Journal of Functional Analysis 273 (2017) 1205–1257

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

A study of conditional spreading sequences

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A R T I C L E I N F O

Article history:
Received 23 November 2016
Accepted 13 April 2017
Available online 23 April 2017
Communicated by G. Schechtman

MSC:
46B03
46B06
46B25
46B45

Keywords:
Spreading sequences
Spreading models
Conditional spreading sequences
Conditional spreading models

A B S T R A C T

It is shown that every conditional spreading sequence can be decomposed into two well behaved parts, one being unconditional and the other being convex block homogeneous, i.e. equivalent to its convex block sequences. This decomposition is then used to prove several results concerning the structure of spaces with conditional spreading bases as well as results in the theory of conditional spreading models. Among other things, it is shown that the space $C(\omega^\omega)$ is universal for all spreading models, i.e., it admits all spreading sequences, both conditional and unconditional, as spreading models. Moreover, every conditional spreading sequence is generated as a spreading model by a sequence in a space that is quasi-reflexive of order one.

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[1] Research of the second author was supported by NSF DMS-1600600.
[2] Research of the third author was supported by grant 208290 from the Simons Foundation.

http://dx.doi.org/10.1016/j.jfa.2017.04.009

Contents

1. Introduction

The notion of a spreading model has been in the heart of the theory of Banach spaces since its conception in 1974 by L. Brunel and A. Sucheston [10]. A bounded sequence $(x_i)_i$ in a Banach space is said to generate a sequence $(e_i)_i$ in a semi-normed space as a spreading models if for any $n \in \mathbb{N}$ and $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with the property that for any $n_0 \leqslant k_1 < \cdots < k_n$ and scalars $a_1, \ldots, a_n$

$$\left| \left\| \sum_{i=1}^n a_i x_{k_i} \right\| - \left\| \sum_{i=1}^n a_i e_i \right\| \right| < \varepsilon. \tag{1}$$

A Banach space, or a subset of that space, is said to admit $(e_i)_i$ as a spreading model if there exists a sequence $(x_i)_i$ in that set generating $(e_i)_i$ as a spreading model. As it was proved in [10] every bounded sequence in a Banach space has a subsequence generating some spreading model. The first and foremost property of a spreading model is that it is a 1-spreading sequence, i.e. it is isometrically equivalent to its subsequences. A spreading sequence need not be Schauder basic, however this paper is focused on those spreading sequences that are conditional Schauder basic. Typical examples of such sequences are the summing basis of c_0 and the boundedly compete basis of James space J from [16]. For a thorough study of basic properties of spreading models and spreading sequences we refer the reader to [8].

There have been several applications of the concept of spreading models, a concept that describes the asymptotic behavior of a sequence in a Banach space. It has been utilized as a tool to prove several important results as it can be used to witness canonical infinite dimensional structure exhibited on finite, yet increasingly large, segments of an infinite sequence. A typical example concerns unconditional structure. Although there exist Banach spaces not containing infinite unconditional sequences (see e.g. [15]), it is well known and not difficult to show that every infinite dimensional Banach space contains a sequence generating an unconditional spreading model. Perhaps surprisingly, there is also a strong relationship between the spreading models admitted by a Banach space and the behavior of strictly singular operators on that space. In [2] it is shown that if a Banach space X contains sequences that generate spreading models with certain properties, then there exists a subspace W of X that fails the scalar-plus-compact property. In [1] the authors expand on this connection by studying the interaction between classes of $\mathcal{S}_\xi$-spreading models and compositions of strictly singular operators. Broadly speaking, understanding the $\mathcal{S}_\xi$-spreading models of a Banach space may provide information about whether compositions of certain strictly singular operators are compact. The strong connection between the theory of spreading models and operator theory is made very clear in [5] where the first two authors constructed the first known example of a reflexive Banach space with the invariant subspace property. The proof of this fact is based on manipulating properties of spreading models in the space that allow to draw conclusions concerning as to when compositions of appropriate operators are compact.

A spreading sequence that is also unconditional is called subsymmetric. As mentioned earlier, the main focus of this paper is to study conditional spreading sequences. Of particular interest is the boundedly complete basis $(e_i)_i$ of James space J, the first known Banach space that is quasi-reflexive of order one. In this space the norm of a vector $x = \sum_{i=1}^\infty a_i e_i$ is given by the formula

$$\|x\| = \sup\left\{\left(\sum_{k=1}^n\left(\sum_{i\in E_k} a_i\right)^2\right)^{1/2}\right\}, \tag{2}$$

where the supremum is taken over all possible choices of successive intervals $(E_k)_{k=1}^n$ of $\mathbb{N}$. Similarly, one may replace the ℓ_2-sum in (2) with the norm over a subsymmetric sequence $(x_i)_i$ to obtain a conditional spreading sequence, called the jamesification of $(x_i)_i$. This was first considered in [9]. A feature shared by all aforementioned spreading sequences is that they are equivalent to their convex block sequences, a property we shall refer to as convex block homogeneity. Sequences with this property are either conditional spreading or they are equivalent to the unit vector basis of ℓ_1. Interestingly, it also turns out to be related to an isometric definition introduced by Brunel and Sucheston in [12], namely that of equal signs additive sequences (see Definition 3.3, Section 3). The aforementioned paper seems to be the first attempt aimed towards studying conditional spreading sequences, or at least a subclass of them.

Theorem I. *Let X be a Banach space with a Schauder basis $(e_i)_i$. Then, the sequence $(e_i)_i$ is convex block homogeneous if and only if X admits an equivalent norm with respect to which $(x_i)_i$ is equal signs additive.*

The most elementary process that leads to conditional spreading bases that are not convex block homogeneous is to take the maximum of a subsymmetric norm and the jamesification of another subsymmetric norm. The natural question to ask is whether this is the unique way by which conditional spreading sequences are obtained. As it is explained in the last section of this paper the answer to this question is negative. Nevertheless, there exists a very nice and useful characterization in terms of decomposing a conditional spreading norm into its constituent parts. In the result below, we naturally identify every Schauder basic sequence with the unit vector basis of $c_{00}(\mathbb{N})$.

Theorem II. *Let $\|\cdot\|$ be a norm on $c_{00}(\mathbb{N})$ with respect to which the unit vector basis $(e_i)_i$ is conditional spreading. Then, there exist two norms $\|\cdot\|_u$ and $\|\cdot\|_c$, both defined on $c_{00}(\mathbb{N})$ satisfying the following.*

(i) *The unit vector basis $(e_i)_i$ is subsymmetric with respect to $\|\cdot\|_u$.*
(ii) *The unit vector basis $(e_i)_i$ is conditional and convex block homogeneous with respect to $\|\cdot\|_c$.*
(iii) *There exist positive constants κ and K so that for every $x \in c_{00}(\mathbb{N})$*

$$\kappa \max\{\|x\|_u, \|x\|_c\} \leqslant \|x\| \leqslant K \max\{\|x\|_u, \|x\|_c\}.$$

The above theorem can be translated into saying that if X is a Banach space with a conditional spreading basis $(e_i)_i$ then there exist Banach spaces U and Z having Schauder bases $(u_i)_i$ and $(z_i)_i$ that are subsymmetric and convex block homogenous respectively so that $(e_i)_i$ is equivalent to the sequence $(u_i, z_i)_i$ in $U \oplus Z$. In other words, X is isomorphic to the diagonal of $U \oplus Z$. We refer to $(u_i)_i$ and $(z_i)_i$ as the unconditional part and the convex block homogeneous part of $(e_i)_i$ respectively. The unconditional part is in fact the sequence $(e_{2i} - e_{2i-1})_i$ whereas the convex block homogeneous part is given by a block sequence of averages of the basis that increase sufficiently rapidly. It was already proved in [14] that the space U is complemented in X, in fact $X \simeq U \oplus X$. It is also true that Z is complemented in X however in interesting cases (i.e. whenever Z is not isomorphic to c_0) $X \not\simeq Z \oplus X$.

The aforementioned analysis can be used to prove regularity results for spaces with conditional spreading bases that are uncommon for such broad classes of Banach spaces. This concerns the behavior of block sequences, non-trivial weak Cauchy sequences, as well as complemented subspaces of a space with a conditional spreading basis.

Proposition III. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. Every seminormalized block sequence $(x_i)_i$ in X either has an unconditional subsequence or it has a convex block sequence that is equivalent to $(z_i)_i$.*

In fact, every convex block sequence of the basis that is equivalent to $(z_i)_i$ defines a bounded linear projection onto its linear span. We actually prove a somewhat more general result that concerns arbitrary sequences that are not necessarily block.

Proposition IV. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. Every sequence in X that is equivalent to $(z_i)_i$ has a subsequence that spans a complemented subspace of X.*

We can combine the above Propositions to obtain a result that concerns complemented subspaces of non-reflexive subspaces of a space X with a conditional spreading basis, namely every non-reflexive subspace of X contains a further subspace that is complemented in X. This complemented subspace can be chosen to be very specific, a fact again manifesting the regularity of the space X.

Theorem V. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. Let $(x_i)_i$ be a non-trivial weak Cauchy sequence in X. Then, $(x_i)_i$ has a convex block sequence $(w_i)_i$ that is either equivalent to the summing basis of c_0 or to $(z_i)_i$. Furthermore, the closed linear span of $(w_i)_i$ is complemented in X.*

The above result demonstrates the highly homogeneous structure of non-reflexive subspaces of spaces with conditional spreading bases. If the convex block homogeneous part of the basis is not equivalent to the summing basis of c_0 one can also deduce the following.

Theorem VI. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. Assume that $(z_i)_i$ is not equivalent to the summing basis of c_0. Then, if $X = Y \oplus W$, exactly one of the spaces Y and W contain a subspace $\tilde{Z}$ that is isomorphic to $Z = [(z_i)_i]$. Furthermore, $\tilde{Z}$ is complemented in X.*

This brings spaces that have a convex block homogeneous basis very close to being primary, a fact that was proved for James space by P. G. Casazza in [13]. Whether such spaces are actually primary or not is unknown to us and it poses an interesting question as to whether such canonical behavior can be witnessed by such a general class of Banach spaces. It is worth mentioning that Rosenthal's dichotomy from [21] plays a very important role in conditional spreading bases. According to this dichotomy, a weakly Cauchy sequence in a Banach space either has a strongly summing subsequence or a convex block sequence that is equivalent to the summing basis of c_0. It easily follows that a conditional spreading sequence is strongly summing if and only if its convex block homogeneous part is not equivalent to the summing basis of c_0. As it is made evident from the above theorem, strongly summing conditional spreading sequences have special properties. For example, if X has such a basis then X is not isomorphic to its square.

Theorem II is not only useful for studying the structure of Banach spaces with conditional spreading bases themselves, it can also be applied to study conditional spreading models of Banach spaces. In fact, the motivation behind the proof of Theorem II originates in questions surrounding conditional spreading models. A fairly simple observation is the following.

Proposition VII. *Every non-reflexive Banach space admits a spreading model that is 1-convex block homogeneous.*

This ought to be compared to [12, Theorem 5, page 296] where it is shown that non-superreflexive Banach spaces that are B-convex have equal signs additive sequences finitely representable in them. In a more specific setting concerning the underlying space, in [18] it is proved that every subsymmetric sequence is admitted as a spreading model by the space $C(\omega^\omega)$. This is interesting because ω^ω is the first infinite ordinal number α for which $C(\alpha)$ is not isomorphic to c_0, a space with much poorer spreading model structure. In fact, every spreading model of c_0 is equivalent to a sequence in c_0, namely either unit vector basis of c_0 or the summing basis of c_0. We extend the result from [18] as described below.

Theorem VIII. *The space $C(\omega^\omega)$ admits all possible spreading models, both the conditional and the unconditional ones.*

The proof involves a construction similar to that of Schreier's space S from [22], a space that has a weakly null basis generating an ℓ_1 spreading model, yet the space S embeds into $C(\omega^\omega)$. A noteworthy fact that was proved in [3] is that c_0 admits every possible spreading sequence as a 2-spreading model. This is a notion of spreading models introduced and studied in [3] and [4] where there is developed an entire theory surrounding the so called ξ-spreading models. This theory examines the idea of higher order spreading models in a direction that allows this notion to be taken with respect transfinite ordinal numbers.

As demonstrated by the results being discussed above, in general, the infinite dimensional structure of the spreading models generated by a space X can be quite different than the structure of the space X itself. In [7] a reflexive Banach space is constructed that admits the unit vector basis of ℓ_1 as a spreading model. This is the first known example of a reflexive Banach space admitting a non-reflexive spreading model. It is well known that whenever a Banach space admits a conditional spreading sequence as a spreading model then that space must be non-reflexive. In the spirit of the aforementioned results we show that every conditional spreading sequence is the spreading model of some quasi-reflexive Banach space, i.e. a non-reflexive Banach space that is "as close as possible to being reflexive". We furthermore show that every subsymmetric sequence is generated as a spreading model by an unconditional basis of some reflexive Banach space. These two results reveal that a Schauder basic spreading sequence is admitted as a spreading model of a sequence spanning in a sense the "smallest" possible type of Banach space, depending on whether it is conditional or unconditional.

Theorem IX. *Let $(x_i)_i$ be a spreading Schauder basic sequence.*

(i) *If $(x_i)_i$ is unconditional, then there exists a reflexive Banach space X with an unconditional Schauder basis $(e_i)_i$ that generates a spreading model equivalent to $(x_i)_i$.*

(ii) *If $(x_i)_i$ is conditional, then there exists a Banach space X that is quasi-reflexive of order one with a Schauder basis $(e_i)_i$ that generates a spreading model equivalent to $(x_i)_i$.*

The solution to this problem involves taking a suitable sequence in $C(\omega^\omega)$ and then applying a saturation method with constraints to the norm on that sequence. This is performed in such a way that the desired spreading model is preserved. Related to Theorems IX and VIII is a result from [6] where a Banach space is constructed admitting all unconditional spreading models in all of its infinite dimensional subspaces. We point out that the space from [6] is hereditarily indecomposable, thus, it does not contain unconditional sequences. This invites the question as to whether there exists a Banach space all subspaces of which admit all possible spreading models, both the conditional and the unconditional ones.

The paper is organized into 13 sections and it can be broken up into four main parts. The first part consists of Sections 2, 3, and 4 in which preliminary facts and definitions are recalled, the definition of convex block homogeneous sequences is studied, and the decomposition of conditional spreading norms is proved. These sections form the foundation for the rest of the paper and all other parts are based on it. In fact, parts two, three, and four can be read independently from one another. The second part consists of Sections 5, 6, 7, 8, and 9. These sections deal with studying the structure of Banach spaces with conditional spreading bases. The third part consists of Sections 10, 11, and 12. In these sections we study conditional spreading sequences as spreading models of certain spaces. The last part consists only of Section 13. It is devoted to showing that a certain convex block homogeneous sequence is not generated via the jamesification process.

2. Preliminaries

We remind some preliminary notions that are integral to the rest of the paper and we state certain known results that will be used repeatedly. We also introduce the concept of the conditional jamesification of a Schauder basic sequence. This is a slight modification of the notion of jamesification introduced in [9].

A sequence $(e_i)_i$ in a Banach space is *spreading* if it is equivalent to all of its subsequences and it is called *1-spreading* if the equivalent constants are one. Not all spreading sequences are Schauder basic, however these are the only ones that we shall consider. The sequence $(e_i)_i$ is *subsymmetric* if it is unconditional and spreading. *1-subsymmetric* means 1-unconditional, that is, the norm of vectors is invariant under changing the signs of coefficients, and 1-spreading. Perhaps, a more often used notion in the theory of spreading sequences is that of *suppression unconditionality*; a property of the basis that the norm of vectors does not increase when any subset of coefficients are deleted. In this paper we are mostly interested in *conditional spreading* sequences, i.e. Schauder basic spreading sequences that are not unconditional. We say $\|\cdot\|$ is a *spreading norm* on $c_{00}(\mathbb{N})$ if the unit vector basis $(e_i)_i$ is spreading with respect to the norm $\|\cdot\|$.

Let X be a Banach space with a conditional and spreading basis $(e_i)_i$. Then the following are satisfied.

(i) The summing functional $s(\sum_i a_i e_i) = \sum_i a_i$ is well defined and bounded. If I is an interval of the natural numbers and P_I is the projection onto I associated to the basis $(e_i)_i$, we shall denote the functional $s \circ P_I$ by s_I.
(ii) Let $d_1 = e_1$ and $d_{i+1} = e_{i+1} - e_i$ for all $i \in \mathbb{N}$. The sequence $(d_i)_i$ forms a Schauder basis for X ([14, Theorem 2.3 b)]) and we shall refer to it as *the difference basis* of X.
(iii) The sequence $(e_i)_i$ is non-trivial weak Cauchy, that is, weak Cauchy and not weakly convergent.

The following two results were proved in [14, Theorem 2.3 a) and Theorem 2.8], we include their statements as we refer to them quite often in the sequel.

Proposition 2.1. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$. If $(x_i)_i$ is a block sequence of the basis with $s(x_i) = 0$ for all $i \in \mathbb{N}$, then $(x_i)_i$ is unconditional. If moreover $(e_i)_i$ is 1-spreading, then $(x_i)_i$ is suppression unconditional.*

If a sequence of successive finite intervals $(I_i)_i$ of $\mathbb{N}$ satisfies $\max I_i + 1 = \min I_{i+1}$ for all i, then we say that the I_i's are consecutive.

Proposition 2.2. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let also $\bar{I} = (I_i)_i$ be a sequence of consecutive intervals of $\mathbb{N}$. The map $P_{\bar{I}} : X \to X$ with*

$$P_{\bar{I}} x = \sum_{i=1}^{\infty} s_{I_i}(x) \left(\frac{1}{\# I_i} \sum_{j \in I_i} e_j \right)$$

is a bounded linear projection. If, moreover, $(e_i)_i$ is 1-spreading and $\cup_i I_i = \mathbb{N}$, then $\|P_{\bar{I}}\| \leqslant 3$.

Conditional jamesification One way to obtain conditional norms is the following. Let X be a Banach space with a Schauder basis $(x_i)_i$. We define a norm on $c_{00}(\mathbb{N})$, called the *conditional jamesification* of X, as follows. For $x \in c_{00}(\mathbb{N})$ set

$$\|x\|_{\tilde{J}(X)} = \max \left\{ \left\| \sum_{i=1}^{n} s_{I_i}(x) x_i \right\| \right\} \tag{3}$$

where the maximum is taken over all consecutive intervals $(I_i)_{i=1}^n$ of $\mathbb{N}$ (i.e. intervals that are successive and have no gaps between them). We also refer to the unit vector basis of $c_{00}(\mathbb{N})$ endowed with $\|\cdot\|_{\tilde{J}(X)}$ as the conditional jamesification of $(x_i)_i$. It is clear that if $(x_i)_i$ is spreading, then the unit vector basis $(e_i)_i$ of $c_{00}(\mathbb{N})$ is spreading with respect to $\|\cdot\|_{\tilde{J}(X)}$.

Remark 2.3. Note that $\|\cdot\|_{\tilde{J}(X)}$ really depends on the basis (x_i) of X and $\|\cdot\|_{\tilde{J}((x_i))}$ would perhaps be a more appropriate notation. However, in what follows the basis in question will always be explicit and will not lead to confusion.

Remark 2.4. We mention that a norm denoted by $\|\cdot\|_{J(X)}$ very similar to the above was considered in [9]. The important difference is that in the definition of $\|\cdot\|_{J(X)}$ the intervals are allowed to have gaps whereas in $\|\cdot\|_{\tilde{J}(X)}$ they are not. We shall adopt the terminology used in that paper and refer to $\|\cdot\|_{J(X)}$ and $\|\cdot\|_{\tilde{J}(X)}$ as *jamesification* and *conditional jamesificiation*, respectively.

Remark 2.5. The conditional jamesification of a subsymmetric sequence $(x_i)_i$ is equivalent to the jamesification of $(x_i)_i$. In fact, if $(x_i)_i$ is suppression unconditional then its jamesification and its conditional jamesification are isometrically equivalent.

Remark 2.6. Iterating the procedure of conditional jamesification will not lead to new norms. That is, if X is a Banach space with a Schauder basis $(x_i)_i$, then the Schauder basis $(e_i)_i$ of the conditional jamesification $\tilde{J}(X)$ of X is isometrically equivalent to its own conditional jamesification, i.e. $\tilde{J}(X) = \tilde{J}(\tilde{J}(X))$.

An easy yet important observation is that conditional spreading sequences dominate their convex block bases.

Lemma 2.7. *Let X be a Banach space with a 1-spreading basis $(e_i)_i$. Then, for every $c_1, \dots, c_n \in \mathbb{R}$ and convex block vectors $x_1, \dots, x_n$ of the basis $(e_i)_i$, we have*

$$\left\|\sum_{i=1}^n c_i x_i\right\| \leqslant \left\|\sum_{i=1}^n c_i e_i\right\|.$$

Proof. Assume that $x_k = \sum_{i\in F_k} \lambda_i^k e_i$ for $k = 1, \dots, n$ and let x^* be a functional with $\|x^*\| = 1$. Choose $i_k \in F_k$ so that $c_k x^*(e_{i_k}) = \max\{c_k x^*(e_i) : i \in F_k\}$. An easy computation yields the following:

$$x^*\left(\sum_{k=1}^n c_k x_k\right) \leqslant x^*\left(\sum_{k=1}^n c_k e_{i_k}\right) \leqslant \left\|\sum_{k=1}^n c_k e_{i_k}\right\| = \left\|\sum_{k=1}^n c_k e_k\right\|. \qquad \square$$

3. Convex block homogeneous bases

In this section we discuss the central concept introduced in this paper that is also the main tool used herein to study conditional spreading sequences. This is the notion of a convex block homogeneous sequence. Interestingly, it turns out that this is an isomorphic formulation of an isometric property, that of an equal signs additive (ESA) sequence, introduced by A. Brunel and L. Sucheston in [12].

Definition 3.1. Let X be a Banach space and $(x_i)_i$ be a Schauder basic sequence in X.

(i) If $(x_i)_i$ is equivalent to all of its convex block sequences then we say that it is convex block homogeneous.
(ii) If $(x_i)_i$ is isometrically equivalent to all of its convex block sequences then we say that it is 1-convex block homogeneous.

