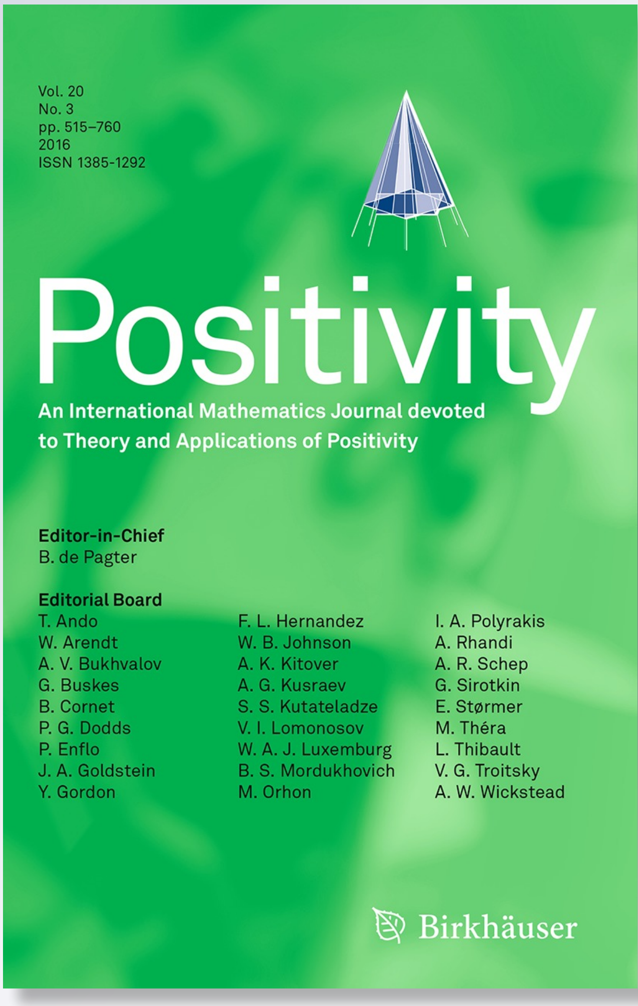
A dual method of constructing hereditarily indecomposable Banach spaces

Spiros A. Argyros & Pavlos Motakis

Positivity
An International Mathematics Journal devoted to Theory and Applications of Positivity

ISSN 1385-1292
Volume 20
Number 3

Positivity (2016) 20:625-662
DOI 10.1007/s11117-015-0378-9



Positivity (2016) 20:625–662
DOI 10.1007/s11117-015-0378-9

Positivity

A dual method of constructing hereditarily indecomposable Banach spaces

Spiros A. Argyros[1] · **Pavlos Motakis**[2]

Received: 22 April 2015 / Accepted: 6 October 2015 / Published online: 17 October 2015

Abstract A new method of defining hereditarily indecomposable Banach spaces is presented. This method provides a unified approach for constructing reflexive HI spaces and also HI spaces with no reflexive subspace. All the spaces presented here satisfy the property that the composition of any two strictly singular operators is a compact one. This yields the first known example of a Banach space with no reflexive subspace such that every operator has a non-trivial closed invariant subspace.

Keywords Spreading models · Strictly singular operators · Invariant subspaces · Hereditarily indecomposable spaces

Mathematics Subject Classification Primary 46B03 · 46B06 · 46B25 · 46B45 · 47A15

1 Introduction

Defining a hereditarily indecomposable (HI) Banach space is not an easy task. It requires the definition of a subset W of $c_{00}(\mathbb{N})$ (the space of real sequences which are

This research was supported by program ΑΡΙΣΤΕΙΑ-1082.

✉ Pavlos Motakis
pavlos@math.tamu.edu

Spiros A. Argyros
sargyros@math.ntua.gr

1 Department of Mathematics, Faculty of Applied Sciences, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece

2 Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA

eventually zero), which in turn, acting as a set of functionals on $c_{00}(\mathbb{N})$, defines an HI norm. In all classical constructions the resulting space admits the unit vector basis of $c_{00}(\mathbb{N})$ as a boundedly complete Schauder basis. This appears to be an inevitable consequence of the saturation of the set W under certain operations which yield, for every n in $\mathbb{N}$, a lower bound C_n of $\|\sum_{k=1}^n x_k\|$, for every sequence of successive normalized block vectors $(x_k)_{k=1}^n$, and $\lim_n C_n = \infty$.

