f(t)dt = 0 \right\}.$$

Note that the Haar system $(h_I)_{I \in \mathcal{D}}$ ordered by \mathcal{O} is a monotone basis for L_0^p , whenever $1 \leq p < \infty$. Moreover, $(h_I)_{I \in \mathcal{D}}$ is unconditional in L_0^p , whenever $1 < p < \infty$. The only reason for considering L_0^p rather than L^p is a notational one. Otherwise we would have to consider the first basis element of L^p , namely $\chi_{[0,1]}$, always separately from the other ones.

Given $\bar{p} = (p_1, \dots, p_d)$, where $1 \leq p_1 \leq \dots \leq p_d \leq \infty$ we define the mixed norm Lebesgue space $L^{\bar{p}}$ by

$$L^{\bar{p}} = L^{p_1}(L^{p_2}(L^{p_3}(\dots(L^{p_d})\dots))).$$

Moreover, we define the closed subspace $L_0^{\bar{p}}$ of $L^{\bar{p}}$ by

$$L_0^{\bar{p}} = \left\{ f \in L^{\bar{p}} : \int_0^1 f(t_1, \dots, t_d) dt_j = 0 \text{ for all } 1 \leq j \leq d \right\}.$$

The d -parameter tensor product Haar system $(h_{\bar{I}})_{\bar{I} \in \mathcal{R}_d}$ is an unconditional basis for $L_0^{\bar{p}}$, whenever $\bar{p} = (p_1, \dots, p_d)$ and $1 < p_1, \dots, p_d < \infty$. The dual of $L_0^{\bar{p}}$ is then $L_0^{\bar{q}}$, where $\bar{q} = (q_1, \dots, q_d)$, and $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for $i = 1, 2, \dots, d$.

In the following, we prove the equivalence of the $L^{\bar{p}}$ norm and the d -parametric square function norm in the reflexive case, i.e., $p = (p_1, \dots, p_d)$ with $1 < p_1, \dots, p_d < \infty$. The content of Proposition 4.1 was known and used by Capon [4]; for the convenience of the reader, we provide a detailed exposition below.

PROPOSITION 4.1: *Let $d \in \mathbb{N}$, $1 < p_1, p_2, \dots, p_d < \infty$ and $\bar{p} = (p_1, p_2, \dots, p_d)$. For $f = \sum_{\bar{I} \in \mathcal{R}_d} a_{\bar{I}} h_{\bar{I}} \in L_0^{\bar{p}}$ we define*

$$(10) \quad \|f\|_{\bar{p}} = \left(\int_0^1 \left(\dots \int_0^1 \left(\int_0^1 \left(\sum_{\bar{I} \in \mathcal{R}_d} a_{\bar{I}}^2 h_{\bar{I}}^2(t_1, \dots, t_d) \right)^{\frac{p_d}{2}} dt_d \right)^{\frac{p_d-1}{p_d}} \dots \right)^{\frac{p_1}{p_2}} dt_1 \right)^{\frac{1}{p_1}}.$$

Then $\|\cdot\|_{\bar{p}}$ is an equivalent norm on $L_0^{\bar{p}}$. The dual norm to $\|\cdot\|_{\bar{p}}$ is equivalent (with constants depending on \bar{p}) to $\|\cdot\|_{\bar{q}}$, where $\bar{q} = (q_1, q_2, \dots, q_d)$, and $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for $i = 1, 2, \dots, d$.

Proof. We show (10) by induction on d . By the unconditionality of the Haar system in L^p , $1 < p < \infty$ (theorem of Paley–Marcinkiewicz [23] and [14]) the statement is true for $d = 1$. Assume that we proved (10) for $L_0^{(p_2, \dots, p_d)}$. In the following, we use the abbreviations $X_j = L_0^{(p_1, \dots, p_j)}$ and $Y_j = L_0^{(p_j, \dots, p_d)}(\ell^2(\mathcal{R}_{d-1}))$, $1 \leq j \leq d$. Since $L_0^{(p_2, \dots, p_d)}$ has the UMD property [16, page II.12] we obtain that

$$\|f\|_{L_0^{\bar{p}}} \sim_{\bar{p}} \left(\int_0^1 \left\| \sum_{\bar{I}} \varepsilon_{I_1} a_{\bar{I}} h_{I_1}(t_1) h_{I_2, \dots, I_d} \right\|_{(p_2, \dots, p_d)}^{p_1} dt_1 \right)^{\frac{1}{p_1}},$$

for all choices of signs ε_{I_1} , $I_1 \in \mathcal{D}$. Hence, by averaging and Kahane’s inequality [12, Theorem 4], we obtain

$$\begin{aligned} \|f\|_{L_0^{\bar{p}}} &\sim_{\bar{p}} \left(\int_0^1 \left(\mathbb{E}_\varepsilon \left\| \sum_{\bar{I}} \varepsilon_{I_1} a_{\bar{I}} h_{I_1}(t_1) h_{I_2, \dots, I_d} \right\|_{(p_2, \dots, p_d)} \right)^{p_1} dt_1 \right)^{\frac{1}{p_1}} \\ &= \left\| \mathbb{E}_\varepsilon \left\| \sum_{\bar{I}} \varepsilon_{I_1} a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right\|_{(p_2, \dots, p_d)} \right\|_{X_1}. \end{aligned}$$

By induction hypothesis, we obtain

$$\|f\|_{L_0^{\bar{p}}} \sim_{\bar{p}} \left\| \mathbb{E}_\varepsilon \left\| \left(\sum_{(I_2, \dots, I_d)} \left(\sum_{I_1} \varepsilon_{I_1} a_{\bar{I}} h_{I_1} \right)^2 h_{I_2, \dots, I_d}^2 \right)^{\frac{1}{2}} \right\|_{(p_2, \dots, p_d)} \right\|_{X_1}.$$

Now, observe that

$$\begin{aligned} &\mathbb{E}_\varepsilon \left\| \left(\sum_{(I_2, \dots, I_d)} \left(\sum_{I_1} \varepsilon_{I_1} a_{\bar{I}} h_{I_1} \right)^2 h_{I_2, \dots, I_d}^2 \right)^{\frac{1}{2}} \right\|_{(p_2, \dots, p_d)} \\ &= \mathbb{E}_\varepsilon \left\| \left\| \sum_{I_1} \varepsilon_{I_1} \left(a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{\ell^2(I_2, \dots, I_d)} \right\|_{(p_2, \dots, p_d)} \\ &= \mathbb{E}_\varepsilon \left\| \left\| \sum_{I_1} \varepsilon_{I_1} \left(a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{Y_2} \right\|. \end{aligned}$$

Kahane’s inequality yields

$$\begin{aligned} &\mathbb{E}_\varepsilon \left\| \left\| \sum_{I_1} \varepsilon_{I_1} \left(a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{Y_2} \right\| \\ &\sim_{\bar{p}} \left(\mathbb{E}_\varepsilon \left\| \left\| \sum_{I_1} \varepsilon_{I_1} \left(a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{Y_2} \right\|^{p_2} \right)^{\frac{1}{p_2}} \\ &= \left(\int \mathbb{E}_\varepsilon \left\| \left\| \sum_{I_1} \varepsilon_{I_1} \left(a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{Y_3} \right\| dt_2 \right)^{\frac{1}{p_2}}. \end{aligned}$$

Combining our estimates yields

$$\begin{aligned} \|f\|_{L_0^{\bar{p}}} &\sim_{\bar{p}} \left\| \left(\int \mathbb{E}_\varepsilon \left\| \left\| \sum_{I_1} \varepsilon_{I_1} \left(a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{Y_3} \right\| dt_2 \right)^{\frac{1}{p_2}} \right\|_{X_1} \\ &= \left\| \mathbb{E}_\varepsilon \left\| \left\| \sum_{I_1} \varepsilon_{I_1} \left(a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{Y_3} \right\|_{X_2}. \end{aligned}$$

With the same argument, we obtain

$$\|f\|_{L_0^{\bar{p}}} \sim_{\bar{p}} \left\| \mathbb{E}_\varepsilon \left\| \sum_{I_1} \varepsilon_{I_1} \left(a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{Y_4} \right\|_{X_3}.$$

Continuing in this fashion yields

$$(11) \quad \|f\|_{L_0^{\bar{p}}} \sim_{\bar{p}} \left\| \mathbb{E}_\varepsilon \left\| \sum_{I_1} \varepsilon_{I_1} \left(a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{\ell^2(\mathcal{R}_{d-1})} \right\|_{L_0^{\bar{p}}}.$$

Applying Kahane’s inequality one last time, we obtain

$$(12) \quad \begin{aligned} & \mathbb{E}_\varepsilon \left\| \sum_{I_1} \varepsilon_{I_1} \left(a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{\ell^2(\mathcal{R}_{d-1})} \\ & \sim_{\bar{p}} \left(\mathbb{E}_\varepsilon \left\| \left(\sum_{I_1} \varepsilon_{I_1} a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right)_{(I_2, \dots, I_d)} \right\|_{\ell^2(\mathcal{R}_{d-1})}^2 \right)^{1/2}. \end{aligned}$$

Note that the last expression is equal to

$$(13) \quad \begin{aligned} \left(\sum_{I_2, \dots, I_d} \mathbb{E}_\varepsilon \left| \sum_{I_1} \varepsilon_{I_1} a_{\bar{I}} h_{I_1} h_{I_2, \dots, I_d} \right|^2 \right)^{1/2} &= \left(\sum_{I_2, \dots, I_d} \sum_{I_1} a_{\bar{I}}^2 h_{I_1}^2 h_{I_2, \dots, I_d}^2 \right)^{1/2} \\ &= \left(\sum_{\bar{I}} a_{\bar{I}}^2 h_{\bar{I}}^2 \right)^{1/2}. \end{aligned}$$

Combining (11) with (12) and (13) yields (10). ■

4.4. HARDY SPACES. We define the dyadic Hardy spaces H^p , $1 \leq p < \infty$ as the completion of

$$\text{span}\{h_I : I \in \mathcal{D}\}$$

under the square function norm

$$\left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_{H^p} = \left\| \mathbb{S} \left(\sum_{I \in \mathcal{D}} a_I h_I \right) \right\|_{L^p},$$

where the square function \mathbb{S} is given by

$$\mathbb{S} \left(\sum_{I \in \mathcal{D}} a_I h_I \right) = \left(\sum_{I \in \mathcal{D}} a_I^2 h_I^2 \right)^{1/2},$$

for all scalar sequences $(a_I)_{I \in \mathcal{D}}$.

The bi-parameter dyadic Hardy spaces $H^p(H^q)$, $1 \leq p, q < \infty$ are defined as the completion of

$$\text{span}\{h_{\bar{I}} : \bar{I} \in \mathcal{R}_2\}$$

under the bi-parameter square function norm

$$\left\| \sum_{\bar{I} \in \mathcal{R}_2} a_{\bar{I}} h_{\bar{I}} \right\|_{H^p(H^q)} = \left\| \mathbb{S} \left(\sum_{\bar{I} \in \mathcal{R}_2} a_{\bar{I}} h_{\bar{I}} \right) \right\|_{L^p(L^q)},$$

where the bi-parameter square function \mathbb{S} is given by

$$\mathbb{S} \left(\sum_{\bar{I} \in \mathcal{R}_2} a_{\bar{I}} h_{\bar{I}} \right) = \left(\sum_{\bar{I} \in \mathcal{R}_2} a_{\bar{I}}^2 h_{\bar{I}}^2 \right)^{1/2},$$

for all scalar sequences $(a_{\bar{I}})_{\bar{I} \in \mathcal{R}_2}$.

The following Lemma is taken from [13].