A convex block homogeneous sequence is clearly spreading, and if it is unconditional, then it is equivalent to the unit vector basis of ℓ_1. A 1-convex block homogeneous sequence is clearly 1-spreading. Furthermore, by a standard argument, if a basis is convex block homogeneous then there exists a constant C so that it is C-equivalent to all of its convex block sequences.

The simplest examples of convex block homogeneous bases are the unit vector basis of ℓ_1 and the summing basis of c_0. Another classical example is the boundedly complete basis of James space [16]. In fact, the jamesification of any subsymmetric sequence yields a convex block homogeneous basis. In a later section we investigate whether these are the only possible convex block homogeneous bases.

The first result of this section is a characterization of convex block homogeneous bases.

Proposition 3.2. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$. The following statements are equivalent.*

(i) *The basis $(e_i)_i$ is equivalent to its conditional jamesification (see (3), page 1212).*
(ii) *The sequence $(s_{[1,n]})_n$ is spreading, i.e. equivalent to its subsequences (for the definition of $(s_{[1,n]})_n$ see page 1212).*
(iii) *The basis $(e_i)_i$ is convex block homogeneous.*
(iv) *Every block sequence $(x_n)_n$ of averages of the basis is equivalent to $(e_i)_i$.*

Proof. We may assume without loss of generality that $(e_i)_i$ is 1-spreading.

(i) ⇒ (ii): Choose $C > 0$ such that for every $x \in c_{00}$ and consecutive intervals $(I_i)_{i=1}^n$ of $\mathbb{N}$ we have $\|\sum_{i=1}^n s_{I_i}(x)e_i\| \leqslant C\|x\|$. Let now $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $k_1 < \cdots < k_n \in \mathbb{N}$. An easy argument using the spreading property of the basis $(e_i)_i$ yields that $\|\sum_{i=1}^n \lambda_i s_{[1,i]}\| \leqslant \|\sum_{i=1}^n \lambda_i s_{[1,k_i]}\|$. Let now $x \in c_{00}(\mathbb{N})$ with $\|x\| = 1$, set $I_1 = \{1, \dots, k_1\}, I_i = \{k_{i-1}+1, \dots, k_i\}$ for $i \geqslant 2$ and set $\tilde{x} = \sum_{i=1}^n s_{I_i}(x)e_i$. We have that $\|\tilde{x}\| \leqslant C$. A calculation yields the following:

$$\sum_{i=1}^n \lambda_i s_{[1,k_i]}(x) = \sum_{i=1}^n \lambda_i s_{[1,i]}(\tilde{x}) \leqslant C \left\| \sum_{i=1}^n \lambda_i s_{[1,i]} \right\|.$$

(ii) ⇒ (iii): Choose $C > 0$ so that for every $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $k_1 < \cdots < k_n \in \mathbb{N}$ we have $\|\sum_{i=1}^n \lambda_i s_{[1,k_i]}\| \leqslant C\|\sum_{i=1}^n \lambda_i s_{[1,i]}\|$. Let $(x_k)_k$ be a convex block sequence of the basis, where for each k we have $x_k = \sum_{i\in F_k} \lambda_i^k e_i$ and let $c_1, \dots, c_n \in \mathbb{R}$. By Lemma 2.7 we immediately obtain that $\|\sum_{i=1}^n \lambda_i x_i\| \leqslant \|\sum_{i=1}^n \lambda_i e_i\|$. On the other hand, let $x^* = \sum_{k=1}^m \mu_k s_{[1,k]}$, with $\|x^*\| = 1$. Set $j_k = \max F_k$. If $y^* = \sum_{k=1}^m \mu_k s_{[1,j_k]}$, then $\|y^*\| \leqslant C$. A calculation yields the following.

$$x^*\left(\sum_{k=1}^n c_k e_k\right) = \sum_{k=1}^n c_k x^*(e_k) = \sum_{k=1}^n c_k y^*(x_k) \leqslant C\left\|\sum_{k=1}^n c_k x_k\right\|.$$

(iii) ⇒ (iv): This is trivial.

(iv) ⇒ (i): For simplicity, we assume that $(e_i)_i$ is bimonotone. Choose $C > 0$ such that for every $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and block vectors $x_1, \dots, x_n$ that are averages of the basis, we have that $\|\sum_{i=1}^n \lambda_i e_i\| \leqslant C\|\sum_{i=1}^n \lambda_i x_i\|$. Let $x \in c_{00}(\mathbb{N})$ and $I_1 < \cdots < I_n$ be consecutive intervals of $\mathbb{N}$. Set $x_i = (\#I_i)^{-1}\sum_{j\in I_i} e_j$ and let $\tilde{x}$ the restriction of x onto the interval $\cup_i I_i$. Then $\|\tilde{x}\| \leqslant \|x\|$. Using Proposition 2.2 we conclude:

$$\begin{aligned}\left\|\sum_{i=1}^n s_{I_i}(x)e_i\right\| &\leqslant C\left\|\sum_{i=1}^n s_{I_i}(x)x_i\right\| = C\left\|\sum_{i=1}^n s_{I_i}(\tilde{x})\left(\frac{1}{\#I_i}\sum_{j\in I_i} e_j\right)\right\| \\ &\leqslant 3C\|\tilde{x}\| \leqslant 3C\|x\|. \qquad \square\end{aligned}$$

Equal signs additive (ESA) bases The following isometric definitions are from [12, p. 288]. As it is proved there, [12, Lemma 1], they are equivalent.

Definition 3.3. Let X be a Banach space with a Schauder basis $(e_i)_i$.

(i) The basis $(e_i)_i$ is called equal signs additive, if for any sequence of scalars $(a_k)_k$, for any natural number k so that $a_k a_{k+1} \geqslant 0$:

$$\left\|\sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1})e_k + \sum_{i=k+2}^{\infty} a_i e_i\right\| = \left\|\sum_{i=1}^{\infty} a_i e_i\right\|.$$

(ii) The basis $(e_i)_i$ is called subadditive, if for any sequence of scalars $(a_k)_k$, for any natural number k:

$$\left\|\sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1})e_k + \sum_{i=k+2}^{\infty} a_i e_i\right\| \leqslant \left\|\sum_{i=1}^{\infty} a_i e_i\right\|.$$

As it turns out, convex block homogeneity characterizes sequences that admit equivalent equal signs additive norms. Actually, a sequence is equal signs additive if and only if it is 1-convex block homogeneous.

Proposition 3.4. *Let X be a Banach space with a Schauder basis $(e_i)_i$. Then, the sequence $(e_i)_i$ is equal signs additive if and only if it is 1-convex block homogeneous.*

Proof. Assume that $(e_i)_i$ is 1-convex block homogeneous and let $(a_i)_i$ be a sequence of scalars. Let $k \in \mathbb{N}$ with $a_k a_{k+1} \geqslant 0$. We will show that Definition 3.3 (i) is satisfied. If $a_k + a_{k+1} = 0$ then this is trivial. Otherwise, define the sequence $(x_i)_i$ with $x_i = e_i$ for $i < k$, $x_k = (a_k/(a_k + a_{k+1}))e_k + (a_{k+1}/(a_k + a_{k+1}))e_{k+1})$ and $x_i = e_{i+1}$ for $i > k$. Then, $(x_i)_i$ is a convex block sequence of the basis and hence isometrically equivalent to it. This yields

$$\begin{aligned}&\left\|\sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1})e_k + \sum_{i=k+2}^{\infty} a_i e_i\right\| \\ &= \left\|\sum_{i=1}^{k-1} a_i x_i + (a_k + a_{k+1})x_k + \sum_{i=k+2}^{\infty} a_i x_i\right\| = \left\|\sum_{i=1}^{k+1} a_i e_i + \sum_{i=k+2}^{\infty} a_i e_{i+1}\right\| \\ &= \left\|\sum_{i=1}^{\infty} a_i e_i\right\| \text{ (by the spreading property of } (e_i)_i).\end{aligned}$$

Conversely, assume that $(e_i)_i$ is equal signs additive. A finite induction on $m \in \mathbb{N}$ applied to Definition 3.3 (i) yields that for $k \in \mathbb{N}$ so that $a_k, \dots, a_{k+m}$ are either all non-negative or all non-positive we have

$$\left\|\sum_{i=1}^{k-1} a_i e_i + (a_k + \cdots + a_{k+m})e_k + \sum_{i=k+m+1}^{\infty} a_i e_i\right\| = \left\|\sum_{i=1}^{\infty} a_i e_i\right\|.$$

This implies that if $(E_k)_k$ is a partition of $\mathbb{N}$ into successive intervals, then for any choice of scalars so that for each $k \in \mathbb{N}$ the scalars a_i, $i \in E_k$ are either all non-negative or all non-positive we have

$$\left\|\sum_{k=1}^{\infty}\left(\sum_{i\in E_k} a_i\right) e_k\right\| = \left\|\sum_{i=1}^{\infty} a_i e_i\right\|. \tag{4}$$

Now, if $(x_k)_k$ is a convex block sequence of $(e_i)_i$, $(E_k)_k$ is a partition of $\mathbb{N}$ into successive intervals of $\mathbb{N}$, and $(c_i)_{i\in E_k}$, $k \in \mathbb{N}$ are finite sequences of non-negative scalars summing up to one with $x_k = \sum_{i\in E_k} c_i e_i$, then for any choice of scalars $(a_k)_k$ we have

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} a_k e_k\right\| &= \left\|\sum_{k=1}^{\infty}\left(a_k \sum_{i\in E_k} c_i\right) e_k\right\| = \left\|\sum_{k=1}^{\infty} a_k \left(\sum_{i\in E_k} c_i e_i\right)\right\| \text{ (by (4))} \\ &= \left\|\sum_{k=1}^{\infty} a_k x_k\right\|,\end{aligned}$$

i.e. $(x_k)_k$ isometrically equivalent to the basis $(e_i)_i$ and the proof is complete. □

Corollary 3.5. *Let X be a Banach space with a convex block homogeneous basis $(e_i)_i$. Then X admits an equivalent norm with respect to which $(e_i)_i$ is bimonotone and equal signs additive. In particular, X admits an equivalent norm with which $(e_i)_i$ is bimonotone and 1-convex block homogeneous.*

Proof. By passing to a first equivalent norm we may assume that $(e_i)_i$ is 1-spreading. We then pass to the conditional jamesification of X which, by Proposition 3.2 is equivalent to the initial norm. It is also clear that this is a bimonotone norm and also that it is 1-spreading. By Remark 2.6 one can easily deduce that the norm is also subadditive and hence, by [12, Lemma 1] it is equal signs additive. By Proposition 3.4 the sequence $(e_i)_i$ is 1-convex block homogeneous. □

As a consequence we obtain the following equivalence between these notions as mentioned above.

Theorem 3.6. *Let X be a Banach space with a Schauder basis $(e_i)_i$. The following are equivalent.*

(i) *The basis $(e_i)_i$ is convex block homogeneous.*
(ii) *The space X admits an equivalent norm with respect to which $(e_i)_i$ is equal signs additive.*

Remark 3.7. Similar arguments as those used above imply that the basis $(e_i)_i$ of X is bimonotone and equal signs additive if and only if it is 1-spreading and equal to its conditional jamesification in the sense of (3).

The following result is of no interest from an isomorphic scope, however it removes the annoying factor three in Proposition 2.2 in case one has a 1-convex block homogeneous basis instead of just a 1-spreading Schauder basic sequence.

Proposition 3.8. *Let X be a Banach space with a 1-convex block homogeneous Schauder basis $(e_i)_i$ and let $\bar{I} = (I_k)_k$ be a partition of $\mathbb{N}$ into successive intervals of natural numbers. Then, the map $P_{\bar{I}} : X \to X$ with*

$$P_{\bar{I}}x = \sum_{k=1}^{\infty} s_{I_k}(x)\left(\frac{1}{\#I_k}\sum_{i\in I_k} e_i\right)$$

is a norm-one linear projection.

Proof. By Proposition 2.2 $P_{\bar{I}}$ is indeed bounded and it has norm at most three. By [12, Lemma 1] the basis $(e_i)_i$ satisfies Definition 3.3 (ii). An argument identical to that used in the of Proposition 3.4 yields that for any scalars $(a_i)_i$ we have

$$\left\|\sum_{k=1}^{\infty}\left(\sum_{i\in I_k} a_i\right) e_k\right\| \leqslant \left\|\sum_{i=1}^{\infty} a_i e_i\right\|. \tag{5}$$

Let now $x = \sum_{i=1}^{\infty} a_i e_i$ be a vector in X. Then,

$$\begin{aligned} \|P_{\bar{I}} x\| &= \left\| \sum_{k=1}^{\infty} s_{I_k}(x) \left(\frac{1}{\# I_k} \sum_{i \in I_k} e_i \right) \right\| \\ &\leqslant \left\| \sum_{k=1}^{\infty} s_{I_k}(x) e_k \right\| \text{ (by Lemma 2.7)} \\ &= \left\| \sum_{k=1}^{\infty} \left(\sum_{i \in I_k} a_i \right) e_k \right\| \leqslant \left\| \sum_{i=1}^{\infty} a_i e_i \right\| = \|x\| \text{ (by (5))}. \qquad \square \end{aligned}$$

We record the following duality result that can be proved directly without the use of equal signs additive bases, however, this tool makes the proof immediate.

Proposition 3.9. *Let X be a Banach space with a spreading basis $(e_i)_i$. Then, $(e_i)_i$ is convex block homogeneous if and only if $(s_{[1,n]})_n$ is convex block homogeneous.*

Proof. If $(s_{[1,n]})_n$ is convex block homogeneous then it is spreading and hence by Proposition 3.2 (ii) ⇒ (iii) $(e_i)_i$ is convex block homogeneous. If, on the other hand, $(e_i)_i$ is convex block homogeneous then by Corollary 3.5 we may assume that it is equal signs additive. By [12, Proposition 2, page 291] $(s_{[1,n]})_{n\geqslant 2}$ is equal signs additive and hence, convex block homogeneous. By Proposition 3.2 (iii) ⇒ (ii) $(s_{[1,n]})_n$ is equivalent to $(s_{[1,n]})_{n\geqslant 2}$. □

4. A characterization: decomposing conditional spreading norms

We devote this section to the statement and proof of the main result used throughout the paper, namely the decomposition of conditional spreading norms into two parts. The first part is subsymmetric and it is simply defined by taking skipped successive differences of the basis, whereas the second part is convex block homogeneous and it is obtained by taking averages of the basis that are growing sufficiently rapidly.

Theorem 4.1. *Let X be a Banach space with a conditional 1-spreading basis $(e_i)_i$. Let also $(u_i)_i$ denote the subsymmetric skipped difference sequence of $(e_i)_i$, i.e. $u_i = e_{2i} - e_{2i-1}$ for all $i \in \mathbb{N}$. Then, there exists a Banach space Z with a 1-convex block homogeneous basis $(z_i)_i$ so that for all scalars $a_1, \ldots, a_n$ we have*

$$\begin{aligned} \frac{1}{2} \max \left\{ \left\| \sum_{i=1}^{n} a_i u_i \right\|, \left\| \sum_{i=1}^{n} a_i z_i \right\| \right\} &\leqslant \left\| \sum_{i=1}^{n} a_i e_i \right\| \\ &\leqslant 2 \max \left\{ \left\| \sum_{i=1}^{n} a_i u_i \right\|, \left\| \sum_{i=1}^{n} a_i z_i \right\| \right\}. \end{aligned} \tag{6}$$

Furthermore, the sequence $(u_i)_i$ is suppression unconditional and the norm on Z is given by

$$\begin{aligned}\left\|\sum_{i=1}^{n} a_i z_i\right\| &= \lim_{N\to\infty}\left\|\sum_{i=1}^{n} a_i\left(\frac{1}{N}\sum_{j=(i-1)N+1}^{iN} e_j\right)\right\| \\ &= \inf\left\{\left\|\sum_{i=1}^{n} a_i w_i\right\| : (w_i)_{i=1}^{n} \text{ is a conv. block seq. of } (e_i)_i\right\}.\end{aligned} \tag{7}$$

Remark 4.2. It is useful to restate the theorem in an isomorphic setting as follows. Let X be a Banach space with a conditional spreading basis $(e_i)_i$. The sequence $(e_i)_i$ may be assumed to be 1-spreading. Let $(u_i)_i$ and $(z_i)_i$ be as above, and set $U = [(u_i)_i]$ and $Z = [(z_i)_i]$. Then, the theorem asserts that the map

$$\begin{aligned} T &: X \to U \oplus Z \\ Te_i &= (u_i, z_i) \end{aligned}$$

is an isomorphic embedding. In this decomposition we shall refer to $(u_i)_i$ and $(z_i)_i$ as unconditional and convex block homogenous parts of $(e_i)_i$, respectively.

Moreover, as we shall observe in next section, (z_i) is equivalent to a block basis of (e_i) (Proposition 5.1), and it is unique in the following sense. If $(\tilde{z}_i)$ is a convex block homogeneous sequence spanning a space $\tilde{Z}$ so that the map $\tilde{T} : X \to U \oplus \tilde{Z}$ defined by $\tilde{T}e_i = (u_i, \tilde{z}_i)$ is an isomorphic embedding, then $(\tilde{z}_i)$ is equivalent to (z_i).

The proof of Theorem 4.1, which is postponed until the end of the section, is based on two lemmas given below.

Lemma 4.3. *Let X be a Banach space with a 1-spreading basis $(e_i)_i$. Then, for every scalars $c_1, \dots, c_n$ and convex block vectors $x_1, \dots, x_n$ of the basis:*

$$\left\|\sum_{k=1}^{n} c_k e_k\right\| \leqslant \left\|\sum_{k=1}^{n} c_k x_k\right\| + \left\|\sum_{k=1}^{n} c_k (e_{2k-1} - e_{2k})\right\|.$$

Proof. Assume that $x_k = \sum_{j\in F_k} \lambda_j^k e_j$ for $k = 1, \dots, n$. Using the spreading property of the basis, we may clearly assume that there exist natural numbers $i_1 < \cdots < i_n$ with $i_1 < F_1 < i_2 < F_2 < \cdots < i_n < F_n$. Let x^* be a functional with $\|x^*\| = 1$. Choose $j_k \in F_k$ with $c_k x^*(e_{j_k}) = \min\{c_k x^*(e_j) : j \in F_k\}$. An easy computation yields the following.

$$\begin{aligned} x^*\left(\sum_{k=1}^{n} c_k e_{i_k}\right) &= x^*\left(\sum_{k=1}^{n} c_k (e_{i_k} - e_{j_k})\right) + x^*\left(\sum_{k=1}^{n} c_k e_{j_k}\right) \\ &\leqslant x^*\left(\sum_{k=1}^{n} c_k (e_{i_k} - e_{j_k})\right) + x^*\left(\sum_{k=1}^{n} c_k x_k\right) \end{aligned}$$

$$\leqslant \left\| \sum_{k=1}^{n} c_k (e_{i_k} - e_{j_k}) \right\| + \left\| \sum_{k=1}^{n} c_k x_k \right\|$$
$$= \left\| \sum_{k=1}^{n} c_k (e_{2k-1} - e_{2k}) \right\| + \left\| \sum_{k=1}^{n} c_k x_k \right\|.$$

The fact that $(e_i)_i$ is 1-spreading finishes the proof. □

Lemma 4.4. *Let X be a Banach space with a 1-spreading basis $(e_i)_i$ and let $w_1, \dots, w_n$ be convex block vectors of the basis. Then for every $\varepsilon > 0$ there exists $M_0 \in \mathbb{N}$, such that for every $c_1, \dots, c_n \in [-1, 1]$ and block averages of the basis $y_1, \dots, y_n$, with $\# \operatorname{supp} y_k \geqslant M_0$ for $k = 1, \dots, n$, the following holds:*

$$\left\| \sum_{k=1}^{n} c_k y_k \right\| < \left\| \sum_{k=1}^{n} c_k w_k \right\| + \varepsilon.$$

Proof. Allowing some error, say $\varepsilon/2$, we may assume that the w_k's have rational coefficients, i.e. there are some $N \in \mathbb{N}$, successive sets $H_1, \dots, H_n$ and $(n_j)_{j \in H_k}$ so that $\sum_{j \in H_k} (n_j/N) = 1$ and $w_k = \sum_{j \in H_k} (n_j/N) e_j$ for $k = 1, \dots, n$. Note that $N = \sum_{j \in H_k} n_j$ for $k = 1, \dots, n$. Choose $M_0 > 4n\varepsilon^{-1}N$.

Let now $y_k = (1/m_k) \sum_{i \in G_k} e_i$, with $\#G_k = m_k$, $m_k \geqslant M_0$, for $k = 1, \dots, n$ and $G_1 < \dots < G_n$. Set $r_k = \lfloor m_k/N \rfloor$, choose $G'_k \subset G_k$ with $\#G'_k = N r_k$ and set $y'_k = (1/(N r_k)) \sum_{j \in G'_k} e_j$. Some computations yield that $\|y_k - y'_k\| < \varepsilon/(2n)$. It is therefore sufficient to prove that $\| \sum_{k=1}^{n} c_k y'_k \| \leqslant \| \sum_{k=1}^{n} c_k w_k \|$.

Partition each G'_k into further successive sets $(E_j^k)_{j \in H_k}$ with $\#E_j^k = r_k n_j$ and define $z_j^k = (1/n_j r_k) \sum_{i \in E_j^k} e_i$ for $k = 1, \dots, n$ and $j \in H_k$.

Applying Lemma 2.7 we finally conclude the following:

$$\left\| \sum_{k=1}^{n} c_k y'_k \right\| = \left\| \sum_{k=1}^{n} c_k \left(\sum_{j \in H_k} \frac{r_k n_j}{N r_k} z_j^k \right) \right\| \leqslant \left\| \sum_{k=1}^{n} c_k \sum_{j \in H_k} \frac{n_j}{N} e_j \right\| = \left\| \sum_{k=1}^{n} c_k w_k \right\|. \qquad \square$$

Proof of Theorem 4.1. Lemma 4.4 clearly yields that any pair of block sequences $(x_k)_k, (y_k)_k$ of averages of the basis with $\# \operatorname{supp}(x_k), \# \operatorname{supp}(y_k)$ tending to infinity must admit the same spreading model. We name this spreading model $(z_i)_i$ and denote its closed linear span Z. Lemma 4.4 also clearly implies (7). The second line of (7) and Lemma 2.7 easily imply that $(z_i)_i$ is 1-convex block homogeneous.