There are two known types of HI spaces whose basis is not boundedly complete. The first one concerns the $\mathcal{L}_\infty$ HI space $\mathfrak{X}_K$ which appeared in [6] and is the result of mixing the Bourgain-Delbaen method [12] of constructing $\mathcal{L}_\infty$-spaces and the Gowers-Maurey corresponding one [15] of constructing HI spaces. The basis of the space is shrinking but not boundedly complete. However, this is a consequence of the $\mathcal{L}_\infty$ structure and not of the HI property of the space. In particular, every block sequence in the space has a boundedly complete subsequence, hence the space is reflexively saturated.

The second type concerns HI spaces with no reflexive subspace. All such spaces whose norm is induced by a norming set W have a boundedly complete Schauder basis. This class includes spaces such as the Gowers Tree space [14] and the spaces which appeared in [2]. The predual of one of the spaces presented in [2] is also an HI space without reflexive subspaces. This space admits a shrinking basis and none of its subspaces admits a boundedly complete basis. This predual is essentially different to a space which is induced by a saturated norming set W. The latter, as we have explained, always yields spaces with a boundedly complete basis.

The preceding discussion leads to the following question. Does there exist a method of defining a norming set W such that the resulting space admits a shrinking Schauder basis and no subspace admits a boundedly complete one? This problem is directly related to the problem of the existence of a $\mathcal{L}_\infty$-space which is HI and has no reflexive subspace. Indeed, any HI $\mathcal{L}_\infty$-space must have separable dual [19,22] and if moreover it does not contain reflexive subspaces, then it does not contain a boundedly complete basic sequence. More generally, every Banach space with a boundedly complete basis and separable dual is reflexively saturated [18].

The aim of the present paper is to answer the first problem by providing a new method of defining a norming set W, which yields an HI space with a shrinking basis with no boundedly complete basic sequence. We perceive this method as the dual method of the classical one. This new approach allows us to affirmatively answer the second problem. Namely, there exists a $\mathcal{L}_\infty$ HI space with no reflexive subspace. This result will appear in a forthcoming paper. Our goal is to use a more classical setting in order to present the definition of the norming set and its consequences, some of which are rather unexpected.

The definition of the norming set W uses an unconditional frame, namely the Tsirelson-like space with constraints $T(1/2^n, \mathcal{S}_n, \alpha)_n$. Norms which are saturated under constraints were introduced in [3] and [8] and are rooted in the earlier work of Odell and Schlumprecht [20,21]. The norm of $T(1/2^n, \mathcal{S}_n, \alpha)_n$ is described by the following implicit formula: if $x \in c_{00}(\mathbb{N})$ then

$$\|x\| = \max\left\{\|x\|_\infty, \sup \frac{1}{2^n}\sum_{q=1}^{d}\|E_q x\|_{m_q}\right\} \tag{1}$$

where the supremum is taken over all $n \in \mathbb{N}$, $\mathcal{S}_n$-admissible successive subsets $(E_q)_{q=1}^d$ of $\mathbb{N}$ and sequences $(m_q)_{q=1}^d$ of $\mathbb{N}$ so that $m_q > 2^{\max E_{q-1}}$ for $q = 2, \ldots, d$. The m-norms appearing in (1) are defined as follows. For $m \in \mathbb{N}$ and $x \in c_{00}(\mathbb{N})$:

$$\|x\|_m = \frac{1}{m} \sup \sum_{i=1}^{m} \|G_i x\|$$

where the supremum is taken over all successive subsets $(G_i)_{i=1}^m$ of $\mathbb{N}$.

The $\|\cdot\|_m$ norms, $m \in \mathbb{N}$, which appear in the definition above, do not contribute to the norm of the element x, in fact they act as constraints. This results in the neutralization of the operations $(1/2^n, \mathcal{S}_n)$ on certain sequences and thus, c_0 spreading models become abundant. As a consequence, every Schauder basic sequence in the space admits either an ℓ_1 or a c_0 spreading model and both of them are admitted by every infinite dimensional subspace. This norm and its variants have been recently established as an effective tool for answering certain problems on the structure of Banach spaces and their spaces of operators [3,8,9,11].

