Spring 2018, Math 304 **Final Exam**

Sample

Last name:

First name:

UIN:

"An Aggie does not lie, cheat or steal or tolerate those who do."

This exam consists of **thirteen** problems.

The answer to each question must be **justified in detail**.

The duration of this exam is two hours.

The use of electronic devices, such as cellphones, tables, laptops, and calculators is prohibited.

Good luck!

Problem 1. Consider the matrix
$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ -1 & 2 & 1 & -3 \end{bmatrix}.$$

(i) Find the rank and nullity of A. *2 pt.*
(ii) Write a basis for the row space of A. *2 pt.*
(iii) Write a basis for the column space of A. *3 pt.*
(iv) Write a basis for the null space of $A^\top$. *3 pt.*
(v) Consider $S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \right\}$ as a subspace of $\mathbb{R}^3$. What are $\dim(S)$ and $\dim(S^\perp)$? *2 pt.*
(vi) Write a basis for S and a basis for $S^\perp$. *2 pt.*

Problem 2.
Consider the transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b+c \\ b+c \\ a+2b+2c \end{bmatrix}.$$

(i) Show that L is linear. *2 pt.*
(ii) Find the standard matrix representation A of L. *2 pt.*
(iii) Find all vectors x for which $\|L(x) - b\|$ is minimized, where $\mathbf{b} = [1, 1, 1]^\top$. *3 pt.*
(iv) Find the kernel of L. *3 pt.*
(v) If $E = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ find the matrix B representing L with respect to E. *3 pt.*
(vi) If $v = 2v_1 - v_2 + 3v_3$ find $L(v) = c_1v_1 + c_2v_2 + c_3v_3$. *1 pt.*

Problem 3.

Consider the vectors in $\mathbb{R}^4$: $x_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, x_2 = \begin{bmatrix} 1/6 \\ 1/6 \\ 1/2 \\ 5/6 \end{bmatrix}, x_3 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \\ 0 \end{bmatrix}, x_4 = \begin{bmatrix} \sqrt{2}/3 \\ \sqrt{2}/3 \\ -\sqrt{2}/2 \\ \sqrt{2}/6 \end{bmatrix}$.

(i) Find $\|x_1\|$, $\|x_2\|$, $\|x_3\|$, and $\|x_4\|$. *1 pts.*
(ii) Find $\langle x_1, x_2\rangle$, $\langle x_1, x_3\rangle$, $\langle x_1, x_4\rangle$, $\langle x_2, x_3\rangle$, $\langle x_2, x_4\rangle$, and $\langle x_3, x_4\rangle$. *1 pts.*
(iii) If $w = [1\ 1\ 1\ 1]^\top$ write this vector as $w = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$. *2 pts.*
(iv) Write the transition matrix from the basis $F = \{x_1, x_2, x_3, x_4\}$ to the usual basis $E = \{e_1, e_2, e_3, e_4\}$ and the transition matrix from E to F. *2 pts.*

Take $u = (-\sqrt{2}/2)x_1 + (1/2)x_2 + (\sqrt{2}/3)x_3 + (1/6)x_4$ and $z = (\sqrt{2}/2)x_2 + (2/3)x_3 + (\sqrt{2}/6)x_4$.

(v) Find $\|u\|$ and $\|z\|$. *2 pts.*
(vi) Find $\langle u, z\rangle$. *1 pts.*
(vii) Find the angle θ between u and z. *1 pts.*

Problem 4.

Let A by an $m \times n$ matrix with $\text{rank}(A) = n$.

(i) What is $\text{nullity}(A)$? *1 pt.*

(ii) Let x be a vector in $\mathbb{R}^n$ for which $(Ax)^\top(Ax) = 0$. Show that x must be the zero vector. *3 pt.*

(iii) Consider for each x, y in $\mathbb{R}^n$ the quantity $\langle x, y\rangle = (Ax)^\top(Ay)$. Show that it defines an inner product. *3 pt.*

Consider the vector space $C[0,1]$ equipped with the inner product given by $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. If $f(x) = \sqrt{3}x$, $g(x) = \sqrt{15}x^7$ find

(iv) The norms of f and g. *3 pt.*

(v) The cosine of the angle of f and g. *3 pt.*

(vi) The vector projection $p_g(f)$ of f onto g. *1 pt.*

Problem 5. Let S be the subspace of $\mathbb{R}^3$ spanned by $x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \\ 2/\sqrt{5} \end{bmatrix}$.