To finish the proof we need to show (6). It follows from Lemmas 2.7 and 4.3 that for all real scalars $(c_k)_{k=1}^n$

$$\left\| \sum_{k=1}^{n} c_k z_k \right\| \leqslant \left\| \sum_{k=1}^{n} c_k e_k \right\| \leqslant \left\| \sum_{k=1}^{n} c_k z_k \right\| + \left\| \sum_{k=1}^{n} c_k (e_{2k-1} - e_{2k}) \right\|,$$

which implies conclusion. □

Remark 4.5. It is tempting to conjecture that every convex block homogeneous norm is equivalent to the jamesification of a subsymmetric norm, and consequently every conditional spreading norm is up to equivalence generated by two subsymmetric norms. However, this turned out to be false. We present a counterexample in Section 13.

5. Block sequences of conditional spreading bases

This section is centered around understanding the structure of block sequences in a space with a conditional spreading basis. Although the basic statements are included in the proposition below, more precise information is given the subsequent lemmas.

Proposition 5.1. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(u_i)_i$, $(z_i)_i$ be the unconditional and convex block homogeneous parts of $(e_i)_i$ respectively. The following hold.*

(i) *There exists a block sequence $(\tilde{z}_i)_i$ of averages of the basis $(e_i)_i$ that is equivalent to $(z_i)_i$.*
(ii) *The closed linear span of every convex block sequence $(x_i)_i$ of the basis $(e_i)_i$ that is equivalent to $(z_i)_i$ is complemented in X. In particular, $Z = [(z_i)_i]$ is isomorphic to a complemented subspace of X.*
(iii) *Every convex block sequence $(x_i)_i$ of the basis has a subsequence that is equivalent to either $(e_i)_i$ or to $(z_i)_i$.*
(iv) *Every convex block sequence $(x_i)_i$ of the basis $(e_i)_i$ has a further convex block sequence that is equivalent to $(z_i)_i$.*
(v) *Every seminormalized block sequence $(x_i)_i$ of the basis $(e_i)_i$ either has a subsequence that is unconditional or it has a convex block sequence that is equivalent to $(z_i)_i$.*

The following lemma is well known. Its proof, which we omit, is based based on a counting argument.

Lemma 5.2. *Let $(u_i)_i$ be a subsymmetric sequence in some Banach space, that is not equivalent to the unit vector basis of ℓ_1. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any real numbers $(a_i)_i$ with $\sum_i |a_i| \leqslant 1$ and $\sup_i |a_i| < \delta$ we have that $\|\sum_i a_i u_i\| < \varepsilon$.*

The following lemma proves Proposition 5.1 (i).

Lemma 5.3. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. Then, there exists a sequence of natural numbers $(n_i)_i$ so that for every sequence $(A_i)_i$ of successive subsets of $\mathbb{N}$ with $\#A_i \geqslant n_i$ for all $i \in \mathbb{N}$ the following holds: if $\tilde{z}_i = (1/\#A_i)\sum_{j\in A_i} e_j$ for $i \in \mathbb{N}$, then the sequence $(\tilde{z}_i)_i$ is equivalent to $(z_i)_i$.*

Proof. Let $(u_i)_i$ be the unconditional part of $(e_i)_i$ and applying Lemma 5.2, for every $i \in \mathbb{N}$ choose $n_i \in \mathbb{N}$ so that for every convex combination u of $(u_i)_i$ with $\|u\|_\infty \leqslant n_i^{-1}$, we have that $\|u\| < 2^{-i}$.

Let now $(A_i)_i$ be successive subsets of $\mathbb{N}$ with $\#A_i \geqslant n_i$ for all $i \in \mathbb{N}$ and set $\tilde{z}_i = (1/\#A_i)\sum_{j\in A_i} e_j$ for $i \in \mathbb{N}$. Let T be the isomorphic embedding from Remark 4.2. By the choice of the sequence $(n_i)_i$ we have that $\sum_i \|P_{U_X} T\tilde{z}_i\| < 1$ which implies that $(\tilde{z}_i)_i$ is equivalent to $(P_{Z_X} T\tilde{z}_i)_i$ which is a convex block sequence of $(z_i)_i$ and hence equivalent to $(z_i)_i$. □

Remark 5.4. The proof of Lemma 5.3 implies that the sequence $(z_i)_i$ is unique in the sense explained in Remark 4.2. Indeed, if $(w_i)_i$ is a convex block homogeneous sequence so that the map $\tilde{T} : X \to U \oplus \tilde{W}$ with $W = [(w_i)_i]$ and $\tilde{T}e_i = (u_i, w_i)$ is an isomorphic embedding, repeating the argument from the proof of Lemma 5.3, if $(\tilde{z}_i)_i$ is as in the statement of that lemma, we conclude that $(\tilde{z}_i)_i$ is equivalent to $(z_i)_i$ as well as to $(w_i)_i$.

The following Lemma proves Proposition 5.1 (ii).

Lemma 5.5. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. Let also $(x_i)_i$ be a convex block sequence of the basis which is equivalent to $(z_i)_i$. Then for every sequence of consecutive intervals $(I_i)_i$ of $\mathbb{N}$ with $\operatorname{supp} x_i \subset I_i$ for all i, we have that the map $P : X \to X$ with $Px = \sum_{i=1}^{\infty} s_{I_i}(x)x_i$ is a bounded linear projection.*

Proof. We may assume that $(e_i)_i$ is 1-spreading. If we define for $i \in \mathbb{N}$ the vector $y_i = (1/\#I_i)\sum_{j\in I_i} e_j$ then Proposition 2.2 yields that the map $P_{\bar{I}} : X \to X$ with $P_{\bar{I}}x = \sum_{i=1}^{\infty} s_{I_i}(x)y_i$ is a bounded linear projection. Observe that by (7) the map $R : [(y_i)] \to [(z_i)_i]$ with $Ry_i = z_i$ has norm one. We easily conclude that if $S : [(z_i)] \to [(x_i)]$ is the isomorphism given by $Sz_i = x_i$ then $Q = SRP$ is a bounded linear projection. □

Remark 5.6. As it is implied by Proposition 5.1 (ii), if X is Banach space with a convex block homogeneous basis $(e_i)_i$, then every convex block sequence of the basis spans a complemented subspace of X. In particular, the space spanned by every subsequence of the basis is complemented, without the basis being unconditional.

Proof of Proposition 5.1 (iii) and (iv). We first observe that statements (i) and (iii) of Proposition 5.1 immediately imply statement (iv), i.e. we only need to prove statement (iii). Let $(x_i)_i$ be a convex block sequence of $(e_i)_i$ and let T be the isomorphic embedding from Remark 4.2. Assume first that $\|x_i\|_\infty \to 0$. Lemma 5.2 implies that $\|P_{U_X} Tx_i\| \to 0$ and therefore $(x_i)_i$ has a subsequence equivalent to $(P_{Z_X} Tx_i)_i$ which is a convex block sequence of $(z_i)_i$ and hence equivalent to $(z_i)_i$.

Otherwise we may assume that there exists $\varepsilon > 0$ such that $\|x_i\|_\infty > \varepsilon$ for all $i \in \mathbb{N}$. In this case if $v_i = P_{U_X} Tx_i$ for $i \in \mathbb{N}$, then $(v_i)_i$ is a convex block sequence of $(u_i)_i$ with $\|v_i\|_\infty > \varepsilon$ for all $i \in \mathbb{N}$. Lemma 2.7 and unconditionality imply that $(v_i)_i$ is equivalent to $(u_i)_i$. Moreover, if $w_i = P_{Z_X} Tx_i$, then $(w_i)_i$ is a convex block sequence of $(z_i)_i$ and hence equivalent to $(z_i)_i$. We conclude that $(v_i, w_i)_i$ is equivalent to $(u_i, z_i)_i$ which is equivalent to $(e_i)_i$. Since $Tx_i = (v_i, w_i)$ for all $i \in \mathbb{N}$ and T is an isomorphic embedding, we have that $(x_i)_i$ is equivalent to $(e_i)_i$. □

Proof of Proposition 5.1 (v). If $(x_i)_i$ has a subsequence equivalent to the unit vector basis of ℓ_1, then obviously there is nothing more to prove. We may therefore assume that $(x_i)_i$ is weak Cauchy.

If $\lim_i s(x_i) = 0$, then passing to a subsequence and perturbing we may assume that $s(x_i) = 0$ for all $i \in \mathbb{N}$. By Proposition 2.1 $(x_i)_i$ is then unconditional. Otherwise, by passing to some subsequence of $(x_i)_i$, perturbing and scaling, we may assume that $s(x_i) = 1$ for all $i \in \mathbb{N}$. Choose $k_i \in \operatorname{supp} x_i$ for all $i \in \mathbb{N}$ and set $y_i = x_i - e_{k_i}$. Observe that $s(y_i) = 0$ for all $i \in \mathbb{N}$ and therefore by Proposition 2.1 $(y_i)_i$ is unconditional. Moreover, as both $(x_i)_i$ and $(e_i)_i$ are weak Cauchy, the same is true for $(y_i)_i$, which implies that $(y_i)_i$ is weakly null. Using Mazur's Theorem we conclude that there exists a convex block sequence of $(x_i)_i$ which is equivalent to some convex block sequence of $(e_i)_i$. Finally, Proposition 5.1 (iv) yields that (ii) is satisfied. □

Remark 5.7. In [11] it is shown that for every non-superreflexive Banach space X there is a space with an ESA basis that is finitely representable in X. Interestingly, their proof implies that if $(e_i)_i$ is a conditional spreading basis and $(z_i)_i$ is its convex block homogeneous part then $(z_i)_i$ is finitely representable in $(e_i)_i$. This may be viewed as a precursor of Proposition 5.1 (i).

6. Characterizing strongly summing conditional spreading sequences

As it will become clear in the next section, within the class of conditional spreading sequences there is a distinction between those that are strongly summing and those that are not. When proving various results, each case may have to be treated separately and sometimes they satisfy different properties. In this relatively brief section we prove a useful criterion for deciding when a given conditional spreading sequence is strongly summing. The concept of a strongly summing sequence is due to H. P. Rosenthal and it first appeared in [21, Definition 1.1]

Definition 6.1. A Schauder basic sequence $(e_i)_i$ is called strongly summing, if for every sequence of real numbers $(a_i)_i$ such that the sequence $(\|\sum_{i=1}^n a_i e_i\|)_n$ is bounded, the real series $\sum_i a_i$ is convergent.

The following is proved in [21, Theorem 1.1].

Theorem 6.2. *Let $(x_i)_i$ be a non-trivial weak Cauchy sequence in some Banach space. Then one of the following holds.*

(i) *There exists a subsequence of $(x_i)_i$ that is strongly summing.*
(ii) *There exists a convex block sequence of $(x_i)_i$ that is equivalent to the summing basis of c_0.*

Lemma 6.3. *Let X be a Banach space with a strongly summing conditional spreading basis $(e_i)_i$ and let $x^{**} \in X^{**}$. If the series $\sum_i x^{**}(e_i^*)e_i$ does not converge in norm, then there exists a strictly increasing sequence of natural numbers $(n_i)_i$, such that if $y_i = \sum_{j=1}^{n_i} x^{**}(e_j^*)e_j$ for all $i \in \mathbb{N}$, we have that $(y_i)_i$ is equivalent to the summing basis of c_0.*

Proof. Since $(e_i)_i$ is strongly summing we have that the series $\sum_i x^{**}(e_i^*)$ is convergent. Combining this with the fact that the series $\sum_i x^{**}(e_i^*)e_i$ does not converge in norm, we may choose a strictly increasing sequence of natural numbers $(n_i)_i$, such that if $x_i = \sum_{j=n_{i-1}+1}^{n_i} x^{**}(e_j^*)e_j$ then $(x_i)_i$ is seminormalized and $\sum_i |s(x_i)| < \infty$. Then, by Proposition 2.1, the sequence $(y_i)_i$ with $y_i = x_i - s(x_i)e_{\min \operatorname{supp} x_i}$ is unconditional. As the sequences $(x_i)_i$ and $(y_i)_i$ are equivalent (or at least, they have equivalent tails), $(x_i)_i$ must be unconditional as well. Since the sequence $(\|\sum_{j=1}^{i} x_i\|)_i$ is bounded we conclude that $(x_i)_i$ is equivalent to the unit vector basis of c_0 and hence, if $y_i = \sum_{j=1}^{i} x_i = \sum_{j=1}^{n_i} x^{**}(e_j^*)e_j$, we have that $(y_i)_i$ is equivalent to the summing basis of c_0. □

Lemma 6.4. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. Let $s_{[1,i]} = \sum_{j=1}^{i} e_j^*$ and $\overline{s}_{[1,i]} = \sum_{j=1}^{i} z_j^*$ for all $i \in \mathbb{N}$. Then, there exist a sequence of natural numbers $(n_i)_i$ so that for every strictly increasing sequence of natural numbers $(k_i)_i$ with $k_i - k_{i-1} \geqslant n_i$ for all $i \in \mathbb{N}$ (where $k_0 = 0$), the sequence $(s_{[1,k_i]})_i$ is equivalent to $(\overline{s}_{[1,i]})_i$. In particular, every subsequence of $(s_{[1,i]})_i$ has a further subsequence equivalent to $(\overline{s}_{[1,i]})_i$.*

Proof. Let $(n_i)_i$ be the sequence provided by Lemma 5.3 and let $(k_i)_i$ be a strictly increasing sequence of natural numbers with $k_i - k_{i-1} \geqslant n_i$ for all $i \in \mathbb{N}$. Set $I_i = \{k_{i-1}+1, \dots, k_i\}$, $x_i = (1/\#I_i)\sum_{j\in I_i} e_j$ and $x_i^* = \sum_{j\in I_i} e_j^*$ for all $\in \mathbb{N}$. By the choice of the sequence $\bar{I} = (I_i)_i$ we have that $(x_i)_i$ is equivalent to $(z_i)_i$ and the map $P : X \to X$ with $P_{\bar{I}}x = \sum_{i=1}^{\infty} s_{I_i}(x)x_i$ is a bounded linear projection. Observe that $P_{\bar{I}}^* x_i^* = x_i^*$ for all $i \in \mathbb{N}$ and that $x_i^*(x_j) = \delta_{i,j}$ for all $i, j \in \mathbb{N}$.

The above imply that $(x_i^*)_i$ is equivalent to $(z_i^*)_i$. Finally, observe that $\sum_{j=1}^{i} x_j^* = s_{[1,k_i]}$ for all $i \in \mathbb{N}$. □

The following is the main result of this section.

Proposition 6.5. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. The following assertions are equivalent.*

(i) *The space X does not embed into a space with an unconditional basis.*
(ii) *The sequence $(z_i)_i$ is not equivalent to the summing basis of c_0.*
(iii) *The basis $(e_i)_i$ is strongly summing.*
(iv) *The sequence $(s_{[1,n]})_n$ is weak Cauchy.*
(v) *For every sequence of real numbers $(a_i)_i$ so that $(\sum_{i=1}^{n} a_i e_i)_n$ is bounded, the series $\sum_i a_i e_i$ is weak Cauchy.*

Proof. (i) ⇒ (ii): If $(z_i)_i$ is equivalent to the summing basis of c_0, then by Remark 4.2 we have that X embeds into $U \oplus c_0$ which has an unconditional basis.

(ii) ⇒ (iii): If $(z_i)_i$ is not equivalent to the summing basis of c_0, Proposition 5.1 (iv) yields that no convex block sequence of $(e_i)_i$ can be equivalent to the summing basis of c_0. The spreading property of $(e_i)_i$ and Theorem 6.2 yield that $(e_i)_i$ is strongly summing.

(iii) ⇒ (i): Assume that the basis $(e_i)_i$ is strongly summing and X embeds into a space with an unconditional basis. It is well known that a non-trivial weak Cauchy sequence in a space with an unconditional basis has a convex block sequence equivalent to the summing basis of c_0. Hence, this is true for $(e_i)_i$. It is straightforward to check that if a sequence, in this case $(e_i)_i$, has a convex block sequence equivalent to the summing basis of c_0, then it cannot be strongly summing.

(iv) ⇒ (ii): Assume that $(s_{[1,i]})_i$ is weak Cauchy. If $(z_i)_i$ is equivalent to the summing basis of c_0, Lemma 6.4 implies that $(s_{[1,i]})_i$ has a subsequence equivalent to the unit vector basis of ℓ_1 and it cannot be weak Cauchy.

(iii) ⇒ (iv): Assume that the basis $(e_i)_i$ is strongly summing, we will show that $(s_{[1,i]})_i$ is weak Cauchy. Let $x^{**} \in X^{**}$. The fact that $(e_i)_i$ is strongly summing implies that the series $\sum_j x^{**}(e_j^*)$ is convergent. Observe that $x^{**}(s_{[1,i]}) = \sum_{j=1}^{i} x^{**}(e_j^*)$ for all $i \in \mathbb{N}$ and therefore the sequence $(x^{**}(s_{[1,i]}))_i$ is convergent.

(v) ⇒ (iv): Let $x^{**} \in X^{**}$ and set $a_i = x^{**}(e_i^*)$. Then $(x^{**}(s_{[1,i]}))_n = (s(\sum_{i=1}^{n} a_i e_i))_n$, which by (v) is a convergent sequence.

(iii) ⇒ (v): If $\sum_i a_i e_i$ is convergent in norm then there is nothing more to prove. Otherwise, it is sufficient to show that if $(n_k)_k$ and $(m_k)_k$ are strictly increasing, then they have subsequences $(n'_k)_k$ and $(m'_k)_k$ so that the sequence $(\sum_{i=m'_k+1}^{n'_k} a_i e_i)_k$ is weak null. Combining (iii) with Lemma 6.3, choose $(n'_k)_k$ and $(m'_k)_k$ so that $m'_k < n'_k$, $\sum_k |\sum_{i=m'_k+1}^{n'_k} a_i| < \infty$ and both $(\sum_{i=1}^{m'_k} a_i e_i)_k$ and $(\sum_{i=1}^{n'_k} a_i e_i)_k$ are equivalent to the summing basis of c_0. We conclude that the sequence $x_k = \sum_{i=m'_k+1}^{n'_k} a_i e_i$ is unconditional and weak Cauchy, i.e. it is weakly null. □

7. Non-trivial weak Cauchy sequences in spaces with conditional spreading bases

In this section we study the behavior of non-trivial weak Cauchy sequences in a space with a conditional and spreading basis. As it turns out, such sequences always have convex block sequence that are very well behaved. The main result is the following theorem that we prove in several steps.

Theorem 7.1. *Let X be a Banach space with a conditional spreading Schauder basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. Let also $(x_i)_i$ be a non-trivial weak Cauchy sequence in X. The following statements hold.*

(i) *The sequence $(x_i)_i$ has a convex block sequence $(w_i)_i$ that is either equivalent to the summing basis of c_0 or equivalent to $(z_i)_i$.*

(ii) *The sequence* $(x_i)_i$ *has a convex block sequence* $(w_i)_i$ *the closed linear span of which is complemented in* X.

Lemma 7.2. *Let* X *be a Banach space and* $(x_i)_i$ *be a Schauder basic sequence in* X *so that the summing functional* $s(\sum_i a_i x_i) = \sum_i a_i$ *is bounded on the space spanned by* $(x_i)_i$. *If* $x \notin [(x_i)_i]$ *then the sequence* $(x_i - x)_i$ *is equivalent to* $(x_i)_i$.

Proof. If $\delta = \mathrm{dist}(x, [(x_i)_i])$, then the Hahn–Banach theorem yields that for a sequence of scalars $a_1, \dots, a_n$ we have

$$\left(\frac{\delta}{\|x\| + \delta}\right) \left\| \sum_{i=1}^n a_i x_i \right\| \leqslant \left\| \sum_{i=1}^n a_i (x_i - x) \right\|.$$

On the other hand,

$$\left\| \sum_{i=1}^n a_i (x_i - x) \right\| \leqslant \left\| \sum_{i=1}^n a_i x_i \right\| + \left| \sum_{i=1}^n a_i \right| \|x\| \leqslant (1 + \|s\| \|x\|) \left\| \sum_{i=1}^n a_i x_i \right\|. \qquad \square$$

Proof of Theorem 7.1 (i). Assume first that the basis $(e_i)_i$ is not strongly summing. Proposition 6.5 yields that X embeds into a space with an unconditional basis and therefore there exists a convex block sequence of $(x_i)_i$ that is equivalent to the summing basis of c_0. Assume now that the basis $(e_i)_i$ is strongly summing and let x^{**} be the weak star limit of $(x_i)_i$. We distinguish two cases.

Case 1: The series $\sum_i x^{**}(e_i^*) e_i$ converges in norm. Take the vector $x = \sum_{i=1}^\infty x^{**}(e_i^*) e_i$. Using Lemma 7.2 we may assume that $(x_i)_i$ is equivalent to $(x_i - x)_i$. Observe that $(x_i - x)_i$ is point-wise null, with respect to the basis $(e_i)_i$, and hence we may assume that $(x_i - x)_i$ is a block sequence. If $(x_i - x)_i$ had an unconditional subsequence then, since it is weak Cauchy, it would have to be weakly null, which is absurd. Proposition 5.1 (v) yields that there exists a convex block sequence of $(x_i - x)_i$, hence also of $(x_i)_i$, that is equivalent to $(z_i)_i$.

Case 2: The series $\sum_i x^{**}(e_i^*) e_i$ does not converge in norm. Using Lemma 6.3, choose a strictly increasing sequence of natural numbers $(n_i)_i$, such that if $y_i = \sum_{j=1}^{n_i} x^{**}(e_j^*) e_j$ for all $i \in \mathbb{N}$, then $(y_i)_i$ is equivalent to the summing basis of c_0 and set $w_i = x_i - y_i$. Observe that $(w_i)_i$ is weak Cauchy and that it is point-wise null, with respect to the basis $(e_i)_i$. Passing to a subsequence, we have that $(w_i)_i$ is equivalent to a block sequence. If $(w_i)_i$ has an unconditional subsequence then, since it is weak Cauchy, it is weakly null. Mazur's Theorem implies that there exists a convex block sequence of $(w_i)_i$ that converges to zero in norm which further yields that there exists a convex block sequence of $(x_i)_i$ that is equivalent to the summing basis of c_0. If $(w_i)_i$ does not have an unconditional subsequence, Proposition 5.1 (v) yields that there exists a convex block sequence $(w_i')_i$ of $(w_i)_i$ that is equivalent to $(z_i)_i$.