The norm on $T(1/2^n, \mathcal{S}_n, \alpha)_n$ is induced by the norming set W_α which is the minimal subset of $c_{00}(\mathbb{N})$ containing the basis $(e_i^*)_i$, all α-averages of its elements, i.e. averages of successive elements of W_α, and it is closed under the operations $(1/2^n, \mathcal{S}_n, \alpha)$ for every $n \in \mathbb{N}$. The latter means that for every very fast growing family $(\alpha_q)_{q=1}^d$ of successive α-averages, which is $\mathcal{S}_n$-admissible, the functional $f = (1/2^n)\sum_{q=1}^d \alpha_q$ is in W_α. Any such f is called a weighted functional with $w(f) = n$. Hence, the set W_α includes the elements of the basis, α-averages and weighted functionals.

The norming set W will be chosen to be a subset of W_α and its definition is based on a tree $\mathcal{U}$, called the universal tree. This tree consists of finite sequences $\{(f_k, x_k)\}_{k=1}^d$, where $(f_k)_{k=1}^d$ is a sequence of successive non-zero weighted functionals in W_α, $(x_k)_{k=1}^d$ is a sequence of successive non-zero vectors in $c_{00}(\mathbb{N})$ with rational coefficients and for each $1 < m \leqslant d$ the weight of f_m is uniquely defined by the sequence $\{(f_k, x_k)\}_{k=1}^{m-1}$.

We will consider a class of subtrees $\mathcal{T}$ of the universal tree $\mathcal{U}$. Each tree $\mathcal{T}$ in this class is either well founded and satisfies certain additional properties or $\mathcal{T} = \mathcal{U}$. For such a tree $\mathcal{T}$ we define the norming set $W_{\mathcal{T}}$. It is worth pointing out that for a well founded tree $\mathcal{T}$ the space $\mathfrak{X}_{\mathcal{T}}$, induced by the set $W_{\mathcal{T}}$, is a reflexive HI space, while for $\mathcal{T} = \mathcal{U}$ the space $\mathfrak{X}_{\mathcal{U}}$ admits a shrinking basis and does not contain a reflexive subspace. It is also interesting, and rather unexpected, that the reflexive and non-reflexive cases have a unified approach, as it is presented in the rest of the paper. Note that the Gowers Tree type HI spaces with no reflexive subspace [2,14] have substantially increased complexity, concerning their definition as well as their proofs, compared to the corresponding reflexive HI spaces.

For a subtree $\mathcal{T}$ of the universal tree $\mathcal{U}$, as above, we define the norm of the space $\mathfrak{X}_{\mathcal{T}}$, which is very similar to the norm of the space $T(1/2^n, \mathcal{S}_n, \alpha)_n$. Namely, the norm of $\mathfrak{X}_{\mathcal{T}}$ is described by the implicit formula (1), the difference lying in the definition of $\|\cdot\|_m$ norms, where

$$\|x\|_m = \sup\left\{\frac{1}{m}\sum_{i=1}^m g_i(x) : \ \frac{1}{m}\sum_{i=1}^m g_i \text{ is an } \alpha_c\text{-average}\right\}$$

and α_c-averages are α-averages which are inductively defined. In other words, to define the norm of $\mathfrak{X}_\mathcal{T}$ we impose some further restrictions on the α-averages used as constraints. Alternatively, the norming set $W_\mathcal{T}$ is the minimal subset of $c_{00}(\mathbb{N})$ containing the basis, the α_c-averages and all $f = (1/2^n)\sum_{q=1}^d \alpha_q$ where $(\alpha_q)_{q=1}^d$ is a very fast growing and $\mathcal{S}_n$-admissible family of α_c-averages.