LEMMA 4.2: For $m \in \mathbb{N}$, let \mathcal{X}_m and \mathcal{Y}_m be non-empty, finite families of pairwise disjoint dyadic intervals, define $f_m = \sum_{I \in \mathcal{X}_m, J \in \mathcal{Y}_m} h_{I \times J}$, and let $1 \leq p, q < \infty$. Suppose in addition that:

- $\mathcal{X}_m \cap \mathcal{X}_n = \emptyset$ or $\mathcal{Y}_m \cap \mathcal{Y}_n = \emptyset$ whenever $m, n \in \mathbb{N}$ are distinct;
- $\bigcup \mathcal{X}_m = \bigcup \mathcal{X}_n$ and $\bigcup \mathcal{Y}_m = \bigcup \mathcal{Y}_n$ for all $m, n \in \mathbb{N}$.

Then for each $\gamma \in \ell^\infty(\mathcal{R})$ with $\|\gamma\|_\infty \leq 1$, the operator M_γ defined as the linear extension of the map $h_{I \times J} \mapsto \gamma_{I \times J} h_{I \times J}$ is bounded by 1, both as a map from $H^p(H^q)$ to itself and from $H^p(H^q)^*$ to itself. Moreover,

- (i) for each $g \in H^p(H^q)^*$, $\sup_{\gamma \in B_{\ell^\infty(\mathcal{R})}} |\langle M_\gamma f_m, g \rangle| \rightarrow 0$ as $m \rightarrow \infty$;
- (ii) for each $g \in H^p(H^q)$, $\sup_{\gamma \in B_{\ell^\infty(\mathcal{R})}} |\langle M_\gamma g, f_m \rangle| \rightarrow 0$ as $m \rightarrow \infty$.

4.5. COLLECTIONS OF DYADIC INTERVALS. We introduce convenient notation and gather basic facts of collections of dyadic intervals.

Notation 4.3:

- (i) For $\mathcal{A} \subset \mathcal{D}$ set

$$\mathcal{G}_0(\mathcal{A}) = \{I \in \mathcal{A} : I \text{ is maximal with respect to inclusion}\}.$$

- (ii) For $\mathcal{A} \subset \mathcal{D}$, recursively define for $n \in \mathbb{N}$ the collection

$$\mathcal{G}_n(\mathcal{A}) = \mathcal{G}_0(\mathcal{A} \setminus (\bigcup_{k=0}^{n-1} \mathcal{G}_k(\mathcal{A}))).$$

For every $n \in \mathbb{N}$ and $I \in \mathcal{G}_{n+1}(\mathcal{A})$ there is a unique $J \in \mathcal{G}_n(\mathcal{A})$ with $I \subset J$. In particular, $\mathcal{G}_{n+1}(\mathcal{A})^* \subset \mathcal{G}_n(\mathcal{A})^*$.

- (iii) For $\mathcal{A} \subset \mathcal{D}$ set $\limsup \mathcal{A} = \bigcap_n \mathcal{G}_n(\mathcal{A})^*$.

- (iv) For a finite $\mathcal{H} \subset \mathcal{D}$, consisting of pairwise disjoint intervals, and $\bar{\varepsilon} = (\varepsilon)_{I \in \mathcal{H}} \in \{-1, 1\}^{\mathcal{H}}$, we define

$$\mathcal{H}_{\bar{\varepsilon}}^* = \left[\sum_{I \in \mathcal{H}} \varepsilon_I h_I = 1 \right] \quad \text{and} \quad \mathcal{H}_{-\bar{\varepsilon}}^* = \left[\sum_{I \in \mathcal{H}} \varepsilon_I h_I = -1 \right].$$

These two sets have measure $|\mathcal{H}^*|/2$ and they form a partition of \mathcal{H}^* .

- (v) For $\mathcal{A} \subset \mathcal{D}$, $n, k \in \mathbb{N}$, with $k \geq n$ and a finite $\mathcal{H} \subset \mathcal{G}_n(\mathcal{A})$, define the collection

$$\mathcal{H}_k^{\text{succ}} = \{I \in \mathcal{G}_k(\mathcal{A}) : I \subset J \text{ for some } J \in \mathcal{H}\}.$$

For any $\bar{\varepsilon} \in \{-1, 1\}^{\mathcal{H}}$ define the sets

$$\mathcal{H}_{\bar{\varepsilon}, k}^{\text{succ}} = \{I \in \mathcal{G}_k(\mathcal{A}) : I \subset \mathcal{H}_{\bar{\varepsilon}}^*\} \quad \text{and} \quad \mathcal{H}_{-\bar{\varepsilon}, k}^{\text{succ}} = \{I \in \mathcal{G}_k(\mathcal{A}) : I \subset \mathcal{H}_{-\bar{\varepsilon}}^*\}.$$

The sets $\mathcal{H}_{\bar{\varepsilon}, k}^{\text{succ}}$, $\mathcal{H}_{-\bar{\varepsilon}, k}^{\text{succ}}$ form a partition of \mathcal{H}_k . We point out that the definitions of $\mathcal{H}_k^{\text{succ}}$, $\mathcal{H}_{\bar{\varepsilon}, k}^{\text{succ}}$, and $\mathcal{H}_{-\bar{\varepsilon}, k}^{\text{succ}}$ depend on the ambient collection \mathcal{A} .

Lemmas 4.4 and 4.5 will be used in Section 6.

LEMMA 4.4: *Let $\mathcal{A} \subset \mathcal{D}$. Then for any $\kappa > 0$ there is $\tilde{\mathcal{A}} \subset \mathcal{A}$ so that for each $k \in \mathbb{N}$ we have*

- (i) $\mathcal{G}_k(\tilde{\mathcal{A}})$ is finite and $\mathcal{G}_k(\tilde{\mathcal{A}}) \subset \mathcal{G}_k(\mathcal{A})$ and
- (ii) $|\limsup(\tilde{\mathcal{A}})| \geq |\limsup(\mathcal{A})| - \kappa$.

Proof. Pick a finite $\mathcal{B}_0 \subset \mathcal{G}_0(\mathcal{A})$ with $|\mathcal{G}_0(\mathcal{A})^* \setminus \mathcal{B}_0^*| < \kappa/2$. Recursively for $k \in \mathbb{N}$, if $\mathcal{C} = \mathcal{B}_{k-1}$, pick $\mathcal{B}_k \subset \mathcal{C}_k^{\text{succ}}$ with $|(\mathcal{C}_k^{\text{succ}})^* \setminus \mathcal{B}_k^*| < \kappa/2^{k+1}$. Set $\tilde{\mathcal{A}} = \bigcup_{k=0}^{\infty} \mathcal{B}_k$. One can check by induction that for all $k \in \mathbb{N}$ we have $\mathcal{G}_k(\tilde{\mathcal{A}}) = \mathcal{B}_k \subset \mathcal{G}_k(\mathcal{A})$ and that $|\mathcal{G}_k(\mathcal{A})^* \setminus \mathcal{B}_k^*| \leq \sum_{i=1}^k \kappa/2^i$. The conclusion easily follows. ■

LEMMA 4.5: *Let $\mathcal{A} \subset \mathcal{D}$, $n \in \mathbb{N}$, $\kappa \in (0, 1/2)$, and $\mathcal{H} \subset \mathcal{G}_n(\mathcal{A})$ be a non-empty finite collection so that if $A = \limsup \mathcal{A}$ then $|\mathcal{H}^* \cap A| > (1 - \kappa)|\mathcal{H}^*|$. The following hold:*

- (i) For any $\bar{\varepsilon} \in \{-1, 1\}^{\mathcal{H}}$, if $C = \mathcal{H}_{\bar{\varepsilon}}^*$ or $C = \mathcal{H}_{-\bar{\varepsilon}}^*$ we have

$$\frac{|\mathcal{H}^*|}{2} \geq |C \cap A| > (1 - 2\kappa)|C| = (1 - 2\kappa) \frac{|\mathcal{H}^*|}{2}.$$

- (ii) For any $\delta > 0$ there exists $k_0 \in \mathbb{N}$ so that for all $k \geq k_0$ and $\bar{\varepsilon} \in \{-1, 1\}^{\mathcal{H}}$, if $C = (\mathcal{H}_{\bar{\varepsilon}, k}^{\text{succ}})^*$ or $C = (\mathcal{H}_{-\bar{\varepsilon}, k}^{\text{succ}})^*$ we have $|C \cap A| \geq (1 - \delta)|C|$.

Proof. For the proof of (i) let $C = \mathcal{H}_{\bar{\varepsilon}}^*$. Recall that $|\mathcal{H}_{\bar{\varepsilon}}^*| = |\mathcal{H}_{-\bar{\varepsilon}}^*| = |\mathcal{H}^*|/2$. Now,

$$\begin{aligned} (1 - \kappa)2|\mathcal{H}_{\bar{\varepsilon}}^*| &= (1 - \kappa)|\mathcal{H}^*| < |\mathcal{H}^* \cap A| \\ &= |\mathcal{H}_{\bar{\varepsilon}}^* \cap A| + |\mathcal{H}_{-\bar{\varepsilon}}^* \cap A| \\ &\leq |\mathcal{H}_{\bar{\varepsilon}}^* \cap A| + |\mathcal{H}_{-\bar{\varepsilon}}^*| = |\mathcal{H}_{\bar{\varepsilon}}^* \cap A| + |\mathcal{H}_{\bar{\varepsilon}}^*| \end{aligned}$$

which yields $|\mathcal{H}_{\bar{\varepsilon}}^* \cap A| > (1 - 2\kappa)|\mathcal{H}_{\bar{\varepsilon}}^*|$. The same argument works for $C = \mathcal{H}_{-\bar{\varepsilon}}$.

We now prove (ii). As there are finitely many choices of $\bar{\varepsilon} \in \{-1, 1\}^{\mathcal{H}}$ it suffices to prove it by fixing one of them. Observe the sequence $((\mathcal{H}_{\bar{\varepsilon},k}^{\text{succ}})^*)_k$ is decreasing so we can define

$$(\mathcal{H}_{\bar{\varepsilon},\infty}^{\text{succ}})^* = \bigcap_k (\mathcal{H}_{\bar{\varepsilon},k}^{\text{succ}})^* = \mathcal{H}_{\bar{\varepsilon}}^* \cap A = \bigcap_k ((\mathcal{H}_{\bar{\varepsilon},k}^{\text{succ}})^* \cap A).$$

We obtain $\lim_k |(\mathcal{H}_{\bar{\varepsilon},k}^{\text{succ}})^*| = |\mathcal{H}_{\bar{\varepsilon}}^* \cap A| \geq (1 - 2\kappa)|\mathcal{H}_{\bar{\varepsilon}}^* \cap A| > 0$. Since also $\lim_k |(\mathcal{H}_{\bar{\varepsilon},k}^{\text{succ}})^* \cap A| = |\mathcal{H}_{\bar{\varepsilon}}^* \cap A|$ we obtain $\lim_k (|\mathcal{H}_{\bar{\varepsilon}}^*|/|\mathcal{H}_{\bar{\varepsilon}}^* \cap A|) = 1$ which yields the desired conclusion. ■

5. Strategical reproducibility of the Haar system

We establish that the Haar system is strategically reproducible in the following classical Banach spaces:

- (i) The multi-parameter tensor product Haar system is strategically reproducible in the reflexive mixed norm Lebesgue spaces $L^{(p_1, \dots, p_d)}$, $1 < p_i < \infty$, $1 \leq i \leq d$, $d \in \mathbb{N}$, in the sense of Definition 3.1.
- (ii) The one-parameter Haar system in H^1 is strategically reproducible according to Definition 3.3.
- (iii) The two-parameter tensor product Haar system is strategically reproducible in the two-parameter mixed norm Hardy spaces $H^p(H^q)$, $1 \leq p, q < \infty$ in the sense of Definition 3.3.