If $w_i' = \sum_{j \in F_i} a_j w_j$ with $\sum_{j \in F_i} a_j = 1$, set $x_i' = \sum_{j \in F_i} a_j x_j$ and $y_i' = \sum_{j \in F_i} a_j y_j$ for all $j \in \mathbb{N}$. Since $(x_i')_i$ is non-trivial weak Cauchy, we may assume that it dominates the summing basis of c_0 and, of course, the same is true for $(z_i)_i$. Combining the above with the fact that $(y_i')_i$ is equivalent to the summing basis of c_0 and $(x_i' - y_i')_i$ is equivalent to $(z_i)_i$, a simple argument yields that $(x_i')_i$ is equivalent to $(z_i)_i$. □

Lemma 7.3. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let also $(x_i)_i$ be a non-trivial weak Cauchy sequence in X. If $(x_i)_i$ has no convex block sequence equivalent to the summing basis of c_0, then* $\liminf_i \lim_k |s_{(i,+\infty)}(x_k)| > 0$.

Proof. Assume that the conclusion is false, i.e.

$$\lim_i \lim_k |s_{(i,+\infty)}(x_k)| = 0. \tag{8}$$

If $x^{**} = w^*\text{-}\lim_k x_k$, set $y_m = \sum_{i=1}^m x^{**}(e_i^*)e_i$ for all m. Note that for all m $\lim_k \|P_{[1,m]}x_k - y_m\| = 0$, which in conjunction with (8) and a sliding hump argument yields that there exists subsequences $(x_{k_n})_k$ and $(y_{m_n})_n$ so that $(x_{k_n} - y_{m_n})_n$ is point-wise null (with respect to $(e_i)_i$) and $\lim_n s(x_{k_n} - y_{m_n}) = 0$. By Lemma 6.3 we may assume that (y_{m_n}) is either equivalent to the summing basis of c_0, or norm-convergent. In particular it is weak Cauchy.

Observe that the norm of $(x_{k_n} - y_{m_n})_n$ is eventually bounded from below, otherwise we would obtain that $(x_k)_k$ has a norm convergent subsequence or a subsequence equivalent to the summing basis of c_0, which is absurd. Hence, passing to a subsequence, $(x_{k_n} - y_{m_n})_n$ is equivalent to a weak Cauchy seminormalized block sequence $(w_n)_n$ of $(e_i)_i$ with $s(w_n) = 0$ for all $n \in \mathbb{N}$. By Proposition 2.1, $(w_n)_n$ is unconditional and, being weak Cauchy, it is weakly null. By Mazur's theorem we conclude that $(x_k)_k$ has a convex block sequence equivalent to a convex block sequence of (y_m), i.e. equivalent to the summing basis of c_0, which is absurd. □

Although the following basically proves Theorem 7.1 (ii), it also provides further information about the kernel of the associated projection in certain cases. Its full statement will be required in the sequel.

Proposition 7.4. *Let X be a Banach space with a strongly summing conditional spreading basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. Then, every sequence $(x_i)_i$ in X that is equivalent to (z_i) has a subsequence the closed linear span W of which is complemented in X. Furthermore, there exists a sequence $\bar{I} = (I_i)_i$ of consecutive intervals of $\mathbb{N}$ so that if $Q : X \to X$ is the corresponding projection onto W and $P_{\bar{I}} : X \to X$ is defined by $P_{\bar{I}}(x) = \sum_{i=1}^\infty s_{I_i}(x)((1/\#I_i)\sum_{j\in I_i} e_j)$, then* $\ker Q$ *is isomorphic to* $\ker P_{\bar{I}}$.

Proof. The idea of the proof is the following. We will pass to a subsequence of $(x_n)_n$, again denoted by $(x_n)_n$, find a sequence of consecutive intervals $(I_n)_n$ of the natural numbers with $\#I_n \to \infty$, a vector y_0 in $X\setminus[(x_n)_n]$ (which also means $y_0 \in X\setminus[(x_n - y_0)_n]$) and a sequence $(\tilde{w}_n)_n$ so that the following hold:

(i) there is a non-zero scalar α so that

$$\sum_{n=1}^{\infty} \|\alpha\tilde{w}_n - (x_n - y_0)\| < \infty,$$

(ii) for all natural numbers m, n we have $s_{I_m}(\tilde{w}_n) = \delta_{m,n}$.

We shall first use the above to conclude the proof and describe the construction of the ingredients later. By (i), perhaps omitting the first few terms of each sequence, there is an invertible operator $A : X \to X$ with $A\tilde{w}_n = x_n - y_0$ for all n. We conclude by Lemma 7.2 that $(\tilde{w}_n)_n$ is equivalent to $(z_n)_n$ and let $R : [(z_n)_n] \to [(\tilde{w}_n)_n]$ be the map witnessing this fact. By Proposition 2.2 the map $P_{\bar{I}}x = \sum_{i=1}^{\infty} s_{I_i}(x)((1/\#I_i)\sum_{j\in I_i} e_j)$ is a bounded linear projection and if $v_i = (1/\#I_i)\sum_{j\in I_i} e_j$ for all $i \in \mathbb{N}$ then by (7) the map $S : [(v_i)_i] \to [(z_i)_i]$ with $Sv_i = z_i$ is bounded. We conclude that the map $Px = \sum_{n=1}^{\infty} s_{I_n}(x)\tilde{w}_n$ is a bounded linear projection, as $P = RSP_{\bar{I}}$ and clearly $\ker P = \ker P_{\bar{I}}$. If $\tilde{Q} = APA^{-1}$, then $\tilde{Q}$ is a projection onto $[(x_k - y_0)_k]$ the kernel of which is isomorphic to $\ker P_{\bar{I}}$. As y_0 is not in $\tilde{Q}[X]$, by the Hahn–Banach Theorem, we may choose a norm-one linear functional f on X with $\tilde{Q}[X] \subset \ker f$. Define $V_0 = \ker f \cap \ker \tilde{Q}$ and $W_0 = \tilde{Q}[X] + \langle\{y_0\}\rangle$. Note that $W = [(x_k)_k]$ is of co-dimension one in W_0. We deduce that setting $V = V_0 + \langle\{y_0\}\rangle$ the spaces W and V are complementary and their sum is the space X. It is also immediate that V is isomorphic to $\ker \tilde{Q}$ which isomorphic to $\ker P_{\bar{I}}$.

We now proceed to present how the aforementioned components are constructed. Fix decreasing sequences of positive real numbers $(\delta_n)_n$, $(\varepsilon_i)_i$ so that $\sum_n \delta_n < 1/4$ and

$$\sum_{n=1}^{\infty} \sum_{F\in[n,\infty)^{<\infty}} \prod_{i\in F} \varepsilon_i < 1/2, \tag{9}$$

where $[n,\infty)^{<\infty}$ denotes the set of all finite subsets of the natural numbers with $\min F \geqslant n$. Use Lemma 7.3 to find a positive real number α so that $\liminf_i \lim_k |s_{(i,\infty)}(x_k)| = \alpha$. By perhaps multiplying all terms of the sequence $(x_n)_n$ with -1, we may assume that $\liminf_i \lim_k |s_{(i,\infty)}(x_k) - \alpha| = 0$. If $x^{**} = w^* - \lim_k x_k$, since $(e_i)_i$ is strongly summing, we may choose a sequence $(j_n)_n$ so that for all $j_n \leqslant k \leqslant m$ we have

$$\left|\sum_{i=k}^{m} x^{**}(e_i^*)\right| < \frac{\alpha\varepsilon_n}{2}. \tag{10}$$

Choose a strictly increasing sequence of natural numbers $(i_n)_n$, with $i_n \geqslant j_n$ and $\lim_n(i_{n+1} - i_n) = \infty$, and a subsequence of $(x_n)_n$, again denoted by x_n, so that for all natural numbers $n \leqslant m$

$$\left|s_{(i_n,+\infty)}(x_m) - \alpha\right| < \frac{\alpha\delta_n}{2}. \tag{11}$$

We can simultaneously choose subsequences of $(i_n)_n$ and $(x_n)_n$ that satisfy:

$$\left|s_{(i_n,i_{n+1}]}(x_n)-\alpha\right|<\alpha\delta_n, \tag{12a}$$

$$\left\|P_{[1,i_n]}x_n-\sum_{i=1}^{i_n}x^{**}(e_i^*)e_i\right\|<\delta_n, \text{ and} \tag{12b}$$

$$\left\|P_{(i_{n+1},+\infty)}x_n\right\|<\delta_n. \tag{12c}$$

For all $n\in\mathbb{N}$ define $I_n=(i_n,i_{n+1}]$ and

$$\tilde{x}_n=\sum_{i=1}^{i_n}x^{**}(e_i^*)e_i+P_{I_n}x_n.$$

Define

$$y_0=\sum_{n=1}^{\infty}\left(\sum_{i\in I_n}x^{**}(e_i^*)\right)\frac{1}{s_{I_n}\left(P_{I_n}\tilde{x}_n\right)}P_{I_n}\tilde{x}_n.$$

Note that $P_{I_n}\tilde{x}_n=P_{I_n}x_n$ and hence by (10) and (12a) y_0 is well defined. Choose n_0 so that y_0 is not in the closed linear span of $(x_n)_{n\geqslant n_0}$ and hence, by Lemma 7.2, $(x_n-y_0)_{n\geqslant n_0}$ is equivalent to $(x_n)_n$, i.e. to $(z_n)_n$. To simplify notation we shall assume that $n_0=1$.

Since $P_{I_n}\tilde{x}_n=P_{I_n}x_n$, by (10) and (12a) we obtain:

$$|s_{I_n}(\tilde{x}_n-y_0)-\alpha|=\left|s_{I_n}(x_n)-\sum_{i\in I_n}x^{**}(e_i^*)-\alpha\right|<\alpha(\delta_n+\varepsilon_n/2), \tag{13}$$

in particular $|s_{I_n}(\tilde{x}_n-y_0)|>\alpha/2$, hence for all $n\in\mathbb{N}$ we may define $w_n=(s_{I_n}(\tilde{x}_n-y_0))^{-1}(\tilde{x}_n-y_0)$. It is straightforward to check the following:

$$s_{I_n}(w_m)=\begin{cases}0 & \text{if } m<n \text{ and}\\ 1 & \text{if } m=n.\end{cases} \tag{14a}$$

Also if $m>n$, $s_{I_n}(w_m)=(s_{I_n}(\tilde{x}_n-y_0))^{-1}\sum_{i\in I_m}x^{**}(e_i^*)$. Hence, by $|s_{I_n}(\tilde{x}_n-y_0)|>\alpha/2$ and (10) for $m>n$

$$|s_{I_m}(w_n)|<\varepsilon_n. \tag{14b}$$

For fixed n recursively define a sequence of scalars $(c_j^n)_{j\geqslant n+1}$ as follows: $c_{n+1}^n=-s_{I_{n+1}}(w_n)$ and $c_{j+1}^n=-s_{I_{j+1}}(w_n)-\sum_{i=n+1}^{j}c_i^n s_{I_{j+1}}(w_i)$. Using (14b), keeping n fixed and an induction on $j\geqslant n+1$ we obtain:

$$|c_j^n| < \sum_{\substack{F\subset[n+1,j]\\ \max F=j}} \prod_{i\in F} \varepsilon_i$$

and therefore

$$\sum_{j=n+1}^{\infty} |c_j^n| < \sum_{F\in[n+1,\infty)^{<\infty}} \prod_{i\in F} \varepsilon_i \tag{15}$$

which, in conjunction with (9), yields that for $n \in \mathbb{N}$ the vector

$$\tilde{w}_n = w_n + \sum_{j=n+1}^{\infty} c_j^n w_j$$

is well defined.

It remains to show that (i) and (ii) are satisfied to complete the proof. The fact that (ii) holds is an easy consequence of (14a) and the choice of the sequences $(c_j^n)_{j\geqslant n+1}$ for $n \in \mathbb{N}$. By (15) and (9) we obtain $\sum_{n=1}^{\infty} \|\tilde{w}_n - w_n\| < \infty$, therefore it remains to observe that $\sum_{n=1}^{\infty} \|\alpha w_n - (x_n - y_0)\| < \infty$. Indeed, for $n \in \mathbb{N}$ (13) yields that $\|\alpha w_n - (\tilde{x}_n - y_0)\| < 2\delta_n + \varepsilon_n$ and by (12b) and (12c) $\|\tilde{x}_n - x_n\| < 2\delta_n$. □

Proof of Theorem 7.1 (ii). If $(x_n)_n$ has a convex block sequence equivalent to the summing basis of c_0, then the result follows from the well known fact that c_0 is separably injective. If this is not the case, let $(z_i)_i$ be the convex block homogeneous part of $(e_i)_i$, and apply Theorem 7.1 (i) to find a convex block sequence of $(w_i)_i$ of $(x_i)_i$ that is equivalent to $(z_i)_i$. Note that, in this case, $(z_i)_i$, being equivalent to $(w_i)_i$, cannot be equivalent to the summing basis of c_0, therefore, by Proposition 6.5, the basis $(e_i)_i$ of X must be strongly summing. By Proposition 7.4, $(w_i)_i$ has a subsequence the closed linear span W of which is complemented in X. □

8. Complemented subspaces of spaces with conditional spreading bases

The main results of this section are the following two theorems. The first one characterizes strongly summing spreading bases with respect to squares of spaces and the second one provides information about arbitrary decompositions of a space with a strongly summing spreading basis. Their proofs are based on certain projections and Theorem 7.1 (ii), in fact the more precise statement of Proposition 7.4. This section concludes with a conversation around the question if spaces with a convex block homogeneous basis are primary.

Theorem 8.1. *Let X be a Banach space with a spreading basis $(e_i)_i$, let $(z_i)_i$ be its convex block homogeneous part, and set $Z = [(z_i)_i]$. The following hold.*

(i) *If* $(e_i)_i$ *is strongly summing, then* $Z \oplus Z$ *does not embed into* X*. In particular,* $X \oplus X$ *does not embed into* X*.*

(ii) *The basis* $(e_i)_i$ *is not strongly summing if and only if* X *is isomorphic to* $X \oplus X$*.*

Theorem 8.2. *Let* X *be a Banach space with a strongly summing conditional spreading basis* $(e_i)_i$*, let* $(z_i)_i$ *be its convex block homogeneous part, and* $Z = [(z_i)_i]$*. If* $X = V \oplus W$*, then exactly one of the spaces* V *and* W *contains a subspace* $\tilde{Z}$ *that is isomorphic to* Z*. Furthermore,* $\tilde{Z}$ *is complemented in the whole space.*

UFDD's and skipped unconditionality A tool used to prove the above results are projections like the one from Proposition 2.2 and their kernels which are studied in this subsection.

If $N = \{j_1 < j_2 < \cdots\} \subset \mathbb{N}$, we shall say that a sequence of successive intervals $(E_j)_{j\in N}$ of $\mathbb{N}$ is skipped, if $\max E_{j_i} + 1 < \min E_{j_{i+1}}$ for all i.

Notation. Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(d_i)_i$ be the difference basis of X.

(i) If E is an interval of $\mathbb{N}$ we denote by X_E the linear span of the vectors $(d_i)_{i\in E}$.

(ii) If $\bar{E} = (E_j)_{j\in N}$ is a sequence of skipped intervals of $\mathbb{N}$, where $N \subset \mathbb{N}$, we denote by $X_{\bar{E}}$ or $X_{(E_j)_{j\in N}}$ the closed linear span of $\cup_{j\in N} X_{E_j}$.

(iii) If $\bar{I} = (I_i)_i$ is a sequence of consecutive intervals of $\mathbb{N}$, we denote by ${}^{\backprime}\bar{I}$ the skipped sequence of successive intervals $({}^{\backprime}I_j)_{j\in N}$, where ${}^{\backprime}I_j = I_j \setminus \{\min I_j\}$ and $N = \{j : {}^{\backprime}I_j \neq \varnothing\}$.

Proposition 8.3. *Let* X *be a Banach space with a conditional spreading basis* $(e_i)_i$ *and let* $(E_j)_{j\in N}$ *be a sequence of skipped intervals of* $\mathbb{N}$*. Then* $(X_{E_j})_{j\in N}$ *is an unconditional finite dimensional decomposition (UFDD) of* $X_{(E_j)_{j\in N}}$*. If moreover* $(e_i)_i$ *is 1-spreading, then the FDD* $(X_{E_j})_{j\in N}$ *is suppression unconditional.*

Proof. We may assume that $(e_i)_i$ is 1-spreading. The result easily follows by combining Proposition 2.1 with the fact that any sequence $(x_j)_j$ with x_j in X_{E_j} is a block sequence of $(e_i)_i$ with $s(x_j) = 0$ for all $j \in N$. $\square$

Remark 8.4. Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let also $(E_j)_j$ and $(F_j)_j$ be skipped sequences of successive intervals of $\mathbb{N}$ so that $\#E_j = \#F_j$. The spreading property of $(e_i)_i$ implies that the spaces $X_{(E_j)_j}$ and $X_{(F_j)_j}$ are naturally isomorphic through the map $d_i \to d_{\phi(i)}$, where if i is the k-th element of E_j, then $\phi(i)$ is the k-th element of F_j. If furthermore the basis $(e_i)_i$ is 1-spreading and $\min(E_1) = \min(F_1) = 1$ or $1 < \min\{\min(E_1), \min(F_1)\}$, then the spaces $X_{(E_j)_j}$ and $X_{(F_j)_j}$ are naturally isometric. The condition concerning the minima of E_1 and F_1 is due to $d_1 = e_1$ whereas $d_{i+1} = e_{i+1} - e_i$ for all $i \in \mathbb{N}$.

Proposition 8.5. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$. Let also $\bar{I} = (I_i)_i$ be a sequence of consecutive intervals of the natural numbers with $\cup_i I_i = [i_0, +\infty)$, for some $i_0 \in \mathbb{N}$ and let $P_{\bar{I}}$ be the associated averaging projection. Then, the kernel of $P_{\bar{I}}$ is the space $\langle\{e_1, \dots, e_{i_0-1}\}\rangle \oplus X_{\restriction \bar{I}}$. In particular, the kernel of $P_{\bar{I}}$ admits a UFDD.*

Proof. It is easy to see that $X_{\restriction I_j}$ is in the kernel of P for all j and an elementary argument yields that the spaces $X_{\restriction I_1}, \dots, X_{\restriction I_n}$ together with the vectors $z_j = (1/\#I_j)\sum_{i\in I_j} e_i$, $j = 1, \dots, n$ and $e_1, \dots, e_{i_0-1}$ span the space $\langle\{e_i : i \leqslant \max I_n\}\rangle$. The above implies that $\langle\{e_1, \dots, e_{i_0-1}\}\rangle \oplus X_{\restriction \bar{I}}$ is the entire kernel of $P_{\bar{I}}$. By Proposition 8.3 the kernel of $P_{\bar{I}}$ admits a UFDD. □

Corollary 8.6. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$. Let also $(E_j)_j$ be a skipped sequence of successive intervals of $\mathbb{N}$ so that $\sup_j \#E_j = \infty$. Then every unconditional sequence $(x_i)_i$ in X has a subsequence that is equivalent to some sequence in $X_{(E_j)_j}$.*

Proof. By Remark 8.4 we may assume that $\max E_j + 2 = \min E_{j+1}$ for all $j \in \mathbb{N}$ and $\min E_1 = 2$. For each j set $I_j = \{\min E_j - 1\} \cup E_j$. Then if $\tilde{z}_i = (1/\#I_i)\sum_{j\in s_{I_i}} e_j$, by Proposition 2.2 the map $P : X \to X$ with $Px = \sum_i s_{I_i}(x)\tilde{z}_i$ is a bounded linear projection and ker$P$ is $X_{(E_j)_j}$. Let now $(x_i)_i$ be an unconditional sequence in X. Then, it is either weakly null or it has a subsequence equivalent to the unit vector basis of ℓ_1. In the first case $\lim_i s(x_i) = 0$ and by perturbing and passing to a subsequence we may assume that $(x_i)_i$ is a block sequence with $s(x_i) = 0$ for all $i \in \mathbb{N}$. In the second case by a standard difference argument we may assume the same. Since the basis $(e_i)_i$ is spreading and $\sup_i \#I_i = \infty$, we may moreover assume that for every $i \in \mathbb{N}$ there exists k_i with $\operatorname{supp} x_i \subset I_{k_i}$. This implies that $x_i \in \ker P$ for all $i \in \mathbb{N}$ which is the desired result. □

The following result is not used in the paper however it demonstrates that spaces with a spreading Schauder basis that contain isomorphic copies of ℓ_1 contain further copies of ℓ_1 that are very well behaved.

Proposition 8.7. *Let X be a Banach space with a conditional and spreading basis $(e_i)_i$ containing a subspace Y that is isomorphic to ℓ_1. Then Y contains a further subspace W that is isomorphic to ℓ_1 and complemented in X. In particular, whenever a Banach space X with a conditional spreading basis contains a copy of ℓ_1, then X contains a complemented copy of ℓ_1.*

Proof. If $(y_i)_i$ is a sequence in X equivalent to the unit vector basis of ℓ_1, then by passing to differences and perturbing, we may assume that it is a block sequence with $s(y_i) = 0$ for all $i \in \mathbb{N}$. Choose a sequence of consecutive intervals $(I_i)_i$ of $\mathbb{N}$ so that $\operatorname{ran} y_i \subset I_i$ for all $i \in \mathbb{N}$. Then, by Proposition 2.2, the map $P_{\bar{I}}x = \sum_i s_{I_i}(x)/\#I_i \sum_{j\in I_i} e_j$ is a bounded linear projection. By Proposition 8.5 the kernel of $P_{\bar{I}}$ admits a UFDD. Observe that $(y_i)_i$

is in $\ker P_{\bar{I}}$. It is well known, and not hard to prove, that a copy of ℓ_1 in a space with a UFDD contains a further copy of ℓ_1 complemented in the whole space. That is, there is a copy of ℓ_1 complemented in the kernel of $P_{\bar{I}}$, and thus complemented in X. □

Proposition 8.8. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$, let $(z_i)_i$ be its convex block homogeneous part, and $Z = [(z_i)_i]$. Then there exists a sequence of natural numbers $(m_i)_i$, so that for every skipped sequence of successive intervals $(E_j)_j$ of $\mathbb{N}$ with $\#E_j \geqslant m_j$ for all j, we have that X is isomorphic to $Z \oplus X_{(E_j)_j}$.*

Proof. Choose a sequence of natural numbers $(m_i)_i$ satisfying $m_i \geqslant n_i - 1$, where $(n_i)_i$ is the sequence of Lemma 5.3. Let $(E_j)_j$ be a skipped sequence of successive intervals of $\mathbb{N}$ with $\#E_j \geqslant m_i$ for all i. By Remark 8.4 we may assume that $\max E_j + 2 = \min E_{j+1}$ for all $j \in \mathbb{N}$. For each j set $I_j = \{\min E_j - 1\} \cup E_j$. Then $(I_j)_j$ satisfies the assumptions of Lemma 5.3 and if $\tilde{z}_i = (1/\#I_i)\sum_{j\in s_{I_i}} e_j$ then $(\tilde{z}_i)_i$ is equivalent to $(z_i)_i$. By Proposition 2.2 the map $P : X \to X$ with $Px = \sum_i s_{I_i}(x)\tilde{z}_i$ is a bounded linear projection and ImP is isomorphic to Z and $\ker P$ is $X_{(E_j)_j}$. □

Remark 8.9. If the basis $(e_i)_i$ of a space X is convex block homogeneous, then the sequence $(\tilde{z}_i)_i$ is equivalent to $(z_i)_i$ without imposing any restrictions on the cardinalities of the sets E_i. In particular, for every sequence of successive intervals $(E_j)_j$ of $\mathbb{N}$, the space X is isomorphic to $X \oplus X_{(E_j)_j}$.