Let us observe that in the definition of $W_\mathcal{T}$ the conditional structure, which yields the HI property of the space $\mathfrak{X}_\mathcal{T}$, is contained in the α_c-averages. The space $\mathfrak{X}_\mathcal{T}$ satisfies the following property. If $\mathcal{T}$ is a well founded subtree of $\mathcal{U}$, for every block sequence with rational coefficients $(y_i)_i$ in $\mathfrak{X}_\mathcal{T}$ there exist a further finite block sequence $(x_k)_{k=1}^d$, with $1/2 < \|x_k\| \leqslant 10$, and $(f_k)_{k=1}^d$ in $W_\mathcal{T}$, such that $\{(f_k, x_k)\}_{k=1}^d$ is a maximal element of $\mathcal{T}$ and $\|\sum_{k=1}^d x_k\| \leqslant 27$. If $\mathcal{T} = \mathcal{U}$, the corresponding result holds in the space $\mathfrak{X}_\mathcal{U}$ for a branch $\{(f_k, x_k)\}_{k=1}^\infty$ of $\mathcal{U}$ such that $\|\sum_{k=1}^d x_k\| < 27$, for all $d \in \mathbb{N}$.

Below we summarize the properties of the space $\mathfrak{X}_\mathcal{T}$, in the case the tree $\mathcal{T}$ is well founded.

Theorem A *If $\mathcal{T}$ is well founded, then the space $\mathfrak{X}_\mathcal{T}$ satisfies the following properties.*

(i) *The space $\mathfrak{X}_\mathcal{T}$ has a bimonotone Schauder basis, and it is hereditarily indecomposable and reflexive.*
(ii) *Every Schauder basic sequence in $\mathfrak{X}_\mathcal{T}$ admits either ℓ_1 or c_0 as a spreading model and every infinite dimensional subspace of $\mathfrak{X}_\mathcal{T}$ admits both of these types of spreading models.*
(iii) *For every block subspace X of $\mathfrak{X}_\mathcal{T}$ and every bounded linear operator $T : X \to X$, there is $\lambda \in \mathbb{R}$ so that $T - \lambda I$ is strictly singular.*
(iv) *For every infinite dimensional subspace X of $\mathfrak{X}_\mathcal{T}$ the ideal of the strictly singular operators $\mathcal{S}(X)$ is non separable.*
(v) *For every subspace X of $\mathfrak{X}_\mathcal{T}$ and every strictly singular operators S, T on X, the composition TS is compact.*
(vi) *For every block subspace X of $\mathfrak{X}_\mathcal{T}$, every non-scalar bounded linear operator $T : X \to X$ admits a non-trivial closed hyperinvariant subspace.*

The above should be compared to the main theorem from [8], where a space with very similar properties is presented. The key difference between the aforementioned case and the present one is in property (v), namely in [8] it is only proved for compositions of three strictly singular operators, and not two. In [8] special weighted functionals are used, which impose the necessity to include β-averages in the definition of the norming set. The absence of these two notions in the present construction yields property (v), which is the best possible, as well as simplified proofs, compared to those in [8].

Below we present the main properties of the space $\mathfrak{X}_\mathcal{U}$.

Theorem B *If $\mathcal{T} = \mathcal{U}$, then the space $\mathfrak{X}_\mathcal{U}$ satisfies the following properties.*

(i) *The space $\mathfrak{X}_\mathcal{U}$ has a bimonotone and shrinking Schauder basis, it is hereditarily indecomposable and contains no reflexive subspace.*

(ii) *Every Schauder basic sequence in $\mathfrak{X}_{\mathcal{U}}$ admits either ℓ_1, c_0 or the summing basis of c_0 as a spreading model, and every infinite dimensional subspace of $\mathfrak{X}_{\mathcal{U}}$ admits all three of these types of spreading models.*

(iii) *For every block subspace X of $\mathfrak{X}_{\mathcal{U}}$ and every bounded linear operator $T : X \to X$, there is $\lambda \in \mathbb{R}$ so that $T - \lambda I$ is weakly compact and hence strictly singular.*

(iv) *For every infinite dimensional subspace X of $\mathfrak{X}_{\mathcal{U}}$ the ideal of the strictly singular operators $\mathcal{S}(X)$ is non separable.*

(v) *For every subspace X of $\mathfrak{X}_{\mathcal{U}}$ and every strictly singular operators S, T on X, the composition TS is compact.*

(vi) *For every block subspace X of $\mathfrak{X}_{\mathcal{U}}$, every non-scalar bounded linear operator $T : X \to X$ admits a non-trivial closed hyperinvariant subspace.*

This is the first known example of a Banach space with no reflexive subspace such that the space generated by every block sequence satisfies the invariant subspace property.