5.1. THE HAAR SYSTEM IN MULTI-PARAMETER LEBESGUE SPACES. Here we show that $(h_{\bar{1}})$ is strategically reproducible in $L^{(p_1, \dots, p_d)}$, $1 < p_i < \infty$, $1 \leq i \leq d$, $d \in \mathbb{N}$, in the sense of Definition 3.1. We exploit the fact that $(h_{\bar{1}})$ is an unconditional basis for $L^{(p_1, \dots, p_d)}$, and that $(L^{(p_1, \dots, p_d)})^* = L^{(q_1, \dots, q_d)}$, where $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $1 \leq i \leq d$.

THEOREM 5.1: *Let $d \in \mathbb{N}$, $1 < p_1, p_2, \dots, p_d < \infty$ and put $\bar{p} = (p_1, p_2, \dots, p_d)$. Then $(h_{\bar{I}})$ with an appropriate linear order is strategically reproducible in $L_0^{\bar{p}}$.*

Proof. We linearly order \mathcal{R}_d into $(\bar{I}^{(k)}) = (I_1^{(k)}, I_2^{(k)}, \dots, I_d^{(k)})_{k=1}^\infty$ in a manner which is compatible with “ \subset ”, i.e., we assume that if for $m, k \in \mathbb{N}$, we have $I_i^{(m)} \subset I_i^{(k)}$, for $i = 1, 2, \dots, d$, then $m \geq k$. We also linearly order the Haar basis of $L_0^{\bar{p}}$ into $(h_{\bar{I}^{(k)}})_{k=1}^\infty$. For any $\bar{I} \in \mathcal{R}_d$, $n(\bar{I})$ denotes the number $n \in \mathbb{N}$ so that $\bar{I} = \bar{I}^{(n)}$.

Then a winning strategy for player (II) will look as follows:

Assume he has chosen b_1, b_2, \dots, b_l , and $b_1^*, b_2^*, \dots, b_l^*$ in $L_0^{\bar{p}}$ and $(L_0^{\bar{p}})^*$. Assume that $b_j, 1 \leq j \leq l$ is of the following form:

$$b_j = \sum_{\substack{\bar{J}=(J_1, \dots, J_d) \in \mathcal{D}_{k(j,1)} \times \mathcal{D}_{k(j,2)} \times \dots \times \mathcal{D}_{k(j,d)} \\ J_1 \times J_2 \times \dots \times J_d \subset I_1^{(j)} \times I_2^{(j)} \times \dots \times I_d^{(j)}}} h_{\bar{J}}$$

with $k(j, i) \leq k(j', i)$ if $j \leq j'$, and $k(j, i) < k(j', i)$ if $I_i^{(j)} \supsetneq I_i^{(j')}$, for $i = 1, 2, \dots, d$. We also assume that for all $j = 1, 2, \dots, l$ we have $n(\bar{J}) > n_j$ for all $\bar{J} \in \mathcal{D}_{k(j,2)} \times \dots \times \mathcal{D}_{k(j,d)}$, where n_j was the j -th move of player (I) and, moreover, we assume that

$$b_j^* = \left(\prod_{s=1}^d |I_s^{(j)}| \right)^{-1} b_j.$$

Thus $(b_j^*)_{j=1}^l$ is biorthogonal to $(b_j)_{j=1}^l$ and $|b_j| = |h_{\bar{I}^{(j)}}|$, for $j = 1, 2, \dots, l$ which means that with respect to $\|\cdot\|_{\bar{p}}$ and, using (10) in Proposition 4.1, $(b_j)_{j=1}^l$ and $(b_j^*)_{j=1}^l$ are isometrically equivalent to $(h_{\bar{I}_j})_{j=1}^l$ and $(h_{\bar{I}_j}^*)_{j=1}^l$, respectively.

Assuming now the $(l + 1)$ st move of player (I) is n_{l+1} , player (II) can proceed as follows. For $j = 1, 2, \dots, d$, he chooses $k(l + 1, j) \geq \max_{m \leq l} k(m, j)$ so that $k(l + 1, j) > k(m, j)$ if $I_j^{(l+1)} \subsetneq I_j^{(m)}$ and so that for all $\bar{J} \in \prod_{j=1}^d \mathcal{D}_{k(l+1,j)}$, we have $n(\bar{J}) \geq n_{l+1}$. ■

5.2. THE HAAR SYSTEM IN H^1 . Here, we use the Gamlen–Gaudet construction [10] (see also [18, 21]) to show that the one-parameter Haar system in H^1 is strategically reproducible according to Definition 3.3.

For convenience, we introduce the following notation: We define $e_I = h_I/|I|$, $I \in \mathcal{D}$, thus $(e_I)_{I \in \mathcal{D}}$ forms a 1-unconditional normalized basis for H^1 . Note that $e_I^* = h_I \in (H^1)^*$, $I \in \mathcal{D}$. Finally, we will identify a dyadic interval $I \in \mathcal{D}$

with $\mathcal{O}(I)$, e.g.,

$$e_k \leftrightarrow e_I, \quad x_k \leftrightarrow x_I, \quad x_k^* \leftrightarrow x_I^*, \quad I \in \mathcal{D}, \mathcal{O}(I) = k.$$

Recall that the linear order \mathcal{O} was introduced in Section 4.1.

THEOREM 5.2: *The normalized Haar basis is strategically reproducible in H^1 .*

Proof. Here, we show that $(e_I)_{I \in \mathcal{D}}$ is $\sqrt{2}$ -strategically reproducible in H^1 .

We start the game with turn 1. In step 1, player (I) chooses $\eta_{[0,1]} > 0$, $W_{[0,1]} \in \text{cof}(H^1)$, and $G_{[0,1]} \in \text{cof}_{w^*}((H^1)^*)$. In step 2, player (II) selects one of the sets $\mathcal{E}_{[0,1]}^{(j)} = \mathcal{D}_j$, $j \in \mathbb{N}$. Put

$$d_{[0,1]}^{(j)} = \sum_{K \in \mathcal{E}_{[0,1]}^{(j)}} |K| e_K,$$

$$d_{[0,1]}^{*(j)} = \sum_{K \in \mathcal{E}_{[0,1]}^{(j)}} e_K^*,$$

and note that $(d_{[0,1]}^{(j)})_{j=1}^\infty$ converges to 0 in the weak topology of H^1 and the sequence $(d_{[0,1]}^{*(j)})_{j=1}^\infty$ converges in the w^* topology in $(H^1)^*$. Hence, there exists an index j_0 such that

$$\text{dist}_{H^1}(d_{[0,1]}^{(j_0)}, W_{[0,1]}) < \eta_{[0,1]} \quad \text{and} \quad \text{dist}_{(H^1)^*}(d_{[0,1]}^{*(j_0)}, G_{[0,1]}) < \eta_{[0,1]}.$$

Player (II) concludes step 2 by choosing

$$\mathcal{E}_{[0,1]} = \mathcal{E}_{[0,1]}^{(j_0)}$$

and

$$\lambda_K^{[0,1]} = |K| \quad \text{and} \quad \mu_K^{[0,1]} = 1, \quad K \in \mathcal{E}_{[0,1]}.$$

In step 3, player (I) chooses $(\varepsilon_K^{([0,1])})_{K \in \mathcal{E}_{[0,1]}} \in \{-1, 1\}^{\mathcal{E}_{[0,1]}}$.

Assume that the game has already been played for $k = \mathcal{O}(I) - 1$ turns. We will now play out turn $k + 1 = \mathcal{O}(I)$. In step 1, player (I) chooses $\eta_I > 0$, $W_I \in \text{cof}(H^1)$, and $G_I \in \text{cof}_{w^*}((H^1)^*)$. In step 2, it is player (II)'s choice to select the finite sets $\mathcal{E}_I \subset \mathcal{D}$. We will now describe this procedure. Note that since $W_I \in \text{cof}(H^1)$ and $G_I \in \text{cof}_{w^*}((H^1)^*)$, there exist $f_j \in H^1$, $g_j \in (H^1)^*$, $1 \leq j \leq N_I$, such that $W_I = \{g_1, \dots, g_{N_I}\}_\perp$ and $G_I = \{f_1, \dots, f_{N_I}\}_\perp$. Let \tilde{I} denote the dyadic predecessor of I , i.e., \tilde{I} is the unique dyadic interval that

satisfies $\tilde{I} \supsetneq I$ and $|\tilde{I}| = 2|I|$. Note that $\mathcal{O}(\tilde{I}) \leq k$; specifically, $\mathcal{E}_{\tilde{I}}$ has already been defined. Put

$$X_I = \begin{cases} \{K^\ell : K \in \mathcal{E}_{\tilde{I}}\}, & \text{if } I \text{ is the left successor of } \tilde{I}, \\ \{K^r : K \in \mathcal{E}_{\tilde{I}}\}, & \text{if } I \text{ is the right successor of } \tilde{I}, \end{cases}$$

where K^ℓ denotes the left successor of K and K^r denotes the right successor of K . We note that by induction $|X_I| = |I|$;

$$d_I^{(j)} = \sum_{\substack{K \in \mathcal{D}_j \\ K \subset X_I}} |K|e_K, \quad j \in \mathbb{N},$$

$$d_I^{*(j)} = \sum_{\substack{K \in \mathcal{D}_j \\ K \subset X_I}} e_K^*, \quad j \in \mathbb{N}.$$

Since the sequence $(d_I^{(j)})_{j=1}^\infty$ converges to 0 in the weak topology of H^1 and $(d_I^{*(j)})_{j=1}^\infty$ converges to 0 in the w^* topology of $(H^1)^*$, there exists an index j_0 such that

$$\text{supp}(d_I^{(j_0)}) = X_I, \quad 2^{-j_0} < \min \left\{ |K| : K \in \bigcup_{i=1}^k \mathcal{E}_i \right\},$$

as well as

$$\text{dist}_{H^1}(d_I^{(j_0)}, W_I) < \eta_I, \quad \text{dist}_{(H^1)^*}(d_I^{*(j_0)}, G_I) < \eta_I.$$

Player (II) concludes step 2 by choosing the collection

$$\mathcal{E}_I = \{K \in \mathcal{D}_{j_0} : K \subset X_I\},$$

the numbers

$$\lambda_K^{(I)} = |K|/|I|, \quad K \in \mathcal{E}_I, \quad \text{and} \quad \mu_K^{(I)} = 1, \quad K \in \mathcal{E}_I,$$

and defining

$$x_I = d_I^{(j_0)} \quad \text{and} \quad x_I^* = d_I^{*(j_0)}.$$

Clearly, since $|X_I| = |\mathcal{E}_I^*| = |I|$, we have

$$\sum_{K \in \mathcal{E}_I} \lambda_K^{(I)} \mu_K^{(I)} = 1.$$

We conclude turn $k + 1 = \mathcal{O}(I)$ with player (I) choosing $(\varepsilon_K^{(I)})_{K \in \mathcal{E}_I}$ in $\{-1, 1\}^{\mathcal{E}_I}$ in step 3.