(i) Find an orthonormal basis of S. *3 pt.*
(ii) Write the orthogonal projection matrix P onto S. *2 pt.*
(iii) Find an orthonormal basis for the the orthogonal complement $S^\perp$ of S. *2 pt.*
(iv) Find the orthogonal projection matrix $\tilde{P}$ onto $S^\perp$. *2 pt.*
(v) What is $P + \tilde{P}$? *1 pt.*

Problem 6.
Consider the vector space $C[-1,1]$ endowed with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

If $f(x) = x$ and $g(x) = x^2$ let $S = \text{Span}\{f, g\}$ as a subspace of $C[-1,1]$. Use the the Gram-Schmidt orthogonalization process to find an orthonormal basis for S.*10 pts.*

Problem 7.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}.$$

(i) What is the determinant of A? *3 pt.*

(ii) What is the trace of A? *3 pt.*

(iii) What are the eigenvalues of A. *3 pt.*

(iv) Is A diagonalizable? *Justify your answer.* *1 pt.*

Problem 8.
Consider the matrix

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(i) Write the characteristic polynomial of A and find the eigenvalues of A. *4 pt.*
(ii) What are the eigenspaces of A? *4 pt.*
(iii) Is A diagonalizable? *Justify your answer.* *2 pt.*

Problem 9.

(i) Consider the matrix

$$A = \begin{bmatrix} 3 & 3 & -1 \\ 2 & 3 & 2 \\ 2 & -1 & 6 \end{bmatrix}.$$

It is given that 3 and 5 are eigenvalues of A. Find all eigenvalues of A. *4 pt.*

(ii) If B is a 2×2 matrix with $\operatorname{tr}(B) = 5$ and $\det(B) = 4$ find all the eigenvalues of the matrix B. *4 pt.*

(iii) If C is a 3×3 matrix with $\operatorname{tr}(C) = 5$ and $\lambda_1 = 1 + i$ is an eigenvalue of C find all the eigenvalues of C. *2 pt.*

Problem 10.
(i) If D is an $n \times n$ diagonal matrix all diagonal entries of which are either 1 or -1 show that necessarily $D^2 = I$. *4 pt.*
(ii) If A is an $n \times n$ orthogonal matrix and λ is an eigenvalue of A show that necessarily $|\lambda| = 1$. *4 pt.*
(iii) If A is a diagonalizable $n \times n$ orthogonal matrix and it has only real eigenvalues show that necessarily $A^2 = I$. *2 pt.*

Problem 11. Let A be a 2×2 orthogonal matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.
(i) If x_1 is an eigenvector of A belonging to $\lambda_1 = 1$ and x_2 is an eigenvector of A belonging to $\lambda_2 = -1$ show that $x_1 \perp x_2$. *4 pt.*

We additionally assume that $x_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^\top$ is an eigenvector of A belonging to $\lambda_1 = 1$.

(ii) Find the eigenspace of A belonging to $\lambda_2 = -1$. *4 pt.*
(iii) Find the matrix A. *2 pt.*

Problem 12.

(i) Let A be an $n \times n$ matrix. Show that A is singular if and only if $\lambda = 0$ is an eigenvalue of A. *4 pt.*

(ii) Let A be a non-singular $n \times n$ matrix and let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. Show that $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$. *4 pt.*

(iii) If A is a 2×2 matrix with eigenvalues 2, 3 and corresponding eigenvectors $\begin{bmatrix} 1 & 0 \end{bmatrix}^\top$, $\begin{bmatrix} -1 & 1 \end{bmatrix}^\top$ find A^{-1}. *2 pt.*

Problem 13. A patch of farmland has a total area of 1000 acres. Initially, 30% of the area is covered by shrubs and 70% is clear. Every year $1/10$ of the clear area is reclaimed by the shrubs and $3/10$ of the shrubland is cleared manually.

(i) Find the area of the shrubland and the clear area after one year. *5 pt.*

(ii) Find the area of the shrubland and the clear area after k years. *3 pt.*

(iii) Find the area of the shrubland and the clear area as $k \to \infty$. *2 pt.*