In Theorems A and B property (vi) can be stated for every subspace X of the corresponding space, such that every T in $\mathcal{L}(X)$ is of the form $\lambda I + S$, with S strictly singular. The present construction can also be carried out over the field of complex numbers. The corresponding complex HI spaces satisfy Theorems A and B, in particular property (vi) holds for every closed subspace [15, Theorem 18].

2 The norming set of the space $\mathfrak{X}_{\mathcal{T}}$

This section is devoted to the norming set $W_{\mathcal{T}}$ of the space. We begin with a brief presentation and discussion concerning the main ingredients involved in the definition of $W_{\mathcal{T}}$. As we have mentioned in the introduction we will consider subtrees of the universal tree $\mathcal{U}$. Each such tree $\mathcal{T}$ is downwards closed and for every node which is non-maximal in $\mathcal{T}$, all of its immediate successors in $\mathcal{U}$ are also included in $\mathcal{T}$. For our needs the tree is either well founded, containing at least all elements $\{(f_k, x_k)\}_{k=1}^d$ of $\mathcal{U}$ such that $(f_k)_{k=1}^d$ is $\mathcal{S}_2$-admissible, or otherwise $\mathcal{T} = \mathcal{U}$.

The second ingredient are the α_c-averages which are inductively defined and are described as follows.

To each weight n we assign a unique weight $\phi(n)$ that appears in the tree $\mathcal{T}$. Two different weights n and m are comparable, if there exist $\{(f_1, x_1), \dots, (f_k, x_k)\}$ in $\mathcal{T}$ and $1 \leqslant i < j \leqslant k$ such $\phi(n) = w(f_i)$ and $\phi(m) = w(f_j)$. Otherwise n, m are incomparable.

We consider the following four types of averages. The first one are averages of the basis $(e_i^*)_i$, called basic averages.

The second one are $\mathcal{IC}$-averages, i.e. α-averages of the form $(1/n)\sum_{i=1}^n g_i$ with $\{w(g_i)\}_{i=1}^n$ pairwise incomparable.

The third one are $\mathcal{IR}$-averages, i.e. α-averages of the form $(1/n)\sum_{i=1}^n g_i$ such that there exist $\{(f_1, x_1), \dots, (f_m, x_m)\}$ in $\mathcal{T}$ and $1 \leqslant k_1 < \cdots < k_n \leqslant m$ with $w(f_{k_i}) = \phi(w(g_i))$ and $|g_i(x_{k_i})| > 10$.

The last type are the conditional averages, called $\mathcal{CO}$-averages. Those are α-averages of the form $(1/n)(g_1 - g_2 + g_3 - g_4 + \cdots + (-1)^{n+1} g_n)$ such that there exist

$\{(f_1, x_1), \ldots, (f_m, x_m)\}$ in $\mathcal{T}$ and $1 \leqslant k_1 < \cdots < k_n \leqslant n$ with $w(f_{k_i}) = \phi(w(g_i))$ and $|g_i(x_{k_i}) - g_j(x_{k_j})| < 1/2^i$ for $1 \leqslant i < j \leqslant n$.

The third and fourth types of averages explain why we consider in the universal tree $\mathcal{U}$ families of pairs $\{(f_k, x_k)\}_{k=1}^d$, instead of $(f_k)_{k=1}^d$ which is the approach used in the classical norming sets. We note that the basic averages permit to begin the construction of weighted functionals in the norming set $W_{\mathcal{T}}$. The $\mathcal{CO}$-averages are responsible for the whole conditional structure in the space $\mathfrak{X}_{\mathcal{T}}$. The remaining two types of averages are necessary to exclude the presence of c_0 in the space.

2.1 The Schreier families

The Schreier families is an increasing sequence of families of finite subsets of the natural numbers, which first appeared in [1], and is inductively defined in the following manner. Set

$$\mathcal{S}_0 = \Big\{\{n\} : n \in \mathbb{N}\Big\} \quad \text{and} \quad \mathcal{S}_1 = \{F \subset \mathbb{N} : \#F \leqslant \min F\}.$$

Suppose that $\mathcal{S}_n$ has been defined and set

$$\mathcal{S}_{n+1} = \Big\{F \subset \mathbb{N} : \ F = \cup_{j=1}^k F_j, \quad \text{where } F_1 < \cdots < F_k \in \mathcal{S}_n \text{ and } \ k \leqslant \min F_1\Big\}.$$