For clarity, we restate what Proposition 8.8 and Corollary 8.6 yield in the following.

Proposition 8.10. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$, let $(z_i)_i$ be its convex block homogeneous part, and $Z = [(z_i)_i]$. Then, there exists a skipped sequence of successive intervals $(E_j)_j$ of $\mathbb{N}$ so that the following hold.*

- (i) *The space X is isomorphic to $Z \oplus X_{(E_j)_j}$, i.e. X is the direct sum of a space with a convex block homogeneous conditional spreading basis and a space with a UFDD.*
- (ii) *Every unconditional sequence $(x_i)_i$ in X has a subsequence equivalent to a sequence in $X_{(E_j)_j}$.*

Proof of Theorem 8.1. Let X be a space with a conditional spreading basis $(e_i)_i$, let $(z_i)_i$ be its convex block homogeneous part, and $Z = [(z_i)_i]$. We first prove (i) thus we assume $(e_i)_i$ is strongly summing. Towards a contradiction, assume that there are sequences $(z_i^1)_i$ and $(z_i^2)_i$ in X, both equivalent to $(z_i)_i$, so that if $Z_1 = [(z_i^1)_i]$ and $Z_2 = [(z_i^2)_i]$, then $Z_1 \cap Z_2 = \{0\}$ and $Z_1 + Z_2$ is a closed subspace of X. By Propositions 7.4 and 8.5 there is a subsequence $(w_i)_i$ of $(z_i^1)_i$ and a projection $P : X \to X$ onto the space $W = [(w_i)_i]$ so that the kernel of P has a UFDD. The open mapping theorem implies that Z_2 embeds into ker$P$. Recall that non-trivial weak Cauchy sequences in spaces with a UFDD have convex block sequences equivalent to the summing basis of c_0. We conclude that $(z_i^2)_i$

has a convex block sequence equivalent to the summing basis of c_0, which implies that $(z_i)_i$ is equivalent to the summing basis of c_0 and hence, by Proposition 6.5, $(e_i)_i$ cannot be strongly summing.

We now prove (ii). By (i), it is enough to show that if $(e_i)_i$ is not strongly summing, then X is isomorphic to $X \oplus X$. By Proposition 6.5, the convex block homogeneous part $(z_i)_i$ of $(e_i)_i$ is equivalent to the summing basis of c_0. Let $(m_i)_i$ be the sequence provided by Proposition 8.8 and for $i \in \mathbb{N}$ set $m_i' = \max\{m_i, m_{2i-1}, m_{2i}\}$. Choose a skipped sequence of successive intervals $\overline{E} = (E_j)_j$ of $\mathbb{N}$ so that for all j, $\#E_{2j-1} = \#E_{2j} = m_j'$. If $\overline{E}_{\mathrm{o}} = (E_{2j-1})_j$ and $\overline{E}_{\mathrm{e}} = (E_{2j})_j$, then by Proposition 8.8 we conclude that X is isomorphic to $c_0 \oplus X_{\overline{E}}$ as well as to $c_0 \oplus X_{\overline{E}_{\mathrm{o}}}$. Proposition 8.3 implies that $X_{\overline{E}} = X_{\overline{E}_{\mathrm{o}}} \oplus X_{\overline{E}_{\mathrm{e}}}$ whereas by Remark 8.4 we obtain $X_{\overline{E}_{\mathrm{o}}} \simeq X_{\overline{E}_{\mathrm{e}}}$. We conclude:

$$\begin{aligned} X &\simeq c_0 \oplus X_{\overline{E}} \simeq c_0 \oplus \left(X_{\overline{E}_{\mathrm{o}}} \oplus X_{\overline{E}_{\mathrm{e}}}\right) \simeq (c_0 \oplus c_0) \oplus \left(X_{\overline{E}_{\mathrm{o}}} \oplus X_{\overline{E}_{\mathrm{o}}}\right) \\ &\simeq \left(c_0 \oplus X_{\overline{E}_{\mathrm{o}}}\right) \oplus \left(c_0 \oplus X_{\overline{E}_{\mathrm{o}}}\right) \simeq X \oplus X. \qquad \square \end{aligned}$$

Proof of Theorem 8.2. We shall prove that Z embed isomorphically into at least one of the spaces V or W. The rest of the statement follows from Proposition 7.4 and Theorem 8.1 (i). Let $P : X \to X$ be the projection with $\mathrm{Im}P = V$ and $\ker P = W$. Note that either $(Pe_i)_i$ or $(e_i - Pe_i)_i$ has to be non-trivial weak Cauchy and we shall assume that the first one holds. By Theorem 7.1 (i), $(e_i)_i$ has a convex block sequence $(w_i)_i$, so that $(Pw_i)_i$ is either equivalent to $(z_i)_i$, or to the summing basis of c_0. If the first one holds, then Z embeds into V and there is nothing left to prove.

Otherwise, $(Pw_i)_i$ is equivalent to the summing basis of c_0. This implies that $(w_i - Pw_i)_i$ has no convex block sequence equivalent to the summing basis of c_0. Indeed, assume that there is a convex block sequence $(\tilde{w}_i)_i$ of $(e_i)_i$ so that both $(P\tilde{w}_i)_i$ as well as $(\tilde{w}_i - P\tilde{w}_i)$ are equivalent to the summing basis of c_0. Then $(\tilde{w}_i)_i$, being non-trivial weak Cauchy, has to be equivalent to the summing basis of c_0 as well. By (7) it follows that $(z_i)_i$ is dominated by $(\tilde{w}_i)$ and hence it is equivalent to the summing basis of c_0 as well. Proposition 6.5 (ii) ⇔ (iii) yields a contradiction.

Note that $(w_i - Pw_i)_i$ is non-trivial weak Cauchy. If this were not the case and $w\text{-}\lim_i(w_i - Pw_i) = x_0$, then $((w_i - x_0) - (Pw_i))_i$ is w-null. By Mazur's theorem, $(w_i - x_0)_i$ has a convex block sequence $(\tilde{w}_i - x_0)_i$ equivalent to $(P\tilde{w}_i)_i$, i.e. equivalent to the summing basis of c_0. It follows that $(w_i)_i$ has a convex block sequence equivalent to the summing basis of c_0, which we showed is not possible.

We have concluded that $(w_i - Pw_i)_i$ is non-trivial weak Cauchy and it has no convex block sequence equivalent to the summing basis of c_0. By Theorem 7.1 (i) the space Z embeds into W. □

The most important examples of spaces with a convex block homogeneous basis are c_0, ℓ_1, and James space J. The first two spaces were shown to be prime by A. Pełczyński, i.e. their complemented subspaces are isomorphic to the whole space. This is no longer true for James space, however as it was shown in [13] the space is primary. This means that if $J = X \oplus Y$ then one of the space X or Y is isomorphic to J. This can also be shown for the spaces J_p, $1 < p < \infty$.

Problem. Let X be a Banach space with a convex block homogeneous basis. Is X primary?

Note that a space with a conditional spreading basis that is not convex block homogeneous may fail to be primary. Consider the norm on $c_{00}(\mathbb{N})$ with $\|x\| = \max\{\|x\|_J, \|x\|_p\}$, for some $1 < p < 2$, and denote its completion by X. Then $X \simeq J \oplus V$ where V is a reflexive space with a UFDD that contains ℓ_p, hence X is not primary.

Before concluding this section we present some results concerning spaces with convex block homogeneous bases that hint towards a positive solution of the above mentioned problem. These results are also of independent interest as they exhibit strong similarity between an arbitrary Banach space with a convex block homogeneous basis and James space.

We introduce some further notation that will make stating and proving the subsequent results possible. If X has a conditional convex block homogeneous basis and $(n_j)_{j\in L}$ is a sequence of natural numbers indexed by a subset L of $\mathbb{N}$ denote by $X_{(n_j)_{j\in L}}$ any space $X_{(E_j)_{j\in L}}$ such that there exists a sequence of skipped intervals $(E_j)_{j\in L}$ with $\#E_j = n_j$ for all $j \in L$. By Remark 8.4, the choice of the intervals does not matter. For isometric reasons we can choose $\min(E_1) > 1$. We shall also allow some of the n_j's to be zero and in this case by $X_{(n_j)_{j\in L}}$ we refer to the space $X_{(n_j)_{j\in\tilde{L}}}$ where $\tilde{L} = \{j \in L : n_j \neq 0\}$. For finite $F \subseteq E \subset \mathbb{N}$ we write $F \preceq E$ if F is an initial segment of E.

Lemma 8.11. *Let X be a Banach with a 1-convex block homogeneous basis $(e_i)_i$. If $(E_j)_j$ is a sequence of skipped intervals of $\mathbb{N}$ and $(F_j)_j$ is such that $F_j \preceq E_j$ for all $j \in \mathbb{N}$, then $X_{(F_j)_j}$ is naturally complemented in $X_{(E_j)_j}$. Hence, if $(n_j)_{j\in L}$ and $(s_j)_{j\in L}$ are sequences of non-negative integers with $s_j \leqslant n_j$, then $X_{(n_j)_{j\in L}} \simeq X_{(n_j-s_j)_{j\in L}} \oplus X_{(s_j)_{j\in L}}$.*

Proof. We may assume that $\max(E_j) + 2 = \min(E_{j+1})$ for all $j \in \mathbb{N}$. Define $k_1 = 1$, for $j \geqslant 2$ $k_j = \min(E_j) - 1$, and for all $j \in \mathbb{N}$ set $\ell_j = \max(F_j)$ and $m_j = \max(E_j)$. A calculation yields that if $x = \sum_{j=1}^N \sum_{i\in E_j} a_i d_i$, then

$$y = \sum_{j=1}^{N} \sum_{i\in F_j} a_i d_i = \sum_{j=1}^{n} \left(\left(\sum_{k_j \leqslant i < \ell_j} s_{\{i\}}(x) e_i \right) + s_{[\ell_j, m_j]}(x) e_{\ell_j} \right).$$

By 1-convex block homogeneity we obtain

$$\|y\| = \left\| \sum_{j=1}^{n} \left(\left(\sum_{k_j \leqslant i < \ell_j} s_{\{i\}}(x) e_i \right) + s_{[\ell_j, m_j]}(x) \left(\frac{1}{\#[\ell_j, m_j]} \sum_{i\in[\ell_j, m_j]} e_i \right) \right) \right\|$$
$$\leqslant \|x\| \text{ (by Proposition 3.8).} \qquad \square$$

Proposition 8.12. *Let X be a Banach space with a 1-convex block homogeneous basis $(e_i)_i$ and let $(n_j)_j$ and $(m_j)_j$ be unbounded sequences of natural numbers. Then, $X_{(n_j)_j}$ is 256-isomorphic to $X_{(m_j)_j}$.*

Proof. By induction on $d = 1, 2, \ldots$ choose non-negative integers $(s_j)_{j=1}^d$, $(t_j)_{j=1}^d$, $(k_j)_{j=1}^d$ so that

(i) $1 \leqslant k_1 < \cdots < k_d$,
(ii) for $1 \leqslant j \leqslant d$ we have $0 \leqslant s_j \leqslant m_{k_j}$ and $0 \leqslant t_j \leqslant n_{k_j}$,
(iii) if $1 \leqslant j \leqslant d$ is not in $\{k_1, \ldots, k_d\}$ then $m_j + s_j = n_j + t_j$, and
(iv) if $1 \leqslant j \leqslant d$ is equal to k_i for some $i \leqslant j$ then $m_j - s_i + s_j = n_j - t_i + t_j$.

There is no difficulty to the induction so we omit it. We now perform a decomposition argument. Define $K = \{k_j : j \in \mathbb{N}\}$.

$$\begin{aligned} X_{(m_j)_j} &\simeq X_{(m_j)_{j\notin K}} \oplus X_{(m_j)_{j\in K}} \text{ (by unconditionality)} \\ &= X_{(m_j)_{j\notin K}} \oplus X_{(m_{k_j})_{j\in\mathbb{N}}} \\ &\simeq X_{(m_j)_{j\notin K}} \oplus \left(X_{(m_{k_j}-s_j)_{j\in\mathbb{N}}} \oplus X_{(s_j)_{j\in\mathbb{N}}}\right) \text{ (by Lemma 8.11 and (ii))} \\ &\simeq X_{(m_j)_{j\notin K}} \oplus \left(X_{(m_{k_j}-s_j)_{j\in\mathbb{N}}} \oplus \left(X_{(s_j)_{j\in K}} \oplus X_{(s_j)_{j\notin K}}\right)\right) \\ &= \left(X_{(m_j)_{j\notin K}} \oplus X_{(s_j)_{j\notin K}}\right) \oplus \left(X_{(m_{k_j}-s_j)_{j\in\mathbb{N}}} \oplus X_{(s_{k_j})_{j\in\mathbb{N}}}\right) \\ &\simeq X_{(m_j+s_j)_{j\notin K}} \oplus X_{(m_{k_j}-s_j+s_{k_j})_{j\in\mathbb{N}}}. \end{aligned}$$

An identical argument yields $X_{(n_j)_j} \simeq X_{(n_j+t_j)_{j\notin K}} \oplus X_{(n_{k_j}-s_j+t_{k_j})_{j\in\mathbb{N}}}$ and by (iii) and (iv) we obtain

$$X_{(m_j+s_j)_{j\notin K}} \oplus X_{(m_{k_j}-s_j+s_{k_j})_{j\in\mathbb{N}}} = X_{(n_j+t_j)_{j\notin K}} \oplus X_{(n_{k_j}-s_j+t_{k_j})_{j\in\mathbb{N}}},$$

i.e. $X_{(m_j)_j} \simeq X_{(n_j)_j}$. If all direct sums are taken with the max-norm, then it follows that the spaces $X_{(m_j)_{j\in\mathbb{N}}}$ and $X_{(n_j)_{j\in\mathbb{N}}}$ are 256-isomorphic. □

Remark 8.13. It can also be shown that for a bounded sequence $(m_j)_j$ the space $X_{(m_j)_j}$ is isomorphic to $U = [(u_i)_i]$ where $(u_i)_i$ is the unconditional part of $(e_i)_i$. This is much easier and the isomorphism constant depends on the bound of the sequence $(m_j)_j$.

Given a Banach space with a 1-convex block homogeneous basis, by Proposition 8.12, we may denote by X_∞ the space $X_{(m_j)_j}$ for an arbitrary unbounded sequence of natural numbers. Up to a constant of 256 the choice of the sequence is not relevant.

Proposition 8.14. *Let X be a Banach space with a 1-convex block homogeneous basis and let $(m_j)_j$ be an unbounded sequence of natural numbers. Then, the space $X_\infty = X_{(m_j)_j}$*

admits an unconditional Schauder decomposition $(X_k)_k$ *so that for each* $k \in \mathbb{N}$ *the spaces* X_∞ *and* X_k *are 256-isomorphic. Notationally,*

$$X_\infty = \sum_{k=1}^{\infty} \oplus X_\infty \text{ unconditionally.} \tag{16}$$

Proof. Partition the natural numbers into infinitely many infinite sets $(N_k)_k$ so that $(m_j)_{j\in N_k}$ is unbounded for all $k \in \mathbb{N}$. By unconditionality we obtain that the spaces $(X_{(m_j)_{j\in N_k}})_{k\in\mathbb{N}}$ form an unconditional Schauder decomposition for $X_{(m_j)_{j\in\mathbb{N}}}$. By Proposition 8.12 the result follows. □

Remark 8.15. If X is a Banach space with a 1-convex block homogeneous basis $(e_i)_i$, by using Theorem 8.2 and Remark 8.9 we conclude that if $X = V \oplus W$ and X is complemented in V (it is always complemented in either V or W) then there exists a complemented subspace Y of X_∞ so that $V \simeq X \oplus (X_\infty \oplus Y)$ (this decomposition requires the information about the kernel of certain projections given by Proposition 7.4). This is not quite enough to imply that X is primary however it is very close to having this property. There is hope that a Pełczyński decomposition type argument [19] can be used to show that $X_\infty \oplus Y \simeq X_\infty$. This would imply that X is primary. The problem in this approach is the poor understanding of the “outside” norm in (16) (unless $X = J_p$, the jamesification of ℓ_p, there is no outside norm in the strict sense).

9. The Baire-1 functions of a space with a spreading basis

We denote by $\mathcal{B}_1(X)$ the subspace of X^{**} that consists of all Baire-1 functions, i.e. those x^{**} for which there is a sequence $(x_n)_n$ in X with $x^{**} = w^*\text{-}\lim_n x_n$. In this rather small section we include some observations concerning the position of $\mathcal{B}_1(X)$ in X^{**} and of X in $\mathcal{B}_1(X)$ whenever X is a Banach space with a conditional spreading basis. We do not use any of these results in the rest of the paper, however, we think that they are of independent interest since they witness the highly canonical behavior exhibited by spaces with conditional spreading bases.

Proposition 9.1. *Let* X *be a Banach space with a strongly summing conditional spreading basis* $(e_i)_i$ *and denote* $e^{**} = w^*\text{-}\lim_i e_i$. *Then, the map* $P : X^{**} \to X^{**}$ *with*

$$Px^{**} = w^*\text{-}\sum_i x^{**}(e_i^*)e_i + \left(\lim_i x^{**}\left(s_{(i,\infty)}\right)\right) e^{**}$$

is a bounded linear projection onto $\mathcal{B}_1(X)$.

Proof. By Proposition 6.5 (v) and (iv) the limit $w^*\text{-}\sum_i x^{**}(e_i^*)e_i$ and the limit $\lim_i x^{**}(s_{(i,\infty)})$ exist. Hence, P is well defined, bounded and maps into the space of Baire-1 functions of X^{**}. It remains to show that $Px^{**} = x^{**}$ whenever x^{**} is Baire-1, i.e. there is a sequence $(x_n)_n$ in X with $x^{**} = w^*\text{-}\lim_n x_n$. For each $n \in \mathbb{N}$ define $y_n = \sum_{i=1}^n x^{**}(e_i^*) + x^{**}(s_{(n,\infty)})e_{n+1}$. Then

$$w^*\text{-}\lim_n y_n = Px^{**} \text{ and } s(y_n) = x^{**}(s) \text{ for all } n. \tag{17}$$

If $(x_n - y_n)_n$ has a subsequence that converges to zero in norm, then $x^{**} = w^*\text{-}\lim_n y_n$ and there is nothing left to prove. Otherwise, the sequence $(x_n - y_n)_n$ is seminormalized and point-wise null, with respect to $(e_i)_i$, and $s(x_n - y_n) = s(x_n) - x^{**}(s) \to 0$. A sliding hump argument yields that, passing to a subsequence, $(x_n - y_n)_n$ is equivalent to a block sequence $(w_n)_n$ with $s(w_n) = 0$ for all n. By Proposition 2.1, $(x_n - y_n)_n$ is unconditional, and since it is weak Cauchy, it is weakly null. In conclusion, $P^{**}x^{**} = w^*\text{-}\lim_n y_n = w^*\text{-}\lim_n x_n = x^{**}$. □

It was proved in [14, Theorem 2.3 f), page 4] that a Banach space with a conditional spreading basis not containing c_0 and ℓ_1 is quasi-reflexive of order one. The following can be viewed as a generalization of this result.

Corollary 9.2. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$. If X contains no subspace isomorphic to c_0, then X is of co-dimension one in $\mathcal{B}_1(X)$.*

Proof. If X does not contain c_0 then, by Proposition 6.5, $(e_i)_i$ must be strongly summing. By Proposition 9.1, for each x^{**} in $\mathcal{B}_1(X)$ we have that $x^{**} = w^*\text{-}\sum_i x^{**}(e_i^*)e_i + (\lim_i x^{**}(s_{(i,\infty)}))e^{**}$. By Lemma 6.3 we obtain that $\sum_i x^{**}(e_i^*)e_i$ is norm convergent for all x^{**} in X^{**}, which implies that $X = \mathcal{B}_1(X) \cap \ker(w\text{-}\lim_i s_{(i,\infty)})$. □

10. Spreading models of non-reflexive spaces

In this section we show that every sequence generating a conditional spreading model has a block sequence of averages that generates a convex block homogeneous spreading model. We also observe that non-reflexive Banach spaces always have sequences that generate convex block homogeneous spreading models.

Remark 10.1. Let $(x_i)_i$ be a non-trivial weak Cauchy sequence in some Banach space X that generates a sequence $(e_i)_i$ as a spreading model. If $(y_i)_i$ is a convex block sequence of $(x_i)_i$ that generates a sequence $(\tilde{e}_i)_i$ as a spreading model, then the linear map $T : [(e_i)_i] \to [(\tilde{e}_i)_i]$ with $Te_i = \tilde{e}_i$ has norm at most one. Note that it is important for the sequence $(x_i)_i$ to already generate some spreading model.

