Spring 2017, Math 308, Section 524 **Final Exam**

Sample

Last name:

First name:

UIN:

Signature:

"An Aggie does not lie, cheat or steal or tolerate those who do."

This exam consists of **seven** problems, the total point value of which is 100 points.

The answer to each question must be justified in detail.

The time length of this exam is 120 minutes.

The use of electronic devices, such as cellphones, tables, laptops, and calculators is prohibited.

Good luck!

Pr. 1	Pr. 2	Pr. 3	Pr. 4	Pr. 5	Pr. 6	Pr. 7	Pr. 8	Pr. 9	Total
8	10	10	12	12	12	12	12	12	100

Problem 1. In each of the following questions choose one appropriate answer. No justification for your answer is necessary. (12 pts total)

(i) The differential equation

$$y'' + 5y' - 9y = e^t \cos(y^3)$$

is a (1 pts)

(a) linear differential equation (b) non-linear differential equation.

(ii) The Laplace transform $\mathscr{L}\{f(t)\}$ of a function $f(t)$ is given by (1 pts)

(a) $\displaystyle\int_0^{+\infty} f(e^{st})\mathrm{d}t$ (b) $\displaystyle\int_0^{+\infty} e^{s^2\ln(t)} f(t)\mathrm{d}t$ (c) $\displaystyle\int_0^{+\infty} e^{-st} f(t)\mathrm{d}t$.

(iii) The convolution of two functions $f(t)$ and $g(t)$ is given by (1 pts)

(a) $\displaystyle\int_0^{t} f(x)g(x)\mathrm{d}x$ (b) $\displaystyle\int_0^{+\infty} e^{-sf(t)}g(t)\mathrm{d}t$ (c) $\displaystyle\int_0^{t} f(t-x)g(x)\mathrm{d}x$.

(iv) The characteristic polynomial of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$ is given by: (1 pts)

(a) $\det\left(\begin{bmatrix} 2-r & 3 \\ 1 & 7-r \end{bmatrix}\right)$ (b) $\det\left(\begin{bmatrix} 2-r & 3-r \\ 1-r & 7-r \end{bmatrix}\right)$ (c) $\det\left(\begin{bmatrix} 2-r & 3 \\ 1-r & 7 \end{bmatrix}\right)$.

Furthermore, given an eigenvalue r_0 of A, a vector $\boldsymbol{\xi}_0$ is called an eigenvector of A corresponding to r_0, if it satisfies the equation (2 pts)

(a) $A\boldsymbol{\xi}_0 = \mathbf{0}$ (b) $(A - r_0 I)\boldsymbol{\xi}_0 = \mathbf{0}$ (c) $(r_0 A - I)\boldsymbol{\xi}_0 = \mathbf{0}$.

(v) The Wronskian $W(\mathbf{x}(t), \mathbf{y}(t))$ of a pair of two-dimensional vector functions

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

is given by (2 pts)

(a) $x_1(t)y_2(t) - x_2(t)y_1(t)$ (b) $x_1(t)y_2'(t) - x_2(t)y_1'(t)$ (c) $x_1'(t)y_2'(t) - x_2'(t)y_1'(t)$.

Problem 2. Solve the i.v.p. (10 pts)

$$(*) \qquad y' = \cos(t)\frac{e^{t-y^2}}{y}, \; t \leq \frac{3\pi}{2} \quad y(0) = \sqrt{\ln(3)}$$

Problem 3. Solve the i.v.p. (10 pts)

$$(*) \qquad y'' + 4y' + 13y = 0, \quad y(\pi/6) = e^{-\pi/3},\ y'(\pi/6) = -5e^{-\pi/3}.$$

Problem 4. For this problem you may find helpful: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $\mathscr{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$, and if $\mathscr{L}\{f(t)\} = F(s)$ then $\mathscr{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$. (12 pts total)

(i) Find the inverse Laplace transform of the expression $F(s) = \dfrac{1}{s(s^2+1)}$ (6 pts)

(ii) If $f(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ \cos(t-\pi) & \text{if } \pi \le t < 2\pi \\ -1 & \text{if } 2\pi \le t < \infty, \end{cases}$ solve the initial value problem (6 pts)

$$y'' = f(t), \quad y(0) = 0, y'(0) = 0.$$

Problem 5. For this problem you may find helpful: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ for $s > a$, $\mathscr{L}\{\delta(t-t_0)\} = e^{-t_0 s}$ if $t_0 > 0$, and if $\mathscr{L}\{f(t)\} = F(s)$ then $\mathscr{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$. (12 pts total)

(i) Find the inverse Laplace transform of the expressions $F(s) = \dfrac{1}{(s-1)(s-2)}$ and $G(s) = \dfrac{s}{(s-1)(s-2)}$ (6 pts)

(ii) Solve the initial value problem (6 pts)

$$y'' - 3y' + 2y = \delta(t-7), \quad y(0) = 1, y'(0) = 0.$$

Problem 6. For this problem you may find helpful: $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$ for a positive integer n, $\mathscr{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$, and $\mathscr{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$. (12 pts total)

(i) If $g(t)$ is a given function and ω is a non-zero number, express the solution of the following i.v.p. in the form of a convolution integral. (6 pts)

$$y'' + \omega^2 y = \omega g(t), \quad y(0) = 0, y'(0) = 0.$$

(ii) Find all functions $\phi(t)$ that satisfy the integro-differential equation (6 pts)

$$\phi'(t) - \frac{1}{2}\int_0^t (t-x)^2 \phi(x) \mathrm{d}x = -t, \quad \phi(0) = 1.$$

Problem 7. Consider the matrix function (12 pts total)

$$A(t) = \begin{bmatrix} 0 & 1 \\ \\ -\left(\frac{1}{t} + \frac{2}{t^2}\right) & \left(1 + \frac{2}{t}\right) \end{bmatrix}$$

and the equation

$$(*) \qquad \boldsymbol{x}' = A(t)\boldsymbol{x}.$$

(i) Verify that the vector functions $\boldsymbol{x}_1 = \begin{bmatrix} t \\ 1 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} te^t \\ (1+t)e^t \end{bmatrix}$ form a fundamental set of solutions for $(*)$. (6 pts)

(ii) Solve the i.v.p. given by $(*)$ and $\boldsymbol{x}(1) = \begin{bmatrix} 1+e \\ 1+2e \end{bmatrix}$. (6 pts)

Problem 8. Consider the matrix $A = \begin{bmatrix} 1 & 6 \\ 7 & 2 \end{bmatrix}$ and the equation (12 pts total)

$$(*) \qquad \boldsymbol{x}' = A\boldsymbol{x}.$$

(i) Find a fundamental set of solutions for $(*)$. (6 pts)

(ii) Solve the i.v.p. given by $(*)$ and $\boldsymbol{x}(0) = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$. (6 pts)

Problem 9. Consider the matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ and the equation (12 pts total)

$$(*) \qquad \boldsymbol{x}' = A\boldsymbol{x}.$$

(i) Find a fundamental set of solutions for $(*)$. (6 pts)

(ii) Solve the i.v.p. given by $(*)$ and $\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (6 pts)