For each n, $\mathcal{S}_n$ is a regular family. This means that it is hereditary, i.e. if $F \in \mathcal{S}_n$ and $G \subset F$ then $G \in \mathcal{S}_n$, it is spreading, i.e. if $F = \{i_1 < \cdots < i_d\} \in \mathcal{S}_n$ and $G = \{j_1 < \cdots < j_d\}$ with $i_p \leqslant j_p$ for $p = 1, \ldots, d$, then $G \in \mathcal{S}_n$ and finally it is compact, if seen as a subset of $\{0, 1\}^{\mathbb{N}}$.

If for $n, m \in \mathbb{N}$ we set

$$\mathcal{S}_n * \mathcal{S}_m = \Big\{F \subset \mathbb{N} : F = \cup_{j=1}^k F_j, \text{ where } F_1 < \cdots < F_k \in \mathcal{S}_m \text{ and } \{\min F_j : j = 1, \ldots, k\} \in \mathcal{S}_n\Big\},$$

then it is well known [4] and follows easily by induction that $\mathcal{S}_n * \mathcal{S}_m = \mathcal{S}_{n+m}$.

2.2 The unconditional frame

The norming set of the space $\mathfrak{X}_{\mathcal{T}}$ is a subset of $W_{(1/2^n, \mathcal{S}_n, \alpha)_n}$, a version of the norming set of Tsirelson space, defined with saturation under constraints.

We denote by $c_{00}(\mathbb{N})$ the space of all real valued sequences $(c_i)_i$ with finitely many non-zero terms. We denote by $(e_i)_i$ the unit vector basis of $c_{00}(\mathbb{N})$, while in some cases we shall denote it as $(e_i^*)_i$. For $x = (c_i)_i \in c_{00}(\mathbb{N})$, the support of x is the set $\operatorname{supp} x = \{i \in \mathbb{N} : \ c_i \neq 0\}$ and the range of x, denoted by $\operatorname{ran} x$, is the smallest interval of $\mathbb{N}$ containing $\operatorname{supp} x$. We say that the vectors $x_1, \ldots, x_k$ in $c_{00}(\mathbb{N})$ are successive if $\max\operatorname{supp} x_i < \min\operatorname{supp} x_{i+1}$ for $i = 1, \dots, k-1$. In this case we write $x_1 < \cdots < x_k$. A sequence of successive vectors in $c_{00}(\mathbb{N})$ is called a block sequence.

Notation We remind some notation and terminology which is used constantly throughout this paper.

(i) A sequence of vectors $x_1 < \cdots < x_k$ in $c_{00}(\mathbb{N})$ is said to be $\mathcal{S}_n$-admissible, for given $n \in \mathbb{N}$, if $\{\min\operatorname{supp} x_i : i = 1, \dots, k\} \in \mathcal{S}_n$.

(ii) Let $G \subset c_{00}(\mathbb{N})$. A vector $\alpha_0 \in c_{00}(\mathbb{N})$ is called an α-average of G of size $s(\alpha_0) = n$, if there exist $f_1 < \cdots < f_d \in G$, where $d \leqslant n$, such that

$$\alpha_0 = \frac{1}{n}(f_1 + \cdots + f_d).$$

(iii) A sequence of successive α-averages of G $(\alpha_q)_q$ is called very fast growing if $s(\alpha_q) > 2^{\max\operatorname{supp}\alpha_{q-1}}$ for $q > 1$.

Definition 2.1 We define $W_\alpha = W_{(1/2^n, \mathcal{S}_n, \alpha)_n}$ to be the smallest subset of $c_{00}(\mathbb{N})$ satisfying the following properties:

(i) for every $i \in \mathbb{N}$, $e_i^* \in W_\alpha$ and the set W_α is symmetric,

(ii) the set W_α contains all α-averages of W_α,

(iii) for every $n \in \mathbb{N}$ and every very fast growing and $\mathcal{S}_n$-admissible sequence of α-averages of W_α $(\alpha_q)_{q=1}^d$, the vector $f = (1/2^n)\sum_{q=1}^d \alpha_q$ is also in W_α.