Lemma 10.2. *Let E_ξ, $\xi < \omega_1$ be a transfinite hierarchy of Banach spaces so that each E_ξ has a Schauder basis $(e_i^\xi)_i$. Assume moreover that for every $\xi < \zeta$ the linear map $T_{\xi,\zeta} : E_\xi \to E_\zeta$ defined by $T_{\xi,\zeta}e_i^\xi = e_i^\zeta$ is well defined and has norm at most one. Then there exists $\xi_0 < \omega_1$ such that for every $\xi_0 \leqslant \xi < \omega_1$ the map $T_{\xi_0,\xi}$ is an isometry.*

Proof. If we assume that the conclusion does not hold, then by passing to an uncountable subset and relabeling we may assume that for every $\xi < \omega_1$ there exist $n_\xi \in \mathbb{N}$ and rational numbers $(c_i^\xi)_{i=1}^{n_\xi}$ such that for every $\xi < \zeta < \omega_1$ the following holds.

$$\left\|\sum_{i=1}^{n_\xi} c_i^\xi e_i^\zeta\right\| < \left\|\sum_{i=1}^{n_\xi} c_i^\xi e_i^\xi\right\|. \tag{18}$$

By passing to a further uncountable subset and relabeling once more we may assume that there exist $n \in \mathbb{N}$ and rational numbers $(c_i)_{i=1}^n$ such that for every $\xi < \zeta < \omega_1$ the following holds.

$$\left\|\sum_{i=1}^{n} c_i e_i^\zeta\right\| < \left\|\sum_{i=1}^{n} c_i e_i^\xi\right\|. \tag{19}$$

Setting $\alpha_\xi = \|\sum_{i=1}^n c_i e_i^\xi\|$ for $\xi < \omega_1$, we conclude that $(\alpha_\xi)_{\xi<\omega_1}$ is strictly decreasing which is absurd. □

Proposition 10.3. *Let X be a Banach space and $(x_i)_i$ be a bounded sequence in X without a weakly convergent subsequence. Then, there exists a convex block sequence $(\tilde{x}_i)_i$ of $(x_i)_i$ generating a spreading model $(e_i)_i$ so that the spreading model admitted by any further convex block sequence $(y_i)_i$ of $(\tilde{x}_i)_i$ is isometrically equivalent to $(e_i)_i$.*

Proof. By Rosenthal's ℓ_1 theorem [20], $(x_i)_i$ has either a subsequence that is equivalent to the unit vector basis of ℓ_1 or it has a non-trivial weak Cauchy subsequence. In either case, by passing to a subsequence, every further convex block sequence of $(x_i)_i$ has a subsequence generates a Schauder basic spreading model. Let us assume now that the conclusion is not satisfied. Using a transfinite recursion and Remark 10.1 we may construct a transfinite hierarchy $(x_i^\xi)_i$ of convex block sequences of $(x_i)_i$, satisfying the following:

- (i) For every $\xi < \zeta < \omega_1$ the sequence $(x_i^\zeta)_i$ is eventually a convex block sequence of $(x_i^\xi)_i$ ("eventually" is necessary to pass the properties to limit ordinals).
- (ii) For every $\xi < \omega_1$ the sequence $(x_i^\xi)_i$ generates a Schauder basic sequence $(e_i^\xi)_i$ as a spreading model.
- (iii) For every $\xi < \zeta$ the natural linear map $T : [(e_i^\xi)_i] \to [(e_i^\zeta)_i]$ has norm at most one but it is not an isometry.

Lemma 10.2 and (iii) yield a contradiction. □

Proposition 10.4. *Let X be a Banach space and $(x_i)_i$ be a sequence in X that generates a conditional spreading model $(e_i)_i$ and let also $(z_i)_i$ be the convex block homogeneous part of $(e_i)_i$. Then there exists a block sequence $(y_i)_i$ of averages of $(x_i)_i$ so that:*

(i) *the sequence* $(y_i)_i$ *generates a spreading model isometrically equivalent to* $(z_i)_i$ *and*
(ii) *every convex block sequence* $(w_i)_i$ *of* $(y_i)_i$ *has a subsequence generating a spreading model isometrically equivalent to* $(z_i)_i$.

Proof. By Rosenthal's ℓ_1 theorem [20], passing to a subsequence, the sequence $(x_i)_i$ is non-trivial weak Cauchy. Otherwise, if it had a weakly null subsequence, as it is well known, it would generates an unconditional spreading model. If it had a subsequence converging weakly to a non-zero element it would generate a singular spreading model or an ℓ_1 spreading model (see [3, Theorem 38, page 592]), or if it had a norm convergent subsequence it would generate a trivial spreading model (i.e. a spreading sequence in seminormed space that is not a normed space). Find a convex block sequence $(\tilde{x}_i)_i$ of $(x_i)_i$ that satisfies the conclusion of Proposition 10.3. Denote by $(\tilde{e}_i)_i$ the spreading model generated by this sequence. The sequence $(x_i - \tilde{x}_i)_i$ is weakly null and, passing to a subsequence, it is either norm null or it generates some unconditional spreading model $(v_i)_i$. If it is norm null then the proof is complete. Otherwise, (v_i) is not equivalent to the unit vector basis of ℓ_1. If it were, then either $(e_i)_i$ or $(\tilde{e}_i)_i$ would have to be equivalent to the unit vector basis of ℓ_1 and by Remark 10.1 this cannot be the case. By Lemma 5.2 (applied to (v_i)) and a standard counting argument, passing to a subsequence of $(x_i - \tilde{x}_i)_i$ we have that for every $\varepsilon > 0$ there exists $M_0, n_0 \in \mathbb{N}$ so that for any $F \subset \mathbb{N}$ with $\min(F) \geqslant n_0$ and $\#F \geqslant M_0$ we have

$$\frac{1}{\#F}\left\|\sum_{i\in F}(x_i - \tilde{x}_i)\right\| < \varepsilon. \tag{20}$$

By Theorem 4.1, given $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $M_n \in \mathbb{N}$ so that for all $a_1, \dots, a_n \in [-1, 1]$ and $M \geqslant M_n$ we have

$$\left|\left\|\sum_{i=1}^n a_i z_i\right\| - \left\|\sum_{i=1}^n a_i\left(\frac{1}{M}\sum_{j=((i-1)M+1)}^{iM} e_j\right)\right\|\right| < \varepsilon. \tag{21}$$

Using (20), the fact that the spreading model of $(\tilde{x}_i)_i$ is preserved when taking averages, and a limit argument one may deduce that for any $\varepsilon > 0$ there exists $M_0 \in \mathbb{N}$ so that for any $M \geqslant M_0$ and $a_1, \dots, a_n \in [-1, 1]$

$$\left|\left\|\sum_{i=1}^n a_i \tilde{e}_i\right\| - \left\|\sum_{i=1}^n a_i\left(\frac{1}{M}\sum_{j=((i-1)M+1)}^{iM} e_j\right)\right\|\right| < \varepsilon. \tag{22}$$

It immediately follows that $(z_i)_i$ and $(\tilde{e}_i)_i$ are isometrically equivalent. Using (20) one may clearly now choose a sequence of averages of $(x_i)_i$ that satisfies the conclusion. □

Corollary 10.5. *Let* X *be a non-reflexive Banach space. Then there exists a sequence* $(x_i)_i$ *in* X *that generates a 1-convex block homogeneous spreading model* $(e_i)_i$.

Proof. For a bounded sequence without a weakly convergent subsequence apply Proposition 10.3 to find a sequence $(x_i)_i$ generating a spreading model $(e_i)_i$ so that the conclusion of that proposition is satisfied. The argument used to obtain (21) yields that $(e_i)_i$ is 1-convex block homogeneous. □

Remark 10.6. Note that it is not true that a non-reflexive Banach space X must necessarily admit a conditional convex block homogeneous spreading model. It may very well be the case that X only admits the unit vector basis of ℓ_1 as a spreading model.

In [6] a Banach space $\mathfrak{X}_{\text{usm}}$ is constructed that is hereditarily spreading model universal for all subsymmetric sequences. This means that for every possible subsymmetric sequence $(x_i)_i$ and every subspace Y of $\mathfrak{X}_{\text{usm}}$ there exists a sequence $(y_i)_i$ in Y that generates a spreading model equivalent to $(x_i)_i$. This is a hereditarily indecomposable reflexive Banach space and it is constructed via a Tsirelson-type saturation method with constraints. We restate a problem posed in that paper that is relevant to this section.

Problem. Does there exist a Banach space X that is hereditarily universal for all spreading sequences, i.e. both for conditional and unconditional ones?

A space X with the aforementioned property would have to be saturated with non-reflexive hereditarily indecomposable subspaces. Before closing this section it is worth mentioning that Banach spaces with a conditional spreading basis don't have a large variety of conditional spreading models.

Proposition 10.7. *Let X be a Banach space with a conditional spreading basis $(e_i)_i$ and let $(z_i)_i$ be its convex block homogeneous part. Let also $(x_i)_i$ be a sequence in X that generates a conditional spreading model $(d_i)_i$. Then, the convex block homogeneous part of $(d_i)_i$ is either equivalent to $(z_i)_i$ or to the summing basis of c_0.*

Proof. Apply Proposition 10.4 to find a sequence $(y_i)_i$ that generates the convex block homogeneous part $(\tilde{d}_i)_i$ of $(d_i)_i$ and crucially also satisfies the second conclusion of that proposition. By Theorem 7.1 (i) $(y_i)_i$ has a convex block sequence $(\tilde{y}_i)_i$ that is either equivalent to $(z_i)_i$ or to the summing basis of c_0. By Proposition 10.4 (ii) $(\tilde{d}_i)_i$ is equivalent to $(z_i)_i$ or equivalent to the summing basis of c_0. □

Remark 10.8. The above proposition easily implies that there exists no space with a spreading Schauder basis that admits all conditional spreading models.

11. Universality of $C(\omega^\omega)$ for all spreading models

It was proved by E. Odell in [18, Proposition 5.10, page 419] that every subsymmetric sequence is generated as a spreading model by some sequence in the space $C(\omega^\omega)$. In this section we prove that this can be extended to include conditional spreading sequences as well. This is interesting because the ordinal number ω^ω is the first α for which $C(\alpha)$ is not isomorphic to $c_0(\mathbb{N})$. Whereas the space $c_0(\mathbb{N})$ admits only the unit vector basis of $c_0(\mathbb{N})$ and the summing basis of $c_0(\mathbb{N})$ as spreading models, the "successor" space $C(\omega^\omega)$ admits all possible spreading models. We mention here that in [3, Proposition 63, page 606] it was proved that $c_0(\mathbb{N})$ admits all spreading sequences as 2-spreading models in the sense of [3]. We first show the result for convex block homogeneous sequences. The final statement is then an easy consequence of Theorem 4.1. The proof requires a relatively short construction in the flavor of Schreier space [22]. In this section we shall denote by e_i^* the elements of the unit vector basis of $\mathbb{R}^n$, of $c_{00}(\mathbb{N})$, or of $\ell_\infty(\mathbb{N})$.

We fix a Banach space Z with a normalized bimonotone equal signs additive basis $(z_i)_i$ (recall that by Corollary 3.5 any space with a convex block homogeneous basis may be renormed to have this property). Then, $(z_i)_i$ is 1-spreading and furthermore, by Remark 3.7, for all intervals $E_1 < E_2 < \cdots < E_n$ of $\mathbb{N}$ with $\max(E_k) + 1 = \min(E_k)$, for $1 \leqslant k < n$, and all scalars $a_1, \ldots, a_m$ we have

$$\left\| \sum_{k=1}^{n} \left(\sum_{j \in E_k} a_j \right) z_k \right\| \leqslant \left\| \sum_{j=1}^{m} a_j z_j \right\|. \tag{23}$$

For each $n \in \mathbb{N}$ take a $(1/2n)$-dense finite set F_n in the unit ball of the subspace spanned by the first n vectors of $(z_i^*)_i$, the biorthogonal functionals to (z_i). We require F_n to satisfy the following.

(a) F_n is symmetric and $z_i^* \in F_n$ for $1 \leqslant i \leqslant n$,
(b) F_n is closed under projections onto intervals, and
(c) if $x^* = \sum_{i=1}^{n} a_i z_i^*$ is in F_n and $1 \leqslant k_0 < n$, then $y^* = \sum_{i=1}^{k_0} a_i z_i^* + \sum_{i=k_0+1}^{n-1} a_{i+1} z_i^*$ is in F_n as well.

Note that property (c) makes sense because, using only the fact that $(z_i)_i$ is 1-spreading, if x^*, y^* are as above then $\|y^*\| \leqslant \|x^*\|$. We furthermore require $F_n \subset F_{n+1}$ for all $n \in \mathbb{N}$.

In the definition bellow, by $\sum_{i \in A} e_i^*$ we mean the sequence of scalars $(c_i)_i$ in $\ell_\infty(\mathbb{N})$ with $c_i = 1$ for $i \in A$ and $c_i = 0$ otherwise. The set A may be finite or infinite. Define the subset of the unit ball of $(\ell_\infty(\mathbb{N}), \|\cdot\|_{\ell_\infty})$

$$\begin{aligned} \mathcal{K}_Z = \Bigg\{ \sum_{k=1}^{n} a_k \Bigg(\sum_{m_k \leqslant i < m_{k+1}} e_i^* \Bigg) &: n \leqslant m_1 < \cdots < m_n < m_{n+1} \leqslant \infty, \\ & n \in \mathbb{N} \text{ and } \sum_{k=1}^{n} a_k z_k^* \in F_n \Bigg\}. \end{aligned} \tag{24}$$

It is rather standard to verify that $\mathcal{K}_Z$ is a countable and compact subset of the unit ball of $\ell_\infty(\mathbb{N})$ endowed with the topology of pointwise convergence and that the Cantor–Bendixson index of $\mathcal{K}_Z$ is $\omega + 1$. In fact, the ω'th derivative of $\mathcal{K}_Z$ is a the singleton that contains the zero element of $\ell_\infty(\mathbb{N})$. To see this, note that an element of $\mathcal{K}_Z$ depends on a set $\{m_1 < \cdots < m_n < m_{n+1}\}$ in $\mathbb{N}\cup\{\infty\}$ with $n \leqslant m_1$ and scalar coefficients $a_1, \dots, a_n$ that may be chosen from a finite set that depends on n. Therefore, $\mathcal{K}_Z$ is homeomorphic to ω^ω and the space $C(\mathcal{K}_Z)$ is isometric to $C(\omega^\omega)$.

We define a norm on $c_{00}(\mathbb{N})$ by setting $\|x\|_{\tilde{\mathcal{S}}(Z)} = \sup\{|f(x)| : f \in \mathcal{K}_Z\}$, where for $f \in \mathcal{K}_Z$ and $x \in c_{00}(\mathbb{N})$ the symbol $f(x)$ denotes the usual inner product of f with x. We denote $\tilde{\mathcal{S}}(Z)$ to be the the completion of the space $(c_{00}(\mathbb{N}), \|\cdot\|_{\tilde{\mathcal{S}}(Z)})$. One may refer to the space $\tilde{\mathcal{S}}(Z)$ as the "conditional schreierification of Z". Note that the map $T : \tilde{\mathcal{S}}(Z) \to C(\mathcal{K}_Z)$ defined by $T(e_i)(f) = f(e_i)$ is a linear isometric embedding. We note for later use the following, which follows from $\mathcal{K}_Z^{(\omega)} = \{0\}$.

Remark 11.1. Any onto homeomorphism $\phi : \omega^\omega \to \mathcal{K}_Z$ satisfies $\phi(\omega^\omega) = 0$. Hence, if $T : \tilde{S}(Z) \to C(\mathcal{K}_Z)$ is the above isometry and $\tilde{T} : \tilde{\mathcal{S}}(Z) \to C(\omega^\omega)$ is defined by $\tilde{T}x(\alpha) = Tx(\phi(a))$, then for every $x \in \tilde{\mathcal{S}}(Z)$ we have $\tilde{T}x(\omega^\omega) = 0$. That is, $\tilde{\mathcal{S}}(Z)$ isometrically embeds into $C_0(\omega^\omega)$.

It remains to observe that the sequence $(e_i)_i$ endowed with $\|\cdot\|_{\tilde{\mathcal{S}}(Z)}$ admits $(z_i)_i$ as a spreading model. Indeed, (24) easily implies that for $n < k_1 < \cdots < k_n$ and $c_1, \dots, c_n \in \mathbb{R}$ setting $m_1 = k_1, \dots, m_n = k_n$ we have

$$\left\|\sum_{i=1}^n c_i e_{k_i}\right\| \geqslant \max\left\{x^*\left(\sum_{i=1}^n c_i z_i\right) : x^* \in F_n\right\} \geqslant \frac{2n-1}{2n}\left\|\sum_{i=1}^n c_i z_i\right\|, \tag{25}$$

whereas on the other hand by (24), if $x = \sum_{i=1}^n c_k z_{k_i}$ we also obtain

$$\begin{aligned}\left\|\sum_{i=1}^n c_i e_{k_i}\right\| \leqslant \max\Bigg\{ &\left\|\sum_{j=1}^m s_{E_j}(x) z_i\right\| : (E_j)_{j=1}^m \text{ is a sequence} \\ &\text{of consecutive intervals of } \mathbb{N}\Bigg\} \\ = &\left\|\sum_{i=1}^n c_i z_i\right\| \text{ (by (23))}.\end{aligned} \tag{26}$$

The desired result follows from (25) and (26). We summarize what we have shown in the following statement.

Remark 11.2. Given a Banach space with a bimonotone equal signs additive basis $(z_i)_i$, the unit vector basis $(e_i)_i$ of $c_{00}(\mathbb{N})$ endowed with $\|\cdot\|_{\tilde{\mathcal{S}}(Z)}$

(i) forms a bimonotone Schauder basis for the space $\tilde{\mathcal{S}}(Z)$,
(ii) generates a spreading model isometrically equivalent to $(z_i)_i$, and
(iii) the summing functional s defined on it is bounded, in fact it has norm one.

To see (i), note that in (24), by property (b) of the sets F_n, the coefficients $(a_i)_{i=1}^n$ may be restricted to intervals of $\{1, \dots, n\}$ by replacing the initial part and the tail part with zeros. This is possible because the sets F_n are chosen to be closed under taking projections onto intervals.

Translating the above remark and using the isometric embedding $\tilde{T}$ of $\tilde{\mathcal{S}}(Z)$ into $C(\omega^\omega)$ we obtain the following.

Proposition 11.3. *Let Z be a Banach space with a bimonotone and equal signs additive Schauder basis $(z_i)_i$. Then, $C(\omega^\omega)$ contains a sequence $(f_i)_i$ that generates a spreading model isometrically equivalent to $(z_i)_i$.*

We now state and prove the main result of this section.

Theorem 11.4. *Let X be a Banach space with a spreading Schauder basis $(x_i)_i$. Then there exists a sequence $(y_i)_i$ in $C(\omega^\omega)$ that generates a spreading model equivalent to $(x_i)_i$.*

Proof. As we mentioned earlier, the unconditional case was already proved by E. Odell. In the conditional case and assuming that $(x_i)_i$ is 1-spreading, apply Theorem 4.1 and let $(u_i)_i$, $(z_i)_i$ be the unconditional and convex block homogeneous parts of $(x_i)_i$ respectively. Find sequences $(f_i)_i$ and $(g_i)_i$ in $C(\omega^\omega)$ generating $(u_i)_i$ and $(z_i)_i$ as spreading models. Then, the sequence $(f_i, g_i)_i$ in $C(\omega^\omega) \oplus C(\omega^\omega)$ generates a spreading model equivalent to $(x_i)_i$. Since $C(\omega^\omega)$ is isomorphic to its square the proof is complete. □

Remark 11.5. A somewhat similar proof yields that for a countable ordinal number α, the space $C(\omega^{\omega^\alpha})$ admits all Schauder basic spreading sequences as $\mathcal{S}_\alpha$-spreading models. That is, if $(x_i)_i$ is a Schauder basic spreading sequence then there exists a sequence $(f_i)_i$ in $C(\omega^{\omega^\alpha})$ and positive constants κ, K so that for all $F \in \mathcal{S}_\alpha$ and choice of scalars $(a_i)_{i\in F}$ we have

$$\kappa \left\| \sum_{i\in F} a_i x_i \right\| \leqslant \left\| \sum_{i\in F} a_i f_i \right\| \leqslant K \left\| \sum_{i\in F} a_i x_i \right\|.$$

The proof requires a variation of the set $\mathcal{K}_Z$ with an $\mathcal{S}_\alpha$ condition. This has to do with the convex block homogeneous part of $(x_i)_i$ and a similar set has to be defined for the unconditional part as well.

We shall now orient our attention towards proving a slightly more precise statement of Theorem 11.4. The reason for this is that we will require an extra condition for the sequence in $C(\omega^\omega)$ that generates the desired conditional spreading model, in order to prove the main result of Section 12.

Remark 11.6. Given a Banach space U with a suppression unconditional spreading basis $(u_i)_i$, one may define a norm $\|\cdot\|_{\mathcal{S}(U)}$ on $c_{00}(\mathbb{N})$ with completion $\mathcal{S}(U)$ so that the unit vector basis $(e_i)_i$ of $c_{00}(\mathbb{N})$, endowed with $\|\cdot\|_{\mathcal{S}(U)}$,

(i) forms a suppression unconditional Schauder basis for the space $\mathcal{S}(U)$,
(ii) it generates a spreading model isometrically equivalent to $(u_i)_i$, and
(iii) it is isometrically equivalent to some sequence $(g_i)_i$ in $C_0(\omega^\omega)$.

The definition of the norm $\|\cdot\|_{\mathcal{S}(U)}$ is a simpler version of the definition of $\|\cdot\|_{\tilde{\mathcal{S}}(Z)}$ and it uses a set similar to $\mathcal{K}_Z$ that contains elements of the form $\sum_{i=1}^n a_i e_{m_i}$ where $n \leqslant m_1 < \cdots < m_n$ and $y^* = \sum_{k=1}^n a_i u_i^*$ is in an appropriate subset G_n of the unit ball of U^* that is closed under projections onto subsets. This is a more precise statement than what was proved by Odell in [18, Proposition 5.10, page 419] and we shall use it in the sequel.