We claim that this defines a winning strategy for player (II). In fact, $(x_I)_{I \in \mathcal{D}}$ and $(x_I^*)_{I \in \mathcal{D}}$ were constructed using the Gamlen–Gaudet construction, for which it was verified in [18, Theorem 0] that $(x_I)_I$ is equivalent to $(e_I)_I \in \mathcal{D}$ in H^1 and $(x_I^*)_I$ is equivalent to $(e_I^*)_I \in \mathcal{D}$ in $(H^1)^*$. Moreover, we note that in the text above we already verified $\text{dist}_{H^1}(x_I, W_I) < \eta_I$ and $\text{dist}_{(H^1)^*}(x_I^*, G_I) < \eta_I$, $I \in \mathcal{D}$. ■

5.3. THE HAAR SYSTEM IN $H^p(H^q)$, $1 \leq p, q < \infty$. In [13] it was shown that the two-parameter tensor product Haar system in $H^p(H^q)$ has the factorization property. We use the techniques introduced in [13] to show that, moreover, the two-parameter tensor product Haar system is strategically reproducible $H^p(H^q)$. Hence, by Theorem 3.12 we recover the main result in [13]. Finally, we remark that by exploiting Theorem 7.6 (see Section 7 below) we obtain a simpler construction than the one used in [13].

In the proof below, we will use following notation: The $H^p(H^q)$ -normalized bi-parameter Haar system $(e_I \otimes f_J)_{I, J}$ is given by $e_I \otimes f_J = h_I \otimes h_J / (|I|^{1/p} |J|^{1/q})$, and its bi-orthogonal functionals $((e_I \otimes f_J)^*)_{I, J}$ are given by

$$(e_I \otimes f_J)^* = e_I^* \otimes f_J^* = h_I \otimes h_J / (|I|^{1/p'} |J|^{1/q'}),$$

where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, with the usual convention that $1/\infty = 0$.

THEOREM 5.3: *The normalized bi-parameter Haar system is strategically reproducible in the mixed norm Hardy spaces $H^p(H^q)$, $1 \leq p, q \leq \infty$.*

Proof. Define the subspaces V_1, V_2 of $H^p(H^q)$ by

$$V_1 = \overline{\text{span}\{h_{I \times J} : |I| < |J|\}}^{H^p(H^q)} \quad \text{and} \quad V_2 = \overline{\text{span}\{h_{I \times J} : |I| \geq |J|\}}^{H^p(H^q)},$$

and note that $V_1 \oplus V_2 = H^p(H^q)$. We will now show that the bi-parameter Haar system is strategically reproducible in V_1 and in V_2 , separately.

CASE V_1 . Here, we will show that the bi-parameter Haar system is strategically reproducible in V_1 . Player (I) opens the game by selecting $\eta_{[0,1/2) \times [0,1)} > 0$, $W_{[0,1/2) \times [0,1)} \in \text{cof}(H^p(H^q))$ and $G_{[0,1/2) \times [0,1)} \in \text{cof}_{w^*}((H^p(H^q))^*)$. In step 2, player (II) will select one of the following collections of dyadic rectangles $\mathcal{E}_{[0,1/2) \times [0,1)}^{(j)}$, $j \in \mathbb{N}$, given by

$$\mathcal{E}_{[0,1/2) \times [0,1)}^{(j)} = \{K \times [0, 1) : K \in \mathcal{D}_j, K \subset [0, 1/2)\}, \quad j \in \mathbb{N}.$$

To this end, define

$$d_{[0,1/2) \times [0,1)}^{(j)} = \sum_{K \times L \in \mathcal{E}_{[0,1/2) \times [0,1)}^{(j)}} \frac{|K|^{1/p} |L|^{1/q}}{|[0, 1/2)|^{1/p} |[0, 1)|^{1/q}} e_K \otimes f_L, \quad j \in \mathbb{N},$$

$$d_{[0,1/2) \times [0,1)}^{*(j)} = \sum_{K \times L \in \mathcal{E}_{[0,1/2) \times [0,1)}^{(j)}} \frac{|K|^{1/p'} |L|^{1/q'}}{|[0, 1/2)|^{1/p'} |[0, 1)|^{1/q'}} e_K^* \otimes f_L^*, \quad j \in \mathbb{N},$$

and note that $K \times L \in \mathcal{E}_{[0,1/2) \times [0,1)}^{(j)}$ implies $L = [0, 1)$. By Lemma 4.2 the sequence $(d_{[0,1/2) \times [0,1)}^{(j)})_{j=1}^\infty$ is a null sequence in the weak topology of $H^p(H^q)$ and $(d_{[0,1/2) \times [0,1)}^{*(j)})_{j=1}^\infty$ is a null sequence in the w^* topology of $H^p(H^q)^*$; we can find j_0 such that

$$\text{dist}_{H^p(H^q)}(d_{[0,1/2) \times [0,1)}^{(j_0)}, W_{[0,1/2) \times [0,1)}) < \eta_{[0,1/2) \times [0,1)},$$

$$\text{dist}_{(H^p(H^q))^*}(d_{[0,1/2) \times [0,1)}^{*(j_0)}, G_{[0,1/2) \times [0,1)}) < \eta_{[0,1/2) \times [0,1)}.$$

Player (II) completes step 2 by selecting

$$\mathcal{E}_{[0,1/2) \times [0,1)} = \mathcal{E}_{[0,1/2) \times [0,1)}^{(j_0)}$$

and the numbers

$$\lambda_{K \times L}^{[0,1/2) \times [0,1)} = \frac{|K|^{1/p} |L|^{1/q}}{|[0, 1/2)|^{1/p} |[0, 1)|^{1/q}}, \quad K \times L \in \mathcal{E}_{[0,1/2) \times [0,1)},$$

$$\mu_{K \times L}^{[0,1/2) \times [0,1)} = \frac{|K|^{1/p'} |L|^{1/q'}}{|[0, 1/2)|^{1/p'} |[0, 1)|^{1/q'}}, \quad K \times L \in \mathcal{E}_{[0,1/2) \times [0,1)}.$$

Finally, in step 3, player (I) chooses

$$(\varepsilon_{K \times L}^{[0,1/2) \times [0,1)})_{K \times L \in \mathcal{E}_{[0,1/2) \times [0,1)}} \in \{-1, +1\}^{\mathcal{E}_{[0,1/2) \times [0,1)}},$$

completing turn 1.

Assume that the turns $1, \dots, k$ of the game have been played out. We will now describe turn $k + 1$. Select $I, J \in \mathcal{D}$ such that $|I| < |J|$ and $I \times J$ is the $(k + 1)^{\text{st}}$ rectangle of the set $\{K \times L : |K| < |L|\}$ in the order \triangleleft . Player (I) starts off the turn by choosing $\eta_{I \times J} > 0$, $W_{I \times J} \in \text{cof}(H^p(H^q))$ and $G_{I \times J} \in \text{cof}_{w^*}((H^p(H^q))^*)$. In step 2, player (II) will select one of the collections $\mathcal{E}_{I \times J}^{(j)}$, $j \in \mathbb{N}$ of dyadic rectangles, which we will now describe in detail. We will distinguish between the following two cases: $J = [0, 1)$ and $J \neq [0, 1)$.

If $J = [0, 1)$, we note that $I \neq [0, 1)$, hence, $\mathcal{E}_{\tilde{I} \times [0,1)}$ has already been defined. Also note that $[0, |I|) \times [0, 1) \subseteq I \times [0, 1)$, and choose the integer $\kappa(I \times [0, 1))$ such that

$$2^{-\kappa(I \times [0,1))} < \min \left\{ |K| : K \times L \in \bigcup_{i=1}^k \mathcal{E}_i \right\}.$$

For all $j \in \mathbb{N}$, we define

$$\mathcal{E}_{I \times [0,1)}^{(j)} = \begin{cases} \{K^+ \times [0, 1) : K \in \mathcal{D}_j, K \subset [0, 1) \subset \mathcal{E}_{\tilde{I} \times [0,1)}\}, & \text{if } I = (\tilde{I})^\ell, \\ \{K^- \times [0, 1) : K \in \mathcal{D}_j, K \subset [0, 1) \subset \mathcal{E}_{\tilde{I} \times [0,1)}\}, & \text{if } I = (\tilde{I})^r, \end{cases}$$

where J^ℓ denotes the left successor and J^r the right successor of a dyadic interval $J \in \mathcal{D}$. Define the functions

$$d_{I \times [0,1)}^{(j)} = \sum_{K \times L \in \mathcal{E}_{I \times [0,1)}^{(j)}} \frac{|K|^{1/p} |L|^{1/q}}{|I|^{1/p} |[0, 1)|^{1/q}} e_K \otimes f_L, \quad j \in \mathbb{N},$$

$$d_{I \times [0,1)}^{*(j)} = \sum_{K \times L \in \mathcal{E}_{I \times [0,1)}^{(j)}} \frac{|K|^{1/p'} |L|^{1/q'}}{|I|^{1/p'} |[0, 1)|^{1/q'}} e_K^* \otimes f_L^*, \quad j \in \mathbb{N},$$

and note that by Lemma 4.2, $(d_{I \times [0,1)}^{(j)})_{j=1}^\infty$ converges to 0 in the weak topology of $H^p(H^q)$ and $(d_{I \times [0,1)}^{*(j)})_{j=1}^\infty$ converges to 0 in the w^* topology of $(H^p(H^q))^*$. Hence, there exists an index $j_0 > \kappa(I \times [0, 1))$ such that

$$\text{dist}_{H^p(H^q)}(d_{I \times [0,1)}^{(j_0)}, W_{I \times [0,1)}) < \eta_{I \times [0,1)},$$

$$\text{dist}_{(H^p(H^q))^*}(d_{I \times [0,1)}^{*(j_0)}, G_{I \times [0,1)}) < \eta_{I \times [0,1)}.$$

Player (II) then completes step 2 by putting

$$\mathcal{E}_{I \times [0,1)} = \mathcal{E}_{I \times [0,1)}^{(j_0)}$$

and selecting the numbers

$$\lambda_{K \times L}^{I \times [0,1)} = |K|^{1/p} |L|^{1/q} / (|I|^{1/p} |[0, 1)|^{1/q}),$$

$$\mu_{K \times L}^{I \times [0,1)} = |K|^{1/p'} |L|^{1/q'} / (|I|^{1/p'} |[0, 1)|^{1/q'}).$$

If $J \neq [0, 1)$, then $\mathcal{E}_{I \times [0,1)}$ has already been defined. The collection $\mathcal{E}_{I \times J}$ will be chosen as one of the following collections $\mathcal{E}_{I \times J}^{(j)}$, $j \in \mathbb{N}$, which are given by

$$\mathcal{E}_{I \times J}^{(j)} = \{K \times L : K \in \mathcal{D}_j, K \times L \subset \mathcal{E}_{I \times [0,1)}\}, \quad j \in \mathbb{N}.$$

To this end, define the functions

$$d_{I \times J}^{(j)} = \sum_{K \times L \in \mathcal{E}_{I \times J}^{(j)}} \frac{|K|^{1/p} |L|^{1/q}}{|I|^{1/p} |J|^{1/q}} e_k \otimes f_L, \quad j \in \mathbb{N},$$

$$d_{I \times J}^{*(j)} = \sum_{K \times L \in \mathcal{E}_{I \times J}^{(j)}} \frac{|K|^{1/p'} |L|^{1/q'}}{|I|^{1/p'} |J|^{1/q'}} e_k^* \otimes f_L^*, \quad j \in \mathbb{N},$$

and note that by Lemma 4.2, $(d_{I \times J}^{(j)})_{j=1}^\infty$ converges to 0 in the weak topology of $H^p(H^q)$ and that $(d_{I \times J}^{*(j)})_{j=1}^\infty$ converges to 0 in the w^* topology of $(H^p(H^q))^*$. Consequently, we can find an index j_0 such that

$$2^{-j_0} < \min \left\{ |K| : K \times L \in \bigcup_{i=1}^k \mathcal{E}_i \right\},$$

and

$$\text{dist}_{H^p(H^q)}(d_{I \times J}^{(j_0)}, W_{I \times J}) < \eta_{I \times J} \quad \text{and} \quad \text{dist}_{(H^p(H^q))^*}(d_{I \times J}^{*(j_0)}, G_{I \times J}) < \eta_{I \times J}.$$

Player (II) completes step 2 by selecting

$$\mathcal{E}_{I \times J} = \mathcal{E}_{I \times J}^{(j_0)}$$

and the numbers

$$\lambda_{K \times L}^{I \times J} = \frac{|K|^{1/p} |L|^{1/q}}{|I|^{1/p} |J|^{1/q}} \quad \text{and} \quad \mu_{K \times L}^{I \times J} = \frac{|K|^{1/p'} |L|^{1/q'}}{|I|^{1/p'} |J|^{1/q'}},$$

for all $K \times L \in \mathcal{E}_{I \times J}$.