We note that, as it is usually the case in this type of constructions, the size of an average and the weight of a weighted functional may not be uniquely defined. However, this does not cause any problems.

Remark 2.2 The set W_α satisfies the properties mentioned below. Note that properties (i), (ii) and (iii) follow readily from property (iv).

(i) Every $f \in W_\alpha$ is either of the form $f = \pm e_i^*$, either an α-average of W_α or $f = (1/2^n)\sum_{q=1}^d \alpha_q$, where $(\alpha_q)_{q=1}^d$ is a very fast growing and $\mathcal{S}_n$-admissible sequence of α-averages of W_α. In the last case we shall say that f is a weighted functional of W_α of weight $w(f) = n$.

(ii) For every $f \in W_\alpha$ and subset of the natural numbers E, the functional Ef, i.e. the restriction of f onto E, is also in W_α.

(iii) The coefficients of every $f \in W_\alpha$ are rational numbers. In particular, W_α is a countable set.

(iv) The set W_α can be constructed recursively to be the union of an increasing sequence of sets $(W_m^\alpha)_{m=0}^\infty$, where $W_0^\alpha = \{\pm e_i^* : i \in \mathbb{N}\}$ and if W_m^α has been defined, then W_{m+1}^1 is the set of all α-averages of W_m^α, W_{m+1}^2 is the set of all weighted functionals constructed on very fast growing sequences of elements of W_{m+1}^1 and $W_{m+1}^\alpha = W_m^\alpha \cup W_{m+1}^1 \cup W_{m+1}^2$.

2.3 The universal tree $\mathcal{U}$

We denote by $\mathcal{Q}$ the set of all finite sequences $\{(f_1, x_1), \dots, (f_k, x_k)\}$ satisfying the following:

(i) the $f_1, \dots, f_k$ are successive non-zero weighted functionals of W_α and
(ii) the $x_1, \dots, x_k$ are successive non-zero vectors in $c_{00}(\mathbb{N}, \mathbb{Q})$ (i.e. they are vectors in $c_{00}(\mathbb{N})$ with rational coefficients).

Note that $\mathcal{Q}$ is a subset of $\cup_n (W_\alpha \times c_{00}(\mathbb{N}, \mathbb{Q}))^n$ and hence countable.

Choose an infinite subset $L' = \{\ell_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ satisfying:

(i) $\min L' \geqslant 8$ and
(ii) for every $k \in \mathbb{N}$, $\ell_{k+1} > 2^{2\ell_k}$.

Define a partition of L' into two infinite subsets L_0 and L'_1 and choose a one-to-one function $\sigma : \mathcal{Q} \to L'_1$, called the coding function, so that for every $\{(f_1, x_1), \dots, (f_k, x_k)\} \in \mathcal{Q}$,

$$\sigma(\{(f_1, x_1), \dots, (f_k, x_k)\}) > \|f_k\|_\infty^{-1} \max \operatorname{supp} x_k. \tag{2}$$

A finite sequence $\{(f_k, x_k)\}_{k=1}^d \in \mathcal{Q}$ is called a special sequence if:

(i) $w(f_1) \in L_0$ and
(ii) if $d \geqslant 2$ then $w(f_k) = \sigma(\{(f_1, x_1), \dots, (f_{k-1}, x_{k-1})\})$ for $k = 2, \dots, d$.

Remark 2.3 Note that if $\{(f_k, x_k)\}_{k=1}^d$ is a special sequence, then (2) and (ii) imply that $w(f_1) < \dots < w(f_d)$.

Note that if $\{(f_k, x_k)\}_{k=1}^d$ is a special sequence and $1 \leqslant p \leqslant d$, then $\{(f_k, x_k)\}_{k=1}^p$ is a special sequence as well, hence if we define $\mathcal{U}$ to be the set of all special sequences, then $\mathcal{U}$ is a tree endowed with the natural ordering "$\sqsubseteq$" of initial segments. Note that the tree $\mathcal{U}$ is ill founded, more precisely every maximal chain of $\mathcal{U}$ is infinite. We shall call the tree $\mathcal{U}$, the universal tree associated with the coding function σ.