Lemma 11.7. *Let Z be a Banach space with a bimonotone equal signs additive basis. Let $x = \sum_{i=1}^m c_i e_i$ be a vector in $c_{00}(\mathbb{N})$ and $1 \leqslant p \leqslant q \leqslant m$ be natural numbers so that $\sum_{i=p}^q c_i = 0$. Then, if $y = \sum_{1\leqslant i<p} c_i e_i + \sum_{q<i\leqslant m} c_i e_i$ we have $\|y\|_{\tilde{\mathcal{S}}(Z)} \leqslant \|x\|_{\tilde{\mathcal{S}}(Z)}$. In particular, any block sequence $(y_i)_i$ in $\tilde{\mathcal{S}}(Z)$, with $s(y_i) = 0$ for all $i \in \mathbb{N}$, is suppression unconditional.*

Proof. Let $f = \sum_{k=1}^n a_k \sum_{m_k\leqslant j<m_{k+1}} e_j^*$ be in $\mathcal{K}_Z$, as in (24), i.e. $x^* = \sum_{k=1}^n a_k z_k^* \in F_n$ and $n \leqslant m_1 < \cdots < m_n < m_{n+1} \leqslant \infty$. We will show that there is $g \in \mathcal{K}_Z$ with $f(y) = g(x)$. We distinguish two cases. If there is $1 \leqslant k \leqslant n$ so that $m_k \leqslant p$ and $q < m_{k+1}$ observe that $f(y) = f(x)$. Otherwise, set $k_1 = \min\{1 \leqslant k \leqslant n\colon \text{with } p \leqslant m_k < q\}$ and $k_2 = \max\{1 \leqslant k \leqslant n\colon \text{with } p \leqslant m_k \leqslant q\}$. We shall assume that $k_1 < k_2$ as the case $k_1 = k_2$ is treated slightly differently but very similarly.

By property (c) of F_n (see page 1242) we have that $y^* = \sum_{k=1}^{k_1} a_k z_k^* + \sum_{k=k_1+1}^{n-k_2+k_1+1} a_{k+k_2-k_1-1} z_k^*$ is in F_n. We define $(\tilde{m}_k)_{k=1}^{n+1-k_2+k_1+1}$ as follows.

$$\tilde{m}_k = \begin{cases} m_k & \text{if } 1 \leqslant k < k_1, \\ p & \text{if } k = k_1, \\ q & \text{if } k = k_1 + 1, \text{ and} \\ m_{k+k_2-k_1-1} & \text{if } k_1 + 1 < k \leqslant n + 1 - k_2 + k_1. \end{cases}$$

Define

$$g = \sum_{k=1}^{k_1} a_k \sum_{\tilde{m}_k\leqslant j<\tilde{m}_{k+1}} e_j^* + \sum_{k=k_1+1}^{n-k_2+k_1+1} a_{k+k_2-k_1-1} \sum_{\tilde{m}_k\leqslant j<\tilde{m}_{k+1}} e_j^*.$$

Then, g is in $\mathcal{K}_Z$. Some computations yields that $f(y) = g(x)$. □

In isomorphic terms, the only improvement of the following statement when compared to Theorem 11.4 is conclusion (iii). This is however a very important condition necessary to prove the main result of Section 12.

Proposition 11.8. *Let Z be a Banach space with a bimonotone equal signs additive basis $(z_i)_i$ and U be a Banach space with a suppression unconditional spreading basis $(u_i)_i$. Denote by $(f_i)_i$ the basis of $\tilde{\mathcal{S}}(Z)$ and by $(g_i)_i$ the basis of $\mathcal{S}(U)$. Also denote by $(x_i)_i$ the sequence $(u_i, z_i)_i$ in $(U \oplus Z)_0$ and set $X = [(x_i)_i]$. Finally, by $(h_i)_i$ denote the sequence $(g_i, f_i)_i$ in $(\mathcal{S}(U) \oplus \tilde{\mathcal{S}}(Z))_0$ and set $\tilde{X} = [(h_i)_i]$. Then, the following hold.*

(i) *The sequence $(h_i)_i$ generates a spreading model isometrically equivalent to $(x_i)_i$.*
(ii) *The summing functional defined on $(h_i)_i$ is bounded, in fact it has norm one.*
(iii) *Every block sequence $(w_i)_i$ of $(h_i)_i$ with $s(w_i) = 0$ for all $i \in \mathbb{N}$ is suppression unconditional.*
(iv) *The space $\tilde{X}$ embeds isometrically into $C(\omega^\omega)$. In particular, $\tilde{X}$ is c_0-saturated.*

Proof. The first statement is an immediate consequence of Remark 11.2 (ii) and Remark 11.6 (ii). The second statement follows from Remark 11.2 (iii). The third statement follows from Remark 11.6 (i) and Lemma 11.7. To see the last statement recall that $(C_0(\omega^\omega) \oplus C_0(\omega^\omega))_0$ embeds isometrically into $C(\omega^\omega)$. By Remark 11.1 the space $\tilde{\mathcal{S}}(Z)$ embeds isometrically into $C_0(\omega^\omega)$ and by Remark 11.6 (iii) $\mathcal{S}(U)$ embeds isometrically into $C_0(\omega^\omega)$ as well. We conclude that $(\mathcal{S}(U) \oplus \tilde{\mathcal{S}}(Z))_0$ embeds isometrically into $C(\omega^\omega)$ which yields the desired result. □

12. Spreading models of quasi-reflexive spaces

As it was proved in Section 10, non-reflexive Banach spaces always admit convex block homogeneous spreading models whereas in Section 11 we showed that $C(\omega^\omega)$ admits all possible spreading sequences as spreading models. In this section we show that every conditional spreading sequence is admitted as a spreading model of a Banach space that is quasi-reflexive of order one, i.e. a non-reflexive space that is almost reflexive.

The most difficult part of the present section is to prove the proposition below. It involves a Tsirelson-type construction with saturation under constraints. For now, we state it and use it to prove the main theorem of this section. We present the proof of the proposition later.

Proposition 12.1. *Let X be a Banach space with a normalized bimonotone Schauder basis $(x_i)_i$ so that*

(i) *the space ℓ_1 does not embed into X and*
(ii) *the summing functional $s : X \to \mathbb{R}$ with respect to the basis $(x_i)_i$ is bounded and any block sequence $(y_i)_i$ of $(x_i)_i$, with $s(y_i) = 0$ for all $i \in \mathbb{N}$, is suppression unconditional.*

Then there exists a Banach space $\mathfrak{X}$ with a bimonotone Schauder basis $(e_i)_i$ satisfying the following.

(a) *The linear map $T : \mathfrak{X} \to X$ defined by $Te_i = x_i$ is bounded and has norm one.*
(b) *For any natural numbers $n < i_1 < \cdots < i_n$ and real numbers $c_1, \ldots, c_n$ we have that $\| \sum_{k=1}^n c_k x_{i_k} \| = \| \sum_{k=1}^n c_k e_{i_k} \|$.*
(c) *The basis $(e_i)_i$ is boundedly complete.*
(d) *The summing functional $s : \mathfrak{X} \to \mathbb{R}$ with respect to the basis $(e_i)_i$ is bounded and every block sequence $(y_i)_i$ of $(e_i)_i$ with $s(y_i) = 0$ for all $i \in \mathbb{N}$ spans a reflexive subspace of $\mathfrak{X}$.*

We now state the main result of this section. This result has two statements, one for unconditional spreading sequences and one for conditional ones. We shall give a proof of the second statement, which is also the more difficult one and it uses Proposition 12.1. Afterwards, we provide a proof of Proposition 12.1. At the end of the section we shall point out some of the steps required to achieve the first statement of the following theorem.

Theorem 12.2. *Let $(x_i)_i$ be a spreading Schauder basic sequence.*

(i) *If $(x_i)_i$ is unconditional, then there exists a reflexive Banach space X with an unconditional Schauder basis $(e_i)_i$ that generates a spreading model equivalent to $(x_i)_i$.*
(ii) *If $(x_i)_i$ is conditional, then there exists a Banach space X that is quasi-reflexive of order one with a Schauder basis $(e_i)_i$ that generates a spreading model equivalent to $(x_i)_i$.*

Proof of (ii). We may assume that $(x_i)_i$ is bimonotone 1-spreading. Using Theorem 4.1 and Corollary 3.5, by passing to an equivalent norm on X, there are a suppression unconditional sequence $(u_i)_i$ and a bimonotone equal signs additive sequence $(z_i)_i$ spanning spaces U and Z respectively, so that $(x_i)_i$ is isometrically equivalent to the sequence $(u_i, z_i)_i$ in $(U \oplus Z)_0$. Take the space $\tilde{X}$ with basis $(h_i)_i$ given by Proposition 11.8. Apply Proposition 12.1 to the space $\tilde{X}$ to obtain a space $\mathfrak{X}$ with a Schauder basis $(e_i)_i$ satisfying the conclusion of that Proposition. We claim that this space has the desired properties.

By Propositions 11.8 (i) and 12.1 (b) we deduce that $(e_i)_i$ generates $(x_i)_i$ as a spreading model. Furthermore, Propositions 11.8 (ii) and 12.1 (a) clearly yield that the summing functional is bounded on $(e_i)_i$ which implies that the sequence $(d_i)_i$ with $d_1 = e_1$ and $d_i = e_i - e_{i-1}$ for $i \geqslant 2$ is a Schauder basis for $\mathfrak{X}$. Indeed, if $(P_n)_n$ denotes the sequence of projections associated to $(e_i)_i$, then the sequence of projections $(Q_n)_n$ associated to $(d_i)_i$ is given by $Q_n x = P_n x + s_{[n+1,\infty)}(x) e_n$. By Proposition 12.1 one may conclude that any skipped block sequence of $(d_i)_i$ spans a reflexive subspace of $\mathfrak{X}$. Since $\mathfrak{X}$ generates a conditional spreading model it cannot be reflexive and therefore by [14, Theorem 2.1, page 3] it is quasi-reflexive of order one. □

Proof of Proposition 12.1 We shall brake up the proof of Proposition 12.1 into several steps. Let us from now on fix a Banach space X with a normalized bimonotone Schauder basis $(x_i)_i$ satisfying the assumptions of Proposition 12.1. Let us also fix strictly increasing sequences of natural numbers $(m_j)_j, (n_j)_j$, with $m_1 \geqslant 2, n_1 \geqslant 4$, $m_j < n_j < m_{j+1}$ for all $j \in \mathbb{N}$ and $\sum_j (m_j/n_j) < 1$. A norming set is a symmetric subset G of the unit ball of $c_{00}(\mathbb{N})$ that contains all elements of the unit vector basis $(e_i)_i$.

Notation. If G is a norming set, a functional

$$\alpha = \frac{1}{n}(f_1 + \cdots + f_k),$$

where $k \leqslant n \in \mathbb{N}$ and $f_1 < \cdots < f_k \in G$, will be called an *α-average* of G. The *size of α* is defined to be $s(\alpha) = n$.

A sequence $\alpha_1 < \cdots < \alpha_k < \cdots$ of α-averages will be called *very fast growing* if $s(\alpha_{k+1}) > \max\{s(\alpha_k), 2^{\max \operatorname{supp} \alpha_k}\}$ for all k.

A functional

$$f = \frac{1}{m_j} \sum_{q=1}^{d} \alpha_q,$$

where $j \in \mathbb{N}$, $d \leqslant n_j$ and $\alpha_1 < \cdots < \alpha_d$ is a very fast growing sequence of α-averages of G, will be called a *weighted functional* of G. The *weight of f* is defined to be $w(f) = j$.

We shall recursively define an increasing sequence of subsets of $c_{00}(\mathbb{N})$ and use their union to define a norm on $c_{00}(\mathbb{N})$. Set

$$W_0 = \left\{ \sum_{i=1}^{n} \lambda_i e_i : \left\| \sum_{i=1}^{n} \lambda_i x_i^* \right\| \leqslant 1 \right\} \tag{27}$$

and assuming that W_n has been defined, we set

$$\begin{aligned} W_{n+1}^{\alpha} &= \{\alpha : \ \alpha \text{ is an } \alpha\text{-average of } W_n\} \\ W_{n+1}^{w} &= \{f : \ f \text{ is a weighted functional of } W_n\} \\ W_{n+1} &= W_n \cup W_{n+1}^{\alpha} \cup W_{n+1}^{w} \end{aligned}$$

Finally we set $W = \cup_{n=0}^{\infty} W_n$. For $x \in c_{00}(\mathbb{N})$ we define

$$\|x\| = \sup\{f(x) : f \in W\}$$

and we set $\mathfrak{X}$ to be the completion of $(c_{00}(\mathbb{N}), \|\cdot\|)$.

Remarks 12.3. The following are easy observations that follow from the definition of W.

(i) For every $f \in W$ and E interval of the natural numbers, $f|_E$ is also in W, therefore the unit vector basis of $(e_i)_i$ forms a bimonotone Schauder basis for $\mathfrak{X}$.
(ii) Since $W_0 \subset W$, the linear map $T : \mathfrak{X} \to X$ defined by $Te_i = x_i$ is bounded and it has norm one. It follows that the summing functional $s : \mathfrak{X} \to \mathbb{R}$ is bounded.
(iii) The set W is the smallest norming set such that $W_0 \subset W$ and for every $j \in \mathbb{N}, d \leqslant n_j$ and very fast growing sequence of α-averages $\alpha_1 < \cdots < \alpha_d$ of W, the functional $f = \frac{1}{m_j}\sum_{q=1}^{d}\alpha_q$ is also in W.

Proposition 12.4. *The unit vector basis of $(e_i)_i$ forms a boundedly complete Schauder basis for $\mathfrak{X}$.*

Proof. Towards a contradiction, let us assume that the conclusion fails. Then, there exists a block sequence $(y_k)_k$ and $\varepsilon > 0$, such that the following hold.

(i) For every $k \in \mathbb{N}$ $\|y_k\| > \varepsilon$, i.e. there exists $f_k \in W$ with $\operatorname{ran} f_k \subset \operatorname{ran}(y_k)$ and $f_k(y_k) > \varepsilon$.
(ii) For every $n \in \mathbb{N}$ we have that $\|\sum_{k=1}^{n} y_k\| \leqslant 1$

Set $E_1 = \{1\}$ and choose a sequence $E_1 < E_2 < \cdots$ of intervals of the natural numbers satisfying the following.

(a) For every $k \in \mathbb{N}$, $\max E_k + 1 = \min E_{k+1}$.
(b) For every $k \in \mathbb{N}$, $\#E_{k+1} > \max\{\#E_k, 2^{\max\operatorname{supp} y_{\max E_k}}\}$.

Set $w_k = \sum_{i\in E_k} y_i$ and $\alpha_k = (1/\#E_k)\sum_{i\in E_k} f_i$. The following hold.

(α) For every $n \in \mathbb{N}$, $\|\sum_{k=1}^{n} w_k\| = \|\sum_{i=1}^{\max E_n} y_i\| \leqslant 1$.
(β) For every $k \in \mathbb{N}$, $\alpha_k(w_k) > \varepsilon$.
(γ) The sequence $(\alpha_k)_k$ is very fast growing.

Choose $j \in \mathbb{N}$ such that $(\varepsilon n_j)/m_j > 1$. Then from ($\gamma$) we have that $f = \frac{1}{m_j}\sum_{k=1}^{n_j}\alpha_k$ is in W. From (β) we have that $f(\sum_{k=1}^{n_j} w_k) > 1$, i.e. $\|\sum_{k=1}^{n_j} w_k\| > 1$. This contradicts ($\alpha$) and the proof is complete. □

Proposition 12.5. *For any natural numbers $n < i_1 < \cdots < i_n$ and real numbers $c_1, \ldots, c_n$ we have that*

$$\left\|\sum_{k=1}^{n} c_k x_{i_k}\right\| = \left\|\sum_{k=1}^{n} c_k e_{i_k}\right\|.$$

Proof. Fix $n < i_1 < \cdots < i_n$ and real numbers $c_1, \ldots, c_n$ such that if $y = \sum_{k=1}^{n} c_k x_{i_k}$, then $\|y\| = 1$. Remark 12.3 (ii) yields that $\|\sum_{k=1}^{n} c_k e_{i_k}\| \geqslant 1$. We shall prove that

$\|x\| \leqslant 1$, $x = \sum_{k=1}^n c_k e_{i_k}$, by inductively showing that for $f \in W_m$, $f(x) \leqslant 1$, for $m = 0, 1, \ldots$.

For $f \in W_0$ this trivially follows by the fact that $\|\sum_{k=1}^n c_k x_{i_k}\| = 1$. Assume that the statement holds for every $f \in W_m$ and let $f \in W_{m+1}$. If f is an α-average of W_m, the result follows trivially from the inductive assumption. Otherwise, there exist $j \in \mathbb{N}, d \leqslant n_j$ and $\alpha_1 < \cdots < \alpha_d$ a very fast growing sequence of α-averages of W_m with $f = \frac{1}{m_j} \sum_{q=1}^d \alpha_q$.

Set $q_1 = \min\{q : \min \operatorname{ran} \alpha_q \geqslant n\}$. Then by the inductive assumption $\alpha_{q_1}(x) \leqslant 1$, while for $q < q_1, \alpha_q(x) = 0$. Moreover, by the very fast growing condition, for $q > q_1$ we have that $\|\alpha_q\|_\infty < 1/2^{n+1}$ and therefore $\sum_{q>q_1} \alpha_q(x) < n/2^{n+1} < 1$. Combining the above, we conclude that $f(x) < (1/m_j)(1+1) \leqslant 1$. $\square$

Lemma 12.6. *Let α be an α-average in W and $x_1 < \cdots < x_k$ be block vectors in the unit ball of $\mathfrak{X}$. Then*

$$\left|\alpha\left(\frac{1}{k}\sum_{i=1}^k x_i\right)\right| < \frac{1}{s(\alpha)} + \frac{2}{k}.$$

Proof. Assume that $\alpha = (1/m)\sum_{j=1}^d f_j$, where $f_1 < \cdots < f_d \in W$ and $d \leqslant m = s(\alpha)$. Set

$$\begin{aligned} E_1 &= \{i : \text{ there exists at most one } j \text{ such that } \operatorname{ran}(f_j) \cap \operatorname{ran}(x_i) \neq \varnothing\}, \\ E_2 &= \{1, \ldots, n\} \setminus E_1, \text{ and} \\ J_i &= \{j : \operatorname{ran}(f_j) \cap \operatorname{ran}(x_i) \neq \varnothing\}, \text{ for } i = 1, \ldots, k. \end{aligned}$$

Then it is easy to see that

$$\left|\alpha\left(\frac{1}{k}\sum_{i\in E_1} x_i\right)\right| \leqslant \frac{1}{m}$$

and

$$\left|\alpha\left(\frac{1}{k}\sum_{i\in E_2} x_i\right)\right| \leqslant \frac{1}{m}\sum_{i\in E_2}\frac{1}{k}\left(\sum_{j\in J_i}|f_j(x_i)|\right) < \frac{2m}{m}\frac{1}{k}. \qquad \square$$

Proposition 12.7. *The space $\ell_1(\mathbb{N})$ does not embed into $\mathfrak{X}$.*

Proof. Towards a contradiction, assume that there exists a normalized block sequence $(z_k)_k$ in $\mathfrak{X}$, equivalent to the unit vector basis of $\ell_1(\mathbb{N})$. Define $Y = \overline{\langle z_k : k \in \mathbb{N}\rangle}$ and take $T|_Y : Y \to X$ to be the map from Remark 12.3 (ii), which is bounded (recall, $Te_i = x_i$, so essentially T is the identity map). By assumption $\ell_1(\mathbb{N})$ does not embed into X and hence T is strictly singular. We may therefore choose a further normalized block sequence $(y_k)_k$ of $(z_k)_k$ with $\sum_k \|Ty_k\| < 1$.

Choose increasing subsets of the natural numbers $F_1 < F_2 < \cdots$ satisfying the following. If $j_k = \max\operatorname{supp}(y_{\max F_k})$, then $\#F_{k+1} = 2kn_{j_k}$. Set $w_k = (1/\#F_k)\sum_{i\in F_k} y_i$. Then $(w_k)_k$ is seminormalized, equivalent to the unit vector basis of ℓ_1 and it satisfies the following.

(i) $\sum_k \|Tw_k\| < 1$ and hence for any scalar sequence $(\lambda_k)_k$ in $[-1,1]$ we have $\|\sum_k \lambda_k Tw_k\| < 1$.
(ii) For every α-average α in W and $k \geqslant 2$ we have that

$$|\alpha(w_k)| < \frac{1}{s(\alpha)} + \frac{1}{kn_{j_{k-1}}}.$$

The second statement follows from Lemma 12.6. Consider now natural numbers $m \leqslant k_1 < \cdots < k_m$ and $\lambda_1,\ldots,\lambda_m$ in $[-1,1]$. We shall inductively prove the following. If $x = \sum_{i=1}^m \lambda_i w_{k_i}$, then $|f(x)| < 8$ for every $f \in W_n, n = 0,1,\ldots$. This clearly contradicts the assumption that $(w_k)_k$ is equivalent to the unit vector basis of ℓ_1.