Finally, in both cases ($J = [0, 1)$ and $J \neq [0, 1)$) player (I) completes step 3 (and thereby turn $(k + 1)$) and chooses

$$(\varepsilon_{K \times L}^{I \times J})_{K \times L \in \mathcal{E}_{I \times J}} \in \{-1, +1\}^{\mathcal{E}_{I \times J}}.$$

CASE V_2 . Follows from a completely parallel argument to Case V_1 . ■

6. The Haar system in L^1 is strategically reproducible

In this section we consider the space L^1 of all absolutely integrable functions on $[0, 1]$ instead of L_0^1 . If we additionally define $h_\emptyset = \chi_{[0,1]}$ and $\mathcal{D}^+ = \mathcal{D} \cup \{\emptyset\}$, then $(h_I)_{I \in \mathcal{D}^+}$ is a monotone Schauder basis of L^1 , if ordered lexicographically (i.e., \emptyset is the minimum of \mathcal{D}^+ and the rest of the order is inherited from the lexicographical order of \mathcal{D}). The reason we consider L^1 is that we can prove an

isometric statement in this setting and it is unclear whether this is possible for the space L^1_0 . The purpose is to prove that the normalized Haar system of L^1 is strategically reproducible, and thus has the factorization property. The main difficulty is proving the following statement.

THEOREM 6.1: *The normalized Haar system of L^1 is 1-strategically reproducible.*

The proof of the above will be presented in its own Subsection 6.2. We will also need the following statement.

PROPOSITION 6.2: *The normalized Haar system of L^1 has the $1/\delta$ -diagonal factorization property.*

We will give the proof of Proposition 6.2 in Subsection 6.1. For the time being, we use the above two results to prove the following Corollary.

COROLLARY 6.3: *The normalized Haar system of L^1 has the $1/\delta$ -factorization property.*

Proof. By Theorem 6.1 the normalized Haar system is 1-strategically reproducible and by Proposition 6.2 it has the $1/\delta$ -factorization property. Since the Haar system is monotone, Theorem 3.12 yields that it has the $1/\delta$ -factorization property. ■

The following is a direct consequence of Corollary 6.3.

COROLLARY 6.4: *The normalized bi-parameter Haar system of $L^1(L^1)$ (which is isometrically isomorphic to $L^1([0, 1]^2)$) has the $1/\delta$ -factorization property.*

Proof. Let $T : L^1(L^1) \rightarrow L^1(L^1)$ be a bounded linear operator so that for all $I, J \in \mathcal{D}$ we have

$$\left| \int \int (h_I \otimes h_J) T \left(\frac{1}{|I||J|} h_I \otimes h_J \right) dx dy \right| \geq \frac{1}{\delta}.$$

The space $L^1(L^1)$ is the projective tensor product of L^1 with itself. This means that if we consider any two bounded linear operators R, S on L^1 , then there is a (unique) bounded linear operator $R \otimes S : L^1(L^1) \rightarrow L^1(L^1)$ satisfying

$$(R \otimes S)(f \otimes g) = (Rf) \otimes (Sg) \quad \text{and} \quad \|R \otimes S\| = \|R\| \|S\|.$$

Consider the canonical projection $P_{[0,1]}$ from L^1 onto the linear span of h_\emptyset , which has norm one, and also consider the identity I_{L^1} on L^1 . Take the map

$$P = P_{[0,1]} \otimes I_{L^1} : L^1(L^1) \rightarrow L^1(L^1),$$

which satisfies $\|P\| = 1$. Its image is the space $Y = [h_\emptyset \otimes h_I]_{I \in \mathcal{D}}$, which is naturally isometric to L^1 via the map $h_\emptyset \otimes h_I \mapsto h_I$. It follows that the map $P \circ T : Y \rightarrow Y$ may be identified with a map on L^1 with diagonal bounded below by δ . This means, by Theorem 6.2, that for $\varepsilon > 0$ there are $B : Y \rightarrow L^1$, $A : L^1 \rightarrow Y$ so that $B(PT)A = I_{L^1}$ and $\|A\|\|B\| \leq (1 + \varepsilon)/\delta$. Now, since L^1 and $L^1(L^1)$ are isometrically isomorphic we may take an onto isometry $Q : L^1 \rightarrow L^1(L^1)$ and set

$$\tilde{A} = AQ^{-1} : L^1(L^1) \rightarrow L^1(L^1), \quad \tilde{B} = QBP : L^1(L^1) \rightarrow L^1(L^1).$$

It follows that $\|\tilde{A}\|\|\tilde{B}\| \leq (1 + \varepsilon)/\delta$ and $\tilde{B}\tilde{A} = I$. ■

Remark 6.5: One can use [8, Theorem 4.2] to give a relatively short proof of the following. There is $C \geq 1$, so that if $T : L^1 \rightarrow L^1$ is a bounded linear operator with diagonal bounded below by δ then there are $A, B : L^1 \rightarrow L^1$ with $BTA = I$ and $\|B\|\|A\| \leq C\|T\|/\delta^2$. There are two differences with Theorem 6.2. The first one is the power appearing on δ . The more noticeable one is that in Theorem 6.2 the factorization does not depend on $\|T\|$. This seems to be the case for all known spaces with the factorization property.

Remark 6.6: We point out that Theorem 6.1, Proposition 6.2, and Corollary 6.3 are true for the normalized Haar system of $L^p[0, 1]$, $1 \leq p < \infty$, as well. The proof of Theorem 6.1 requires only minor modifications and in certain cases it is simpler due to reflexivity and unconditionality. The proof of Proposition 6.2 is different and one has to use the details of the proof of Theorem 6.1. The factorization property of these spaces has been known since [2], however existing proofs did not give a sharp factorization estimate; in particular, it was not known whether the Haar system of $L^p[0, 1]$, $1 \leq p < \infty$, has the C/δ -factorization property for a uniform constant $C \geq 1$.

6.1. THE PROOF OF PROPOSITION 6.2. We now turn to the proof of Proposition 6.2. We divide the argument into several steps formulated in Lemmas 6.7 and 6.9 below. Lemma 6.7 is likely to be known. We present a short proof for the sake of completeness and convenience of the reader.

LEMMA 6.7: Let $I_0 \in \mathcal{D}$ and $Y_{I_0} = [(h_I)_{I \subset I_0}]$. Then there are a subspace Z of Y_{I_0} that is isometrically isomorphic to L^1 and a norm one linear projection $P : L^1 \rightarrow Z$.

Proof. Set $m_0 = \inf I_0$, $M_0 = \sup I_0$, and let Z be the subspace consisting of all absolutely integrable functions f that have support in I_0 and satisfy the condition

$$f(x) = -f\left(\frac{m_0 + M_0}{2} + x\right) \quad \text{a.e. in } I_0^+.$$

It follows that W is a subspace of Y_{I_0} . An onto isometry $T : Z \rightarrow L^1$ is given by

$$Tf(x) = |I_0|f\left(m_0 + \frac{|I_0|}{2}x\right).$$

The desired projection $P : L^1 \rightarrow Z$ is defined as follows. For every $f \in L^1$ let $f_1 = f|_{I_0^+}$, $f_2 = f|_{I_0^-}$, let g_1 be the function with support I_0^+ so that $g_1(x) = f(x + |I_0|/2)$, for $x \in I_0^+$, and let g_2 be the function with support I_0^- so that $g_2(x) = f(x - |I_0|/2)$, for $x \in I_0^-$. One can check that

$$\|f_1 + f_2\|_{L^1} = \|g_1 + g_2\|_{L^1} = \|f\|_{L^1}.$$

It is also not hard to see that $(f_1 + f_2) - (g_1 + g_2) = (f_1 - g_1) + (f_2 - g_2)$ is in W . The final step is to observe that

$$Pf = (1/2)[(f_1 + f_2) - (g_1 + g_2)]$$

is a norm one projection onto Z . ■

It is common to call diagonal operators on the Haar system Haar multipliers. Loosely following [25] we use the following notation.

Notation 6.8:

- (i) A chain of \mathcal{D}^+ is a sequence of intervals $\mathcal{C} = (I_n)_n$ so that $I_1 \supseteq I_2 \supseteq \dots$.
- (ii) Given a Haar multiplier D with entries $(c_I)_{I \in \mathcal{D}^+}$ we define the quantity

$$\|D\|_W = \sup \sum_{n=1}^{\infty} |c_{I_n} - c_{I_{n+1}}|,$$

where the supremum is taken over all chains $\mathcal{C} = (I_n)_n$.

- (iii) Given a Haar multiplier D with entries $(c_I)_{I \in \mathcal{D}^+}$ we define the quantity

$$\|D\|_{\infty} = \sup_{I \in \mathcal{D}^+} |c_I|.$$

LEMMA 6.9: Let $D : L^1 \rightarrow L^1$ be a bounded Haar multiplier with entries $(c_I)_{I \in \mathcal{D}}$. Then for every $\varepsilon > 0$ there exists $I_0 \in \mathcal{D}$ so that if $Y_{I_0} = [(h_I)_{I \subset I_0}]$ then

$$\|(D - c_{I_0}I)|_{Y_{I_0}}\| \leq \varepsilon.$$

Proof. According to [25, Theorem 3] (see also [26]), for any Haar multiplier $D : L^1 \rightarrow L^1$ we have

$$(14) \quad \frac{1}{4}\|D\|_W \leq \|D\| \leq \|D\|_W + 3\|D\|_\infty.$$

We use the above to choose $I_0 \in \mathcal{D}$ with the property that for any chain $\mathcal{C} = (I_n)$ with $I_n \subset I_0$ we have $\sum_n |c_{I_n} - c_{I_{n+1}}| \leq \varepsilon/4$. If such an I_0 would not exist then it would easily follow that $\|D\|_W = \infty$, which by (14) contradicts the boundedness of D .

Let P_{I_0} define the canonical projection onto Y_{I_0} , given by

$$P\left(\sum_{I \in \mathcal{D}} a_I h_I\right) = \sum_{I \subset I_0} a_I h_I,$$

which has norm at most two. Clearly, P_{I_0} is a Haar multiplier. Next, define the Haar multiplier S with entries $(\tilde{c}_I)_{I \in \mathcal{D}^+}$, where $\tilde{c}_I = c_I$ if $I \subset I_0$ and $\tilde{c}_I = 0$ otherwise. It is easy to see that $\|S - c_{I_0}P_{I_0}\|_W = \|S\|_W \leq \varepsilon/4$ and that $\|S - c_{I_0}P_{I_0}\|_\infty \leq \varepsilon/4$. Therefore by (14) we deduce $\|S - P_{I_0}\| \leq \varepsilon$. Since $D|_{Y_{I_0}} = S|_{Y_{I_0}}$ and $I|_{Y_{I_0}} = P_{I_0}|_{Y_{I_0}}$ we finally conclude that

$$\|(D - c_{I_0}I)|_{Y_{I_0}}\| = \|(S - c_{I_0}P_{I_0})|_{Y_{I_0}}\| \leq \|S - c_{I_0}P_{I_0}\| \leq \varepsilon. \quad \blacksquare$$

We are ready to conclude this subsection with a proof of Proposition 6.2.

Proof of Proposition 6.2. Let D be a bounded Haar multiplier on L^1 satisfying $\inf_{I \in \mathcal{D}^+} |c_I| \geq \delta > 0$ and fix $\varepsilon > 0$. We will show that the identity $(1 + \varepsilon)/\delta$ -factors through D . Use Lemma 6.9 to find $I_0 \in \mathcal{D}$ so that

$$\|(D - c_{I_0}I)|_{Y_{I_0}}\| \leq \delta\varepsilon/(1 + \varepsilon).$$

By Lemma 6.7 there are a subspace Z of Y_{I_0} , an onto isometry $A : L^1 \rightarrow Z$, and a norm one projection $P : L^1 \rightarrow Z$. Define $B = (1/c_{I_0})A^{-1}P$, which is well defined on L^1 , and $\|B\| = |1/c_{I_0}| \leq 1/\delta$. As the image of A is Z it easily follows that $B(c_{I_0}I)A = I$. We calculate

$$\|BDA - I\| = \|B(D - c_{I_0}I)A\| = \|B(D - c_{I_0}I)|_Z A\| \leq \|B\|\|A\| \frac{\delta\varepsilon}{1 + \varepsilon} \leq \frac{\varepsilon}{1 + \varepsilon}$$

and hence the operator BDA is invertible with $\|(BDA)^{-1}\| \leq 1 + \varepsilon$. Define $\tilde{B} = (BDA)^{-1}B$. Observe that $\tilde{B}DA = I$ and $\|\tilde{B}\| \|A\| \leq (1 + \varepsilon)/\delta$, i.e., the identity almost $1/\delta$ -factors through D . ■

6.2. THE PROOF OF THEOREM 6.1. The following Lemma 6.10 is a well known result, which goes back to Gamlen–Gaudet [10]. For more details, we also refer to [21, page 176 ff.]. It describes the situation player (II) is striving to achieve in order to win the game of strategical reproducibility. Recall that for $\mathcal{A} \subset \mathcal{D}$ we set

$$\mathcal{A}^* = \bigcup \mathcal{A} = \bigcup_{I \in \mathcal{A}} I.$$

LEMMA 6.10: *Let $\kappa \in (0, 1)$. Then, there exists a sequence of positive real numbers $(\delta_n)_n$ so that the following holds. Let $(\mathcal{H}_I)_{I \in \mathcal{D}^+}$ be a collection of non-empty finite subsets of \mathcal{D} , so that for each $I \in \mathcal{D}^+$ the collection \mathcal{H}_I consists of pairwise disjoint intervals, and for each $I \in \mathcal{D}^+$ let $\bar{\varepsilon}_I = (\varepsilon_L^I)_{L \in \mathcal{H}_I} \in \{-1, 1\}^{\mathcal{H}_I}$. Define for each $I \in \mathcal{D}$ the function*

$$b_I = \sum_{L \in \mathcal{H}_I} \varepsilon_L^I h_L$$

and

$$b_\emptyset = \left| \sum_{L \in \mathcal{H}_\emptyset} \varepsilon_L^I h_L \right|.$$

Assume that the following are satisfied.

- (a) For all $I, J \in \mathcal{D}$ with $I \cap J = \emptyset$ we have $\mathcal{H}_I^* \cap \mathcal{H}_J^* = \emptyset$.
- (b) For all $I \in \mathcal{D}$ we have $\text{supp}(b_{I^+}) \subset [b_I = 1]$ and $\text{supp}(b_{I^-}) \subset [b_I = -1]$.
- (c) For all $n \in \mathbb{N}$ and $I \in \mathcal{D}_n$ if $I = J^+$ or $I = J^-$ we have

$$(1 - \delta_n) \frac{|\mathcal{H}_J^*|}{2} \leq |\mathcal{H}_I^*| \leq (1 + \delta_n) \frac{|\mathcal{H}_J^*|}{2}.$$

- (d) $\mathcal{H}_{[0,1]}^* \subset \mathcal{H}_\emptyset^*$ and $|\mathcal{H}_{[0,1]}^*| \geq (1 - \delta_1) |\mathcal{H}_\emptyset^*|$.

Then, if $\lambda = |\mathcal{H}_\emptyset^*|$, the sequences $(h_I/|I|)_{I \in \mathcal{D}^+}$ and $(b_I/\lambda|I|)_{I \in \mathcal{D}^+}$, when they are both viewed as sequences in L^1 , are $(1 + \kappa)$ -impartially equivalent. Furthermore, the sequences $(b_I)_{I \in \mathcal{D}^+}$ and $(h_I)_{I \in \mathcal{D}^+}$, when they are both viewed as a sequence in L^∞ , are isometrically equivalent.

The following Lemma allows player (II) to make the appropriate choice of vectors.

LEMMA 6.11: *Let H be in $\text{cof}(L^1)$, G be in $\text{cof}_{w^*}(L^\infty)$, and $\kappa > 0$. Then there exists $n_0 \in \mathbb{N}$ so that for every f in the linear span of $(h_I)_{I \in \mathcal{D} \setminus \mathcal{D}^{n_0}}$ with $\|f\|_\infty \leq 1$ we have*

- (i) $\text{dist}(f, H) < \kappa$, if f is viewed as an element of L^1 , and
- (ii) $\text{dist}(f, H) < \kappa$, if f is viewed as an element of L^∞ .

Proof. We first show (i). Recall that there are $g_1, \dots, g_N \in L^\infty$ so that $H = \bigcap_{j=1}^N \ker g_j$. It follows, from the Hahn–Banach theorem, that there is $\delta > 0$ so that for every $f \in L^1$ with $|\int g_j(x)f(x)dx| < \delta$ for $1 \leq j \leq N$ we have $\text{dist}(f, H) < \kappa$. If we assume that the conclusion is false, i.e., the desired k_0 does not exist, there is a sequence $(f_k)_k$ with $\|f_k\|_\infty \leq 1$ and $f_k \in \text{span}\{h_I : I \in \mathcal{D} \setminus \mathcal{D}^k\}$, so that $\text{dist}(f_k, H) \geq \kappa$ for all $k \in \mathbb{N}$. As this sequence is uniformly integrable it has a subsequence $(f_{k_i})_i$ that converges weakly to an $f \in \bigcap_k \overline{\text{span}}\{h_I : I \in \mathcal{D} \setminus \mathcal{D}^k\} = \{0\}$. Thus $\lim_i |\int g_j(x)f_{k_i}(x)dx| = 0$, for $1 \leq j \leq N$, i.e., $\lim_k \text{dist}(f_{k_i}, H) = 0$, which is a contradiction.

The second statement follows from a similar argument. Use that there are g_1, \dots, g_N in L^1 so that $G = \{f_1, \dots, f_N\}^\perp$ and that for any sequence $(g_k)_k$ with $\|g_k\|_\infty \leq 1$ and $g_k \in \text{span}\{h_I : I \in \mathcal{D} \setminus \mathcal{D}^k\}$, for all $k \in \mathbb{N}$, we have that $(g_k)_k$ converges to zero in the w^* -topology. ■

We refer to Section 4.5 for the notation employed systematically in the proof below.

Proof of Theorem 6.1. Enumerate \mathcal{D}^+ as $(I_k)_{k \in \mathbb{N}}$ according to lexicographical order. We will describe the winning strategy of player (II) in a game of $\text{Rep}_{(L^1, (h_I))}(1, \eta)$, for fixed $\eta > 0$. Before the game starts player (I) picks a partition $\mathbb{N} = N_1 \cup N_2$, which corresponds to a partition $\mathcal{D}^+ = \mathcal{A}_1 \cup \mathcal{A}_2$. Before proceeding with the game, we claim that $[0, 1) = \limsup(A_1) \cup \limsup(A_2)$. Indeed, if $x \in [0, 1)$ then for every $I \in \mathcal{D}$ with $x \in I$ we have $I \in \mathcal{A}_1$ or $I \in \mathcal{A}_2$. That is, $x \in I$ for infinitely many $I \in \mathcal{A}_1$ or for infinitely many $I \in \mathcal{A}_2$. In the first case $x \in \limsup(A_1)$ and in the second case $x \in \limsup(A_2)$. We proceed by assuming without loss of generality that $|\limsup(\mathcal{A}_1)| \geq 1/2$. By Lemma 4.4 there is $\mathcal{A} \subset \mathcal{A}_1$ so that $\mathcal{G}_n(\mathcal{A})$ is finite for all $n \in \mathbb{N}$ and $|\limsup(\mathcal{A})| = \lambda \geq 2/3$. Henceforth, when we use Notation 4.3 it shall be with respect to the collection \mathcal{A} . This collection \mathcal{A} corresponds to some $N \subset N_1$. Each round k corresponds to an $I \in \mathcal{D}^+$ via the lexicographical identification $\mathcal{D}^+ \leftrightarrow \mathbb{N}$. Player (I) first chooses $\eta_k > 0$, $W_k \in \text{cof}(L^1)$ and $G_k \in \text{cof}_{w^*}(L^\infty)$.

Then player (II) has the right to choose a subset E_k of either N_1 or N_2 . He or she will always choose $E_k \subset N$. This E_k corresponds to a finite $\mathcal{H}_I \subset \mathcal{A}$. We shall describe the choice in detail further below, but let us say for the time being that there is $m_k \in \mathbb{N}$, with $m_k > m_{k-1}$ if $k > 1$, so that $\mathcal{H}_I \subset \mathcal{A}_{m_k}$ and $\mathcal{H}_I^* \subset \mathcal{H}_\emptyset^*$. Next, player (I) chooses signs $(\varepsilon_i^k)_{i \in E_k}$ which we relabel as $(\varepsilon_L^I)_{L \in \mathcal{H}_I}$. We shall put $b_I = \sum_{L \in \mathcal{H}_I} \varepsilon_L^I h_I$ and $\tilde{b}_I = b_\emptyset b_I$ (pointwise). If $k > 1$, then by the fact that $m_k > 1$ and $\mathcal{H}_I^* \subset \mathcal{H}_\emptyset^*$ we have that

$$\tilde{b}_I = \sum_{L \in \mathcal{H}_I} \tilde{\varepsilon}_L^I h_I,$$

for a choice of signs $(\tilde{\varepsilon}_L^I)_{L \in \mathcal{H}_I}$ that does not necessarily coincide with $(\varepsilon_L^I)_{L \in \mathcal{H}_I}$. Of some importance is also the sequence of positive real numbers $(\delta_n)_n$ provided by Lemma 6.10 if we take $\kappa = \eta$.