For $f \in W_0$ this clearly follows from (i) and the obvious fact that $\|Tx\| = \sup\{|f(x)| : f \in W_0\}$. Assume that it holds for every $f \in W_m$ and let $f \in W_{m+1}$. If f is an α-average then there is nothing to prove. Otherwise, there are $t \in \mathbb{N}, d \leqslant n_t$ and a very fast growing sequence of α-averages $\alpha_1 < \cdots < \alpha_d$ of W_m with $f = (1/m_t)\sum_{q=1}^d \alpha_q$. Set $i_0 = \min\{i \leqslant m : j_{k_i} > t\}$. By the choice the sequence $(j_k)_k$, we have that

$$\|f\|_\infty \leqslant \frac{1}{m_t} \leqslant \frac{1}{t} \leqslant \frac{1}{j_{k_{i_0-1}}} = (\max\operatorname{supp} w_{k_{i_0-1}})^{-1}$$

and hence we conclude the following.

$$\left| f\left(\sum_{i<i_0} \lambda_i w_{k_i}\right)\right| \leqslant \|f\|_\infty \max\operatorname{supp} w_{k_{i_0-1}} \leqslant 1. \tag{28}$$

Set $q_1 = \min\{q \leqslant d : \max\operatorname{supp}\alpha_q \geqslant \max\operatorname{supp} w_{k_{i_0+1}}\}$. Then, for $q > q_1$, we have that

$$s(\alpha_q) > 2^{\max\operatorname{supp} w_{k_{i_0+1}}} \geqslant 2^{\#F_{k_{i_0+1}}} = 2^{2k_0\cdot n_{j_{k_{i_0}}}} \geqslant 2k_0 2^{n_{j_{k_{i_0}}}} > m2^{n_t} > mn_t.$$

Using (ii) and the above, for $q > q_1$ and $i > i_0 + 1$ we have that

$$|\alpha_q(w_{k_i})| < \frac{1}{mn_t} + \frac{1}{m\cdot n_{j_{k_{i_0}}}} < \frac{2}{mn_t}.$$

We conclude the

$$\left|\frac{1}{m_t}\sum_{q>q_1}\alpha_q\left(\sum_{i>i_0+1}\lambda_q w_{k_i}\right)\right| < \frac{1}{2}mn_t\frac{2}{mn_t} = 1. \tag{29}$$

The inductive assumption yields that $|\alpha_{q_1}(\sum_{i>i_0+1}\lambda_q w_{k_i})| < 8$. Combining this with (29) we obtain

$$\left|f\left(\sum_{i>i_0+1}\lambda_q w_{k_i}\right)\right| = \left|\frac{1}{m_t}\sum_{q\geqslant q_1}\alpha_q\left(\sum_{i>i_0+1}\lambda_q w_{k_i}\right)\right| \leqslant 5. \tag{30}$$

Finally, using the fact that $|f(\lambda_{i_0}w_{k_{i_0}} + \lambda_{i_0+1}w_{k_{i_0+1}})| \leqslant 2$, (28) and (30) we conclude that $|f(x)| < 8$. $\square$

Proposition 12.8. *Every block sequence $(z_k)_k$ of $(e_i)_i$ in $\mathfrak{X}$ with $s(z_k) = 0$ for all $k \in \mathbb{N}$ spans a reflexive subspace of $\mathfrak{X}$.*

Proof. Let $(z_k)_k$ be a normalized block sequence in $\ker s$. By Proposition 12.4, $(z_k)_k$ is boundedly complete. It remains to prove that it is shrinking as well. Towards a contradiction, assume that this is not the case, i.e. there exist a linear functional $f : \mathfrak{X} \to \mathbb{R}$ with $\|f\| = 1$, a further normalized block sequence $(y_k)_k$ of $(z_k)_k$, and $\varepsilon > 0$ with $f(y_k) > \varepsilon$ for all $k \in \mathbb{N}$.

Choose a summable sequence of positive reals $(\varepsilon_k)_k$ with $\sum_{j>k}\varepsilon_j < \varepsilon_k$ for all $k \in \mathbb{N}$. Moreover, choose increasing subsets $F_1 < F_2 < \cdots$ of the natural numbers with $\#F_k > 2\varepsilon_k^{-1}$ for all $k \in \mathbb{N}$ and set $w_k = (1/\#F_k)\sum_{i\in F_k} y_i$. Then $\varepsilon < f(w_k) \leqslant \|w_k\| \leqslant 1$ for all $k \in \mathbb{N}$. We shall prove that $(w_k)_k$ is unconditional and therefore equivalent to the unit vector basis of ℓ_1. This contradicts Proposition 12.7 and the proof will be finished.

We shall prove by induction on n that for every $f \in W_n$, $\ell \leqslant m \in \mathbb{N}$, $G \subset \{\ell, \ldots, m\}$ and $\lambda_\ell, \ldots, \lambda_m \subset [-1, 1]$, there exists $g \in W_n$ satisfying the following.

(i) $\operatorname{ran}(g) \subset \operatorname{ran}(f)$.
(ii) If f is an α-average, then g is also an α-average of size $s(g) = s(f)$
(iii) $f(\sum_{k\in G}\lambda_k w_k) < g(\sum_{k=\ell}^{m}\lambda_k w_k) + \varepsilon_\ell$

The third assertion of the above statement implies that $(w_k)_k$ is unconditional with suppression constant at most $1 + \varepsilon_1$ which is the desired result.

We proceed to the inductive proof. Let $f \in W_0$, $\ell, m \in \mathbb{N}$ with $\ell \leqslant m$, $G \subset \{\ell, \ldots, m\}$ and $\lambda_\ell, \ldots, \lambda_m \subset [-1, 1]$. Set

$$\begin{aligned} k_1 &= \min\{k \in G : \min\operatorname{supp} f \leqslant \max\operatorname{supp} w_k\} \\ k_2 &= \max\{k \in G : \max\operatorname{supp} f \geqslant \min\operatorname{supp} w_k\} \\ i_1 &= \min\{i \in F_{k_1} : \min\operatorname{supp} f \leqslant \max\operatorname{supp} y_i\} \\ i_2 &= \max\{i \in F_{k_2} : \max\operatorname{supp} f \geqslant \min\operatorname{supp} y_i\} \end{aligned}$$

For $k_1 < k < k_2$ set $\tilde{w}_k = w_k$. Set $\tilde{G} = \{k \in G : k_1 \leqslant k \leqslant k_2\}$, and also set

$$\tilde{w}_{k_1} = \frac{1}{\#F_{k_1}} \sum_{\{i \in F_{k_1} : i > i_1\}} y_i$$

$$\tilde{w}_{k_2} = \frac{1}{\#F_{k_2}} \sum_{\{i \in F_{k_2} : i < i_2\}} y_i$$

Then,

$$\begin{aligned} f\left(\sum_{k \in G} \lambda_k w_k\right) &= f\left(\sum_{k \in \tilde{G}} \lambda_k w_k\right) \\ &= f\left(\sum_{k \in \tilde{G}} \lambda_k \tilde{w}_k\right) + f\left(\lambda_{k_1} \frac{1}{\#F_{k_1}} y_{i_1}\right) + f\left(\lambda_{k_2} \frac{1}{\#F_{k_2}} y_{i_2}\right). \end{aligned}$$

Using the fact that for $k \geqslant \ell$, $\#F_k > 2\varepsilon_\ell^{-1}$ we conclude

$$f\left(\sum_{k \in G} \lambda_k w_k\right) < f\left(\sum_{k \in \tilde{G}} \lambda_k \tilde{w}_k\right) + \varepsilon_\ell \tag{31}$$

Recall that every block sequence of $(x_i)_i$ in X that is in $\ker s$ is suppression unconditional. Therefore

$$f\left(\sum_{k \in \tilde{G}} \lambda_k \tilde{w}_k\right) \leqslant \left\| \sum_{k \in \tilde{G}} \lambda_k T \tilde{w}_k \right\| \leqslant \left\| \sum_{k=k_1}^{k_2} \lambda_k T \tilde{w}_k \right\|.$$

Choose $g' \in W_0$ with $g'(\sum_{k=k_1}^{m} \lambda_k \tilde{w}_k) = \| \sum_{k=k_1}^{k_2} \lambda_k T \tilde{w}_k \|$. Then, using (31) we have that

$$f\left(\sum_{k \in G} \lambda_k w_k\right) < g'\left(\sum_{k=k_1}^{k_2} \lambda_k \tilde{w}_k\right) + \varepsilon_\ell \tag{32}$$

Set g to be $g'|_E$, where

$$E = \{\max \operatorname{supp}(y_{i_1}) + 1, \ldots, \min \operatorname{supp}(y_{i_2}) - 1\} \subset \operatorname{ran}(f).$$

The fact that $(x_i)_i$ is a bimonotone basis for X, yields that g is in W_0. Moreover, $g(\sum_{k=\ell}^{m} \lambda_k w_k) = g'(\sum_{k=k_1}^{k_2} \lambda_k \tilde{w}_k)$. Using (32) we deduce that g is the desired functional.

Assume now that the statement holds for $m \in \mathbb{N}$ and let $f \in W_{m+1}$, $\ell \leqslant m \in \mathbb{N}$, $G \subset \{\ell, \dots, m\}$ and $\lambda_\ell, \dots, \lambda_m \subset [-1,1]$. Consider first the case in which f is an α-average of W_m, i.e. there are d, $p \in \mathbb{N}$ with $d \leqslant p$ and $f_1 < \cdots < f_d \in W_m$ with $f = (1/p)(f_1 + \cdots + f_d)$. By the inductive assumption, for $q = 1, \dots, d$ there exist $g_i \in W_n$ satisfying the following.

(a) $\operatorname{ran}(g_i) \subset \operatorname{ran}(f_i)$.
(b) $f_i(\sum_{k\in G} \lambda_k w_k) < g_i(\sum_{k=\ell}^m \lambda_k w_k) + \varepsilon_\ell$

We conclude that $g = \frac{1}{p}(g_1 + \cdots + g_d)$ is the desired functional.

Consider now that case in which f is a weighted functional in W_m, i.e. there are $j \in \mathbb{N}$, $d \leqslant n_j$, and a very fast growing sequence $\alpha_1 < \cdots < \alpha_d$ of α-averages of W_m with $f = (1/m_j)(\alpha_1 + \cdots + \alpha_d)$. Set

$$H = \{q \leqslant d : \operatorname{ran}(\alpha_q) \cap \operatorname{ran}(w_k) = \varnothing \text{ for some } k \in G\}$$

If $H = \{q_1 < \cdots < q_p\}$, then clearly $(\alpha_{q_i})_{i=1}^p$ is very fast growing and also

$$f\left(\sum_{k\in G} \lambda_k w_k\right) = \frac{1}{m_j} \sum_{i=1}^{p} \alpha_{q_i}\left(\sum_{k\in G} \lambda_k w_k\right) \tag{33}$$

Partition H into two sets as follows.

$$H_1 = \{q_i \in H : \operatorname{ran}(\alpha_{q_i}) \cap \operatorname{ran}(w_k) = \varnothing \text{ for all } k \in \{\ell, \dots, m\} \setminus G\}$$
$$H_2 = H \setminus H_1$$

For $q_i \in H_2$, set $\ell_i = \min\{k \in \{\ell, \dots, m\} : \operatorname{ran}(\alpha_{q_i}) \cap \operatorname{ran}(w_k) \neq \varnothing\}$. Notice that for $q_i < q_j \in H_2$ we have $\ell_1 < \ell_2$. For $q_i \in H_2$, set $G_i = G \cap \{\ell_i, \dots, m\}$. Then $\alpha_{q_i}(\sum_{k\in G} \lambda_k w_k) = \alpha_{q_i}(\sum_{k\in G_i} \lambda_k w_k)$ and by the previous case, there exists an α-average $\tilde{\alpha}_{q_i}$ of W_n with $\operatorname{ran}(\tilde{\alpha}_{q_i}) \subset \operatorname{ran}(\alpha_{q_i})$, $s(\tilde{\alpha}_{q_i}) = s(\alpha_{q_i})$ and $\alpha_{q_i}(\sum_{k\in G_i} \lambda_k w_k) < \tilde{\alpha}_{q_i}(\sum_{k=\ell_i}^m \lambda_k w_k) + \varepsilon_{\ell_i}$. Clearly, $\tilde{\alpha}_{q_i}(\sum_{k=\ell_i}^m \lambda_k w_k) = \tilde{\alpha}_{q_i}(\sum_{k=\ell}^m \lambda_k w_k)$. We have concluded that for $q_i \in H_2$, $\tilde{\alpha}_{q_i}$ is an α-average satisfying the following statements.

(α) $\operatorname{ran}(\tilde{\alpha}_{q_i}) \subset \operatorname{ran}(\alpha_{q_i})$,
(β) $s(\tilde{\alpha}_{q_i}) = s(\alpha_{q_i})$, and
(γ) $\alpha_{q_i}(\sum_{k\in G} \lambda_k w_k) < \tilde{\alpha}_{q_i}(\sum_{k=\ell}^m \lambda_k w_k) + \varepsilon_{\ell_i}$.

For $q_i \in H_1$ set $\tilde{\alpha}_{q_i} = \alpha_{q_i}$. Then $(\tilde{\alpha}_{q_i})_{i=1}^p$ is very fast growing and hence $g = (1/m_j)\sum_{i=1}^p \tilde{\alpha}_{q_i}$ is in W_{n+1} and $\operatorname{ran} g \subset \operatorname{ran} f$. Combining (33), the fact that for $q_i < q_j \in H_2$ we have that $\ell_1 < \ell_2$, the fact that $\sum_{k=\ell}^m \varepsilon_k < 2\varepsilon_\ell$ and (γ), we conclude the following.

$$\begin{aligned}f\left(\sum_{k\in G}\lambda_k w_k\right) &= \frac{1}{m_j}\sum_{i=1}^{p}\alpha_{q_i}\left(\sum_{k\in G}\lambda_k w_k\right)\\ &< \frac{1}{m_j}\sum_{i=1}^{p}\tilde{\alpha}_{q_i}\left(\sum_{k=\ell}^{m}\lambda_k w_k\right) + \frac{1}{m_j}\sum_{k=\ell}^{m}\varepsilon_k\\ &= g\left(\sum_{k=\ell}^{m}\lambda_k w_k\right) + \frac{1}{m_j}\sum_{k=\ell}^{m}\varepsilon_k\\ &< g\left(\sum_{k=\ell}^{m}\lambda_k w_k\right) + \varepsilon_\ell\end{aligned}$$

The inductive step is now complete and so is the proof. □

To the best of our knowledge, it was not known whether every subsymmetric sequence is generated as a spreading model by an unconditional basis of a reflexive space. Recall that in [6] a reflexive Banach space is constructed that admits all subsymmetric sequences as spreading models in all of its subspaces. However, this space is hereditarily indecomposable, i.e. it does not contain unconditional basic sequences and its construction is rather complicated. We give a short description of Theorem 12.2 (i).

Proof of 12.2 (i). We may assume that $(x_i)_i$ 1-subsymmetric and we take the space $\mathcal{S}(U)$ from Remark 11.6 with an unconditional Schauder basis $(e_i)_i$ generating a spreading model isometrically equivalent to $(x_i)_i$. Then, we define W_0 similarly as in (27), using instead the unit ball of $\mathcal{S}(U)$. The sets W_m are then defined in the same way and their union gives a norming set W that then defines the norm of the space $\mathfrak{X}$ with an unconditional basis that generates $(x_i)_i$ as a spreading model. This is proved identically as Proposition 12.5. Proofs identical to those of Propositions 12.4 and 12.7 yield that $\mathfrak{X}$ does not contain c_0 and ℓ_1 and hence it is reflexive. □

13. The diversity of convex block homogeneous bases

In this final section we attempt to give an answer to the question of what different types of convex block homogeneous bases there are. As it was observed in section 3, one way of defining a convex block homogeneous basis is to take the jamesification of a subsymmetric basis. One may ask whether this is the unique way of obtaining such bases. By Theorem 4.1, a positive answer to this question would imply that every conditional spreading sequence is defined via two subsymmetric sequences alone. As it turns out however, this is false as there exist convex block homogeneous bases that cannot be obtained by jamesifying subsymmetric sequences. We provide a fairly simple example obtained by duality.

Remarks 13.1. Let X, be a Banach space with a 1-subsymmetric (i.e. 1-unconditional and 1-spreading) basis $(x_i)_i$. Let $J(X)$ denote the jamesification of X and $(e_i)_i$ its natural Schauder basis. The following can be shown easily.

(i) If $(x_i)_i$ is not equivalent to unit vector basis of $\ell_1(\mathbb{N})$, then $(e_i)_i$ is conditional and spreading.
(ii) The sequence $(e_i)_i$ is 1-convex block homogeneous and bimonotone.
(iii) The unconditional part $(u_i)_i$ of $(e_i)_i$, i.e. the sequence $(e_{2i} - e_{2i-1})_i$, is equivalent to the sequence $(x_i)_i$. In particular, for any real numbers $a_1, \ldots, a_n$ we have

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leqslant \left\| \sum_{i=1}^{n} a_i u_i \right\| \leqslant 2 \left\| \sum_{i=1}^{n} a_i x_i \right\|. \tag{34}$$

Remark 13.2. Remark 13.1 (iii) implies the following. If Z is a Banach space with a convex block homogeneous basis $(z_i)_i$ that is equivalent to the jamesification of some subsymmetric sequence $(x_i)_i$, then actually $(x_i)_i$ is equivalent to the unconditional part $(u_i)_i$ of $(z_i)_i$. In other words, if $(z_i)_i$ is equivalent to the jamesification of some subsymmetric sequence, then it is actually equivalent to the jamesification of its own unconditional part.

A convex block homogeneous basis by duality Let us denote by $(e_i)_i$ the boundedly complete basis of James space J. The term "jamesification" actually stems from this basis and it is the jamesification of the unit vector basis of $\ell_2(\mathbb{N})$, i.e. for a sequence of scalars $(a_i)_i$ one has

$$\left\| \sum_{i} a_i e_i \right\| = \sup \left\{ \left(\sum_{k=1}^{n} \left(\sum_{i \in E_k} a_i \right)^2 \right)^{1/2} \right\}, \tag{35}$$

where the supremum is taken over all possible choices of successive intervals $(E_k)_{k=1}^n$ of $\mathbb{N}$ that are allowed to have gaps between them (although this makes no difference). The basis $(e_i)_i$ of James space is 1-convex block homogeneous and so by Proposition 3.9 the sequence $(s_{[1,n]})_n$, with $s_{[1,n]} = \sum_{i=1}^n e_i^*$ for all $n \in \mathbb{N}$, is convex block homogeneous. Let us denote by Y the closed linear span of the sequence $(s_{[1,n]})_n$. This is a subspace of co-dimension one in J^* and in fact it is isomorphic to J^*. As it was proved by R. C. James in [17], the spaces J and J^* are not isomorphic. This in particular means that $(s_{[1,n]})_n$ is not equivalent to $(e_i)_i$, i.e., $(s_{[1,n]})_n$ is not equivalent to the jamesification of the unit vector basis of $\ell_2(\mathbb{N})$.

Proposition 13.3. *The basis $(s_{[1,n]})_n$ of Y is convex block homogeneous but not equivalent to the jamesification of any subsymmetric sequence.*

Proof. Assume that there exists a subsymmetric sequence $(x_i)_i$ so that $(s_{[1,n]})_n$ is equivalent to the jamesification of $(x_i)_i$. By Remark 13.2, $(x_i)_i$ is equivalent to the sequence $(s_{[1,2n]}-s_{[1,2n-1]})_n$, i.e. the sequence $(e^*_{2n})_n$. We will show that $(e^*_{2n})_n$ is equivalent to the unit vector basis of $\ell_2(\mathbb{N})$. This contradicts the last sentence of the preceding discussion.

Observe that by (35) the basis $(e_i)_i$ of J dominates the unit vector basis of $\ell_2(\mathbb{N})$ with constant one. Hence, $(e^*_n)_n$ and in extend also $(e^*_{2n})_n$ is 1-dominated by the unit vector basis of $\ell_2(\mathbb{N})$. For the inverse domination, let $a_1, \ldots, a_n$ be scalars the squares of which sum up to one, we will evaluate the norm of $x^* = \sum_{i=1}^n a_i e^*_{2i}$. Consider the vector $x = \sum_{i=1}^n a_i(e_{2i} - e_{2i-2})$ in J. Then, by (35) it follows that $1 \leqslant \|x\| \leqslant \sqrt{2}$ and an easy calculation yields $x^*(x) = 1$. In conclusion, $1/\sqrt{2} \leqslant \|x^*\| \leqslant 1$. Therefore, $(e^*_{2n})_n$ is equivalent to the unit vector basis of $\ell_2(\mathbb{N})$ which is absurd. □

References

[1] G. Androulakis, P. Dodos, G. Sirotkin, V.G. Troitsky, Classes of strictly singular operators and their products, Israel J. Math. 169 (2009) 221–250.

[2] G. Androulakis, E. Odell, Th. Schlumprecht, N. Tomczak-Jaegermann, On the structure of the spreading models of a Banach space, Canad. J. Math. 57 (4) (2005) 673–707.

[3] S.A. Argyros, V. Kanellopoulos, K. Tyros, Finite order spreading models, Adv. Math. 234 (2013) 574–617.

[4] S.A. Argyros, V. Kanellopoulos, K. Tyros, Higher order spreading models, Fund. Math. 221 (1) (2013) 23–68.

[5] S.A. Argyros, P. Motakis, A reflexive hereditarily indecomposable space with the hereditary invariant subspace property, Proc. Lond. Math. Soc. (3) 108 (6) (2014) 1381–1416.

[6] S.A. Argyros, P. Motakis, A hereditarily indecomposable Banach space with rich spreading model structure, Israel J. Math. 203 (1) (2014) 341–387.

[7] A. Baernstein II, On reflexivity and summability, Studia Math. 42 (1972) 91–94.

[8] B. Beauzamy, J.-T. Lapresté, Modèles étalés des espaces de Banach, Publ. Dép. Math. (Lyon) (N. S.) 4/A (1983) 1–199.

[9] S.F. Bellenot, R.G. Haydon, E. Odell, Quasi-reflexive and tree spaces constructed in the spirit of R.C. James, in: Banach Space Theory, Iowa City, IA, 1987, in: Contemp. Math., vol. 85, Amer. Math. Soc., Providence, RI, 1989, pp. 19–43.

[10] A. Brunel, L. Sucheston, On B-convex Banach spaces, Math. Syst. Theory 7 (4) (1974) 294–299.

[11] A. Brunel, L. Sucheston, On J-convexity and some ergodic super-properties of Banach spaces, Trans. Amer. Math. Soc. 204 (1975) 79–90.

[12] A. Brunel, L. Sucheston, Equal signs additive sequences in Banach spaces, J. Funct. Anal. 21 (1976) 286–304.

[13] P.G. Casazza, James' quasi-reflexive space is primary, Israel J. Math. 26 (3–4) (1977) 294–305.

[14] D. Freeman, E. Odell, B. Sarı, B. Zheng, On spreading sequences and asymptotic structures, http://arxiv.org/abs/1607.03587.

[15] W.T. Gowers, B. Maurey, The unconditional basic sequence problem, J. Amer. Math. Soc. 6 (4) (1993) 851–874.

[16] R.C. James, Bases and reflexivity of Banach spaces, Ann. of Math. 52 (3) (1950) 518–527.

[17] R.C. James, Banach space quasi-reflexive of order one, Studia Math. 60 (1977) 157–177.

[18] E. Odell, Stability in Banach spaces, Extracta Math. 17 (3) (2002) 385–425.

[19] A. Pełczyński, Projections in certain Banach spaces, Studia Math. 19 (1960) 209–228.

[20] H.P. Rosenthal, A characterization of Banach spaces containing ℓ_1, Proc. Natl. Acad. Sci. USA 71 (1974) 2411–2413.

[21] H.P. Rosenthal, A characterization of Banach spaces containing c_0, J. Amer. Math. Soc. 7 (3) (1994) 707–748.

[22] J. Schreier, Ein Gegenbeispiel zur Theorie der schwachen Konvergenz, Studia Math. 2 (1) (1930) 58–62.