We can now describe how player (II) makes a choice in each round k . Let $I \in \mathcal{D}^+$ correspond to k in the lexicographical enumeration. Then, either $I = \emptyset$ (if $k = 1$), $I = [0, 1)$ (if $k = 2$), or there is $2 \leq k' < k$ so that if $J = I_{k'}$ then $I = J^+$ or $I = J^-$. The round starts by player (I) picking $\eta_k > 0$. Player (II) will pick $E_k \subset N$ that corresponds to an $\mathcal{H}_I \subset \mathcal{A}$ which is chosen as follows:

- (i) There is $m_k \in \mathbb{N}$, with $m_k > m_{k-1}$ if $k > 1$, so that \mathcal{H}_I is one of the following forms:
 - (ia) $\mathcal{H}_I = \mathcal{G}_{m_k}(\mathcal{A})$, if $I = \emptyset$ or $I = [0, 1)$ (i.e., when $k = 1$ or $k = 2$).
 - (ib) $\mathcal{H}_I = (\mathcal{H}_J)_{\tilde{\varepsilon}_J, m_k}^{\text{succ}}$, if $I = J^+ = I_{k'}$ and $\tilde{\varepsilon}_J = (\tilde{\varepsilon}_L^J)_{L \in \mathcal{H}_J}$ coming from \tilde{b}_J .
 - (ic) $\mathcal{H}_I = (\mathcal{H}_J)_{-\tilde{\varepsilon}_J, m_k}^{\text{succ}}$, if $I = J^- = I_{k'}$ and $\tilde{\varepsilon}_J = (\tilde{\varepsilon}_L^J)_{L \in \mathcal{H}_J}$ coming from \tilde{b}_J .
- (ii) If $A = \limsup(\mathcal{A})$ then if $|\mathcal{H}_I^* \cap A| > (1 - \delta_{n+1}/2)|\mathcal{H}_I^*|$, where $I \in \mathcal{D}_n$ and $(\delta_i)_i$ is the sequence mentioned above, provided by Lemma 6.10. If $I = \emptyset$ replace n with 0.
- (iii) For every $f \in \text{span}(h_L)_{L \in \mathcal{D} \setminus \mathcal{D}^{m_k-1}}$ with $\|f\|_\infty \leq 1$ we have

$$\text{dist}_{L^1}(f, W_k) \leq \eta_k |I| \quad \text{and} \quad \text{dist}_{L^\infty}(f, G_k) \leq \eta_k.$$

Having chosen such an \mathcal{H}_I , player (II) picks scalars $(\lambda_L^I)_{L \in \mathcal{H}_I}$, $(\mu_L^I)_{L \in \mathcal{H}_I}$ by taking $\lambda_L^I = |L|/(|\mathcal{H}_\emptyset||I|)$ and $\mu_L^I = 1$, for all $L \in \mathcal{H}_I$.

We must show that player (II) can pick \mathcal{H}_I satisfying (i), (ii), and (iii) as well as that

$$(15) \quad 1 - \eta \leq \sum_{L \in \mathcal{H}_I} \lambda_L^I \mu_L^I \leq 1 + \eta.$$

Note that if m_k is sufficiently large, then by Lemma 6.11 condition (iii) is satisfied. We can focus on showing that we can pick m_k , as large as desired, so that \mathcal{H}_I is one of the forms in (i) and so that (ii) is satisfied. If $I = \emptyset$ and $k = 1$, then $A = \bigcap_m \mathcal{G}_m(\mathcal{A})^*$; so $|A| = \lim_n |\mathcal{G}_m(\mathcal{A})^*|$ which easily yields that we can pick m_1 as large as we wish so that

$$|\mathcal{G}_{m_1}(\mathcal{A})^* \cap A| = |A| > (1 - \delta_2/2)|\mathcal{G}_{m_1}(\mathcal{A})^*|.$$

If $I = [0, 1)$ we act similarly. If $I = J^+$ with $I \in \mathcal{D}_n$, then by assumption player (II) has picked $\mathcal{H}_J \subset \mathcal{A}$ with $|\mathcal{H}_J^* \cap A| > (1 - \delta_n/2)|\mathcal{H}_J^*|$. Let also $\tilde{\varepsilon}_J = (\tilde{\varepsilon}_L^J)_{L \in \mathcal{H}_J}$ denote the signs coming from \tilde{b}_J . By Lemma 4.5 there exists m_0 so that for any $m \geq m_0$ we have

$$|((\mathcal{H}_J)_{\tilde{\varepsilon}_J, m}^{\text{succ}})^* \cap A| > (1 - \delta_{n+1}/2)|((\mathcal{H}_J)_{\tilde{\varepsilon}_J, m}^{\text{succ}})^*|.$$

Thus, if we pick m_k sufficiently large we may set $\mathcal{H}_I = (\mathcal{H}_J)_{\tilde{\varepsilon}_J, m}^{\text{succ}}$ and (i) and (ii) are satisfied. If $I = J^-$ the argument is the same.

The proof of (15) requires an inductive argument. We will show this simultaneously with proving that player (II) has forced the desired winning conditions. Define for $I \in \mathcal{D}^+$ the functions

$$x_I = \sum_{L \in \mathcal{H}_I} \varepsilon_L^I \frac{|L|}{|\mathcal{H}_\emptyset||I|} h_I \in L^1 \quad \text{and} \quad x_I^* = \sum_{L \in \mathcal{H}_I} \varepsilon_L^I h_I \in L^\infty.$$

Our next goal is to show that $(x_I)_{I \in \mathcal{D}^+}$ is $(1 + \eta)$ -impartially equivalent to $(h_I/|I|)_{I \in \mathcal{D}^+}$ in L^1 and $(x_I^*)_{I \in \mathcal{D}^+}$ is isometrically equivalent to $(h_I)_{I \in \mathcal{D}^+}$ in L^∞ . We will first show that $(\tilde{b}_I)_{I \in \mathcal{D}^+}$, where

$$\tilde{b}_\emptyset = b_\emptyset^2 = \left| \sum_{L \in \mathcal{H}_\emptyset} \varepsilon_L^\emptyset h_L \right|$$

and, for each $I \in \mathcal{D}$, $\tilde{b}_I = \sum_{L \in \mathcal{H}_I} \tilde{\varepsilon}_L^I h_I$, satisfies the assumptions of Lemma 6.10 and use that to reach the desired goal.

Assumption (a) follows easily from (i) and assumption (b) follows easily from (ib) and (ic). Let us now show that (c) is satisfied and let $n \in \mathbb{N}$, $I \in \mathcal{D}_n$, with $I = I_k$, so that $I = J^+$ or $I = J^-$ for some $J \in \mathcal{D}_{n-1}$. We shall assume that $I = J^+$ as the other case has the same proof. By (iii), applied to J , we have that $|\mathcal{H}_J^* \cap A| > (1 - \delta_n/2)|\mathcal{H}_J^*|$. This, by the first statement of Lemma 4.5, yields that

$$|(\mathcal{H}_J)_{\tilde{\varepsilon}_J}^* \cap A| > (1 - \delta_n) \frac{|\mathcal{H}_J^*|}{2}.$$

By the definition of $(\mathcal{H}_J)_{\bar{\varepsilon}_J, m_k}^{\text{succ}}$ it follows that

$$(\mathcal{H}_J)_{\bar{\varepsilon}_J}^* \cap A = (\mathcal{H}_J)_{\bar{\varepsilon}_J, m_k}^* \cap A = \mathcal{H}_I^* \cap A.$$

We calculate

$$\frac{|\mathcal{H}_J^*|}{2} = |(\mathcal{H}_J)_{\bar{\varepsilon}_J}^*| \geq |\mathcal{H}_I^*| \geq |\mathcal{H}_I^* \cap A| > (1 - \delta_n) \frac{|\mathcal{H}_J^*|}{2},$$

i.e., (c) holds. Assumption (d) is easier to show.

Now that we know that the assumptions of Lemma 6.10 are satisfied we conclude that the sequences

$$\tilde{x}_I = \sum_{L \in \mathcal{H}_I} \tilde{\varepsilon}_L^I \frac{|L|}{|\mathcal{H}_\emptyset||I|} h_I \in L^1$$

and

$$\tilde{x}_I^* = \sum_{L \in \mathcal{H}_I} \tilde{\varepsilon}_L^I h_I \in L^\infty$$

are $(1 + \eta)$ -impartially equivalent to $(h_I/|I|)_{I \in \mathcal{D}^+}$ in L^1 and isometrically equivalent to $(h_I)_{I \in \mathcal{D}}$ in L^∞ respectively. We next observe that for $I \in \mathcal{D}^+$ we have

$$\tilde{x}_I = b_\emptyset x_I \quad \text{and} \quad \tilde{x}_I^* = b_\emptyset x_I^*.$$

But $|b_\emptyset(t)|$ is one whenever $t \in \mathcal{H}_\emptyset^* \supset \text{supp}(x_I) = \text{supp}(x_I^*)$, and thus $(x_I)_{I \in \mathcal{D}^+}$ is isometrically equivalent to $(\tilde{x}_I)_{I \in \mathcal{D}^+}$ and $(x_I^*)_{I \in \mathcal{D}^+}$ is isometrically equivalent to $(\tilde{x}_I^*)_{I \in \mathcal{D}^+}$. This means that we have reached our goal.

Next, we need to observe that if $I = I_k$, then $\text{dist}_{L^1}(x_I, W_k) < \eta_k$ as well as $\text{dist}_{L^\infty}(x_I^*, G_k) < \eta_k$. Both of these inequalities are an immediate consequence of (iii) and the fact that $x_k, x_k^* \in \text{span}(h_L)_{L \in \mathcal{G}_{n_k}(A)} \subset \text{span}(h_L)_{L \in \mathcal{D} \setminus \mathcal{D}^{m_k-1}}$ and $\|x_I\|_{L^\infty} = |I|, \|x_I^*\|_{L^\infty} = 1$.

It only remains to prove (15), which follows easily from the fact that $(x_I)_I$ is $(1 + \eta)$ -impartially equivalent to $(h_I/|I|)_I$. Indeed,

$$\sum_{L \in \mathcal{H}_I} \lambda_L^I \mu_L^I = \frac{1}{\lambda|I|} \sum_{L \in \mathcal{H}_I} |L| = \|x_I\|_{L^1}$$

and $\|x_I\|_{L^1}$ is between

$$\sqrt{(1 - \eta)}(\|h_I\|_{L^1}/|I|) > 1 - \eta \quad \text{and} \quad \sqrt{(1 + \eta)}(\|h_I\|_{L^1}/|I|) < 1 + \eta.$$

The proof is complete. ■

7. Unconditional sums of spaces with strategical reproducible bases

In this section we determine that the strategical reproducibility is inherited by unconditional sums.

For a Banach space X with a 1-unconditional basis $(e_n)_n$ and a sequence of Banach spaces $(Y_n)_n$, we denote by $Z = (\sum Y_n)_X$ the Banach space of all sequences $z = (y_n)_n$ with $y_n \in Y_n$ for all $n \in \mathbb{N}$ and the quantity

$$\|z\| = \left\| \sum_{n=1}^{\infty} \|y_n\|_{Y_n} e_n \right\|_X$$

is well defined. For each $k \in \mathbb{N}$ let $P_k : Z \rightarrow Y_k$ denote the map given by

$$P_k(y_n)_n = y_k.$$

The space Y_k can be naturally isometrically identified with a subspace of Z , namely the one consisting of all sequences which have all coordinates, except the k 'th one, equal to zero. Thus, with this identification, P_k is a norm one projection.

Remark 7.1: If $A_n : Y_n \rightarrow Y_n$, $n \in \mathbb{N}$, are bounded linear operators and $\sup_n \|A_n\| = C < \infty$, then by 1-unconditionality the map $A : Z \rightarrow Z$ with

$$A(z) = \sum_n A_n P_n(z)$$

is bounded with $\|A\| = C$.

Remark 7.2: If there exists a common $\lambda \geq 1$ such that each Y_n has a Schauder basis $(e_i^{(n)})_i$ whose basis constant is bounded by λ , then there is an enumeration (\tilde{e}_i) of $((e_i^{(n)})_i)_n$ that is Schauder basic whose basis constant is at most λ . In fact, this is satisfied by any enumeration (\tilde{e}_i) with the property that whenever $i < j$, if for some $n \in \mathbb{N}$ we have $e_i^{(n)} = \tilde{e}_{k_i^{(n)}}$, $e_j^{(n)} = \tilde{e}_{k_j^{(n)}}$, then $k_i^{(n)} < k_j^{(n)}$.

LEMMA 7.3: *Let X be a Banach space with a 1-unconditional basis $(e_n)_n$, $(Y_n)_n$ be a sequence of Banach spaces, and $Z = (\sum Y_n)_X$. Assume that there are common $\lambda \geq 1$ and $K : (0, +\infty) \rightarrow \mathbb{R}$ so that each Y_n has a Schauder basis $(e_i^{(n)})_i$ whose constant is at most λ that has the $K(\delta)$ -diagonal factorization property. Then the sequence $((e_i^{(n)})_i)_n$ is a Schauder basis (using the linear order defined in Remark 7.2) whose basis constant is at most λ and it has the $K(\delta)$ -diagonal factorization property.*

Proof. Let $D : X \rightarrow X$ be a diagonal operator with respect to $((e_i^{(n)})_i)_n$ so that

$$\inf_{i,n} |e_i^{(n)*} D(e_i^{(n)})| > \delta.$$

It follows that for each $n \in \mathbb{N}$ the map D restricted on Y_n is a diagonal operator D_n so that $\inf_{i,n} |e_i^{(n)*} (D_n(e_i^{(n)}))| > \delta$. For $\kappa > 0$, by assumption, there exist $B_n, A_n : Y_n \rightarrow Y_n$ with $\|A_n\| \|B_n\| \leq K(\delta) + \kappa$ and $B_n D_n A_n$ is the identity map on Y_n . By scaling, we may assume that $\max\{\|A_n\|, \|B_n\|\} \leq \sqrt{K(\delta) + \kappa}$ and hence the maps $A, B : Z \rightarrow Z$, with $A(z) = \sum_n A_n P_n(z)$ and $B(z) = \sum_n B_n P_n(z)$, are well defined with

$$\|A\| \|B\| \leq K(\delta) + \kappa.$$

It is easily verified that $I = BDA$. ■

LEMMA 7.4: *Let X be a Banach space with a 1-unconditional basis $(e_n)_n$, $(Y_n)_n$ be a sequence of Banach spaces, and $Z = (\sum Y_n)_X$. Fix $n \in \mathbb{N}$ and let A, B be finite subsets of Z and Z^* respectively. Define*

$$A_n = \{P_n(x) : x \in A\}, \quad B_n = \{P_n^*(x^*) : x^* \in B\},$$

$$G = A^\perp, \quad H = \bigcap_{x^* \in B} \ker(x^*), \quad G_n = A_n^\perp, \quad W_n = \bigcap_{x^* \in B_n} \ker(x^*).$$

Then, for every $x \in Y_n$ and $x^* \in Y_n^*$ we have

$$\text{dist}(x, H) \leq \text{dist}(x, W_n) \quad \text{and} \quad \text{dist}(x^*, G) \leq \text{dist}(x^*, G_n).$$

Proof. For every $y \in W_n$ it follows that $P_n(y) \in H$. Hence

$$\|x - y\| \geq \|P_n(x - y)\| = \|x - P_n(y)\| \geq \text{dist}(x, H)$$

and so $\text{dist}(x, W_n) \geq \text{dist}(x, H)$. Similarly, for $f \in G_n$ we have $P_n^* f \in G$ and we conclude in the same manner that $\text{dist}(x^*, G) \leq \text{dist}(x^*, G_n)$. ■

PROPOSITION 7.5: *Let X be a Banach space with a 1-unconditional basis $(e_n)_n$, $(Y_n)_n$ be a sequence of Banach spaces, and $Z = (\sum Y_n)_X$. Assume that there are common $\lambda \geq 1$ and $C \geq 1$ so that each Y_n has a C -strategically reproducible Schauder basis $(e_i^{(n)})_i$ whose basis constant is at most λ . Then the sequence $((e_i^{(n)})_i)_n$ enumerated as (\tilde{e}_n) according to Remark 7.2 is a C -strategically reproducible Schauder basis whose basis constant is at most λ .*

Proof. For each $n \in \mathbb{N}$ let M_n be the infinite subset of \mathbb{N} so that

$$(\tilde{e}_i)_{i \in M_n} = (e_i^{(n)})_i.$$

Here, we will now describe the winning strategy of player (II) in a game $\text{Rep}_{(X, (\tilde{e}_i))}(C, \eta)$. Let player (I) pick a partition $\mathbb{N} = N_1 \cup N_2$. Note that for each $n \in \mathbb{N}$, $M_n = M_n^1 \cup M_n^2$, where

$$M_n^1 = N_1 \cap M_n, \quad M_n^2 = N_2 \cap M_n.$$

The m 'th round is played out as follows. Player (I) selects $\eta_m > 0$ as well as $W_n \in \text{cof}(Z)$, $G_n \in \text{cof}_{w^*}(Z^*)$. Then, there are finite subsets A_m and B_m of Z and Z^* respectively so that $G_m = A_m^\perp$ and $W_m = (B_m)^\perp$. If $m \in M_n$, for some $n \in \mathbb{N}$, and m is the k 'th element of M_n then set $A_k^{(n)} = \{P_n x : x \in A\}$ and $B_k^{(n)} = \{P_n^* f : f \in B\}$. Then set $G_k^{(n)} = (A_k^{(n)})^\perp$, $W_k^{(n)} = (B_k^{(n)})^\perp$. Let player (II) treat this round as the k 'th round of a game $\text{Rep}_{(Y_n, (e_i^{(n)}))}(C, \eta)$ and follow a winning strategy. In the end, for each $n \in \mathbb{N}$, player (II) has chosen $(x_k^{(n)})_k$ in Y_n and $(x_k^{(n)*})_k$ in Y_n^* so that

- (i) the sequences $(x_k^{(n)})_k$ and $(e_k^{(n)})_k$ are impartially $(C + \eta)$ -equivalent,
- (ii) the sequences $(x_k^{(n)*})_k$ and $(e_k^{(n)*})_k$ are impartially $(C + \eta)$ -equivalent,
- (iii) for all $k \in \mathbb{N}$, if the k 'th element of M_n is m , we have $\text{dist}(x_k^{(n)}, W_k^{(n)}) < \eta_m$,
- (iv) for all $k \in \mathbb{N}$, if the k 'th element of M_n is m , we have $\text{dist}(x_k^{(n)*}, G_k^{(n)}) < \eta_m$.

If we relabel $(x_k^{(n)})_k$ as $(\tilde{x}_m)_{m \in M_n}$ and stitch them all together to a sequence $(\tilde{x}_m)_{m \in \mathbb{N}}$ then it easily follows that this sequence is impartially $(C + \eta)$ -equivalent to $(\tilde{e}_m)_m$. Also by Lemma 7.4 we have $\text{dist}(x_m, W_m) < \eta_m$, for all $m \in \mathbb{N}$. Similarly, relabel $(x_k^{(n)*})_k$ as $(\tilde{x}_m^*)_{m \in M_n}$ and take $(\tilde{x}_m^*)_m$, which is $(C + \eta)$ -impartially equivalent to $(\tilde{e}_m^*)_m$. Also, $\text{dist}(x_m^*, G_m) < \eta_m$, for all $m \in \mathbb{N}$. In other words, player two has emerged victorious. ■

THEOREM 7.6: *Let X be a Banach space with a 1-unconditional basis $(e_n)_n$, $(Y_n)_n$ be a sequence of Banach spaces, and $Z = (\sum Y_n)_X$. Assume that there are common $\lambda \geq 1$, $C \geq 1$, and $K : (0, +\infty) \rightarrow \mathbb{R}$ so that each Y_n has a Schauder basis $(e_i^{(n)})_i$ that satisfies the following:*

- (i) its basis constant is at most λ ,
- (ii) it has the $K(\delta)$ -diagonal factorization property, and
- (iii) it is C -strategically reproducible in Y_n .

Then the sequence $((e_i^{(n)})_i)_n$ enumerated as (\tilde{e}_n) (using the linear order defined in Remark 7.2) has the $\lambda C^2 K(\delta)$ -factorization property.

Proof. By Lemma 7.3 the basis of Z is λ -basic and it has the $K(\delta)$ -diagonal factorization property. By Proposition 7.5 the basis of Z is C -strategically reproducible. We finish off the proof by using Proposition 7.5. ■

Remark 7.7: A consequence of the above theorem is that if one takes $X = \ell_p$, $1 \leq p < \infty$ or $X = c_0$ and a sequence of spaces $(Y_n)_n$, with each Y_n being some L^p or ℓ_p , $1 \leq p < \infty$, or c_0 , then $Z = (\sum Y_n)_X$ has a 1-strategically reproducible basis. Of course, X can be any space of an unconditional basis and this produces an interesting example. It was proved in [13] that Gowers' space X with an unconditional basis from [11] does not satisfy the factorization property; however, for this X and any sequence $(Y_n)_n$ as before the space $Z = (\sum Y_n)_X$ does satisfy it.

8. Final comments and open problems

Capon [4] showed that the bi-parameter Haar system in $L^p(L^q)$, $1 < p, q < \infty$, has the factorization property. With refined techniques, Capon's result was extended to $H^1(H^1)$ by [20] and then later in [13] to $H^p(H^1)$ and $H^1(H^p)$, $1 < p < \infty$. In Section 5, we gave a different proof of their results by writing $H^p(H^q)$ as a complemented sum of two spaces, solving the problem in each of the components separately, and then using the fact that strategical reproducibility is inherited by complemented sums. This begs the following question.

Problem 8.1: If X and Y are Banach spaces with bases that have the factorization property, does the union of those two bases (in the right order) have the factorization property in the complemented sum of X and Y ?

As we remarked after Theorem 3.12 the uniform factorization property from Definition 3.10 is formally stronger than the factorization property from Definition 2.3 (iii).

Problem 8.2: Is there a Banach space with a basis that satisfies the factorization property and fails the uniform factorization property?

In Corollary 6.4 we showed that the bi-parameter Haar system has the factorization property. Nevertheless, we do not know the answer to the following problem.

Problem 8.3: Is the normalized bi-parameter Haar system of $L^1(L^1)$ strategically reproducible?

The unit vector basis of Tsirelson space T , i.e., the space constructed by Figiel and Johnson in [9], which is the dual of Tsirelson's original space, is *not* strategically reproducible. This follows from the following two facts. On the one hand, every block base (x_i) is equivalent to a subsequence (e_{n_i}) with $n_i \in \text{supp}(x_i)$, $i \in \mathbb{N}$. Secondly, if the subsequence is an Ackerman sequence (i.e., is increasing fast enough), then $[e_{n_i}]$ is not isomorphic to T . Thus, if player (I) chooses an Ackerman sequence in the game described in Definition 3.3, he wins. This leads to the next problem.

Problem 8.4: Does the unit vector basis in T have the factorization property?

Among all the bi-parametric Lebesgue and Hardy spaces, $L^p(L^1)$ and $L^1(L^p)$, $1 < p < \infty$, seem to resist our approaches.

Problem 8.5: Is the bi-parameter Haar system strategically reproducible or does it at least have the factorization property in $L^p(L^1)$ and $L^1(L^p)$, $1 < p < \infty$?

More generally, if X has a basis which is strategically reproducible, or has the factorization property, does the tensor product of that basis with the Haar system in $L^p(X)$, $1 \leq p < \infty$ have the same property?

In this context it is worth noting that Capon [5] proved $L^p(X)$, $1 \leq p < \infty$ is primary, if X has a symmetric basis.

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