2.4 Subtrees of $\mathcal{U}$

We fix a subtree $\mathcal{T}$ of $\mathcal{U}$ which satisfies the following properties:

(i) for every $\{(f_k, x_k)\}_{k=1}^d$ in $\mathcal{T}$ and $1 \leqslant p \leqslant d$ $\{(f_k, x_k)\}_{k=1}^p$ is also in $\mathcal{T}$, i.e. $\mathcal{T}$ is a downwards closed subtree of $\mathcal{U}$,
(ii) if $\{(f_k, x_k)\}_{k=1}^d$ is a non-maximal node in $\mathcal{T}$, then for every element (f_{d+1}, x_{d+1}) so that $\{(f_k, x_k)\}_{k=1}^{d+1}$ is in $\mathcal{U}$, $\{(f_k, x_k)\}_{k=1}^{d+1}$ is also in $\mathcal{T}$ and
(iii) for every $\{(f_k, x_k)\}_{k=1}^d$ in $\mathcal{U}$ with $(f_k)_{k=1}^d$ being $\mathcal{S}_2$-admissible, we have that $\{(f_k, x_k)\}_{k=1}^d$ is in $\mathcal{T}$.

Definition 2.4 We define $L_1 = \sigma(\mathcal{T})$, which is a subset of L'_1, and $L = L_0 \cup L_1$. Define $\phi : \{i \in \mathbb{N} : i \geqslant \min L\} \to L$ with $\phi(i) = \max\{\ell \in L : \ell \leqslant i\}$.

Observe that the function ϕ is non-decreasing, $\phi(i) \leqslant i$ for all $i \in \mathbb{N}$ and $\lim_i \phi(i) = \infty$.

Definition 2.5 Two natural numbers i and j, both greater than or equal to $\min L$, are called incomparable if one of the following holds:

(i) $\phi(i)$ and $\phi(j)$ are both in L_0 and $\phi(i) \neq \phi(j)$ or
(ii) $\phi(i)$ and $\phi(j)$ are both in L_1 and $\sigma^{-1}(\phi(i)), \sigma^{-1}(\phi(j))$ are incomparable, in the ordering of $\mathcal{T}$.

If i, j are not incomparable they will be called comparable.

2.5 α_c-averages

We shall define very specific types of averages, based on the tree $\mathcal{T}$ and the notion of comparability of natural numbers from Definition 2.5. Alongside averages of elements of the basis $(e_i^*)_i$, in the definition of the norming set $W_{\mathcal{T}}$ we shall only consider these types of averages.

Definition 2.6 Let $g_1 < \cdots < g_d$ be weighted functionals in a subset G of W_α, all of which have weight greater than or equal to $\min L$, satisfying $\phi(w(g_1)) < \cdots < \phi(w(g_d))$.

(i) The sequence $(g_i)_{i=1}^d$ is called incomparable, if the natural numbers $w(g_i)$, $i = 1, \ldots, d$ are pairwise incomparable, in the sense of Definition 2.5. In this case, if $n \in \mathbb{N}$ with $d \leqslant n$ we call the average

$$\alpha_0 = \frac{1}{n} \sum_{i=1}^{d} g_i$$

an $\mathcal{IC}$-average of G.

(ii) The sequence $(g_i)_{i=1}^d$ is called comparable, if there exist $m \in \mathbb{N}$ with $d \leqslant m$, $\{(f_1, x_1), \ldots, (f_m, x_m)\} \in \mathcal{T}$ and $1 \leqslant k_1 < \cdots < k_d \leqslant m$ so that the following are satisfied:
  (a) $w(f_{k_i}) = \phi(w(g_i))$,
  (b) if $d \geqslant 3$ then $|g_i(x_{k_i})| \leqslant 10$ for $i = 2, \ldots, d-1$ and
  (c) if $d \geqslant 4$ then $|g_i(x_{k_i}) - g_j(x_{k_j})| < 1/2^i$ for $2 \leqslant i < j \leqslant d-1$.

In this case, if $n \in \mathbb{N}$ with $d \leqslant n$ and $(\varepsilon_i)_{i=1}^d$ is a sequence of alternating signs in $\{-1, 1\}$ we call the average

$$\alpha_0 = \frac{1}{n} \sum_{i=1}^{d} \varepsilon_i g_i$$

a $\mathcal{CO}$-average of G.

(iii) The sequence $(g_i)_{i=1}^d$ is called irrelevant, if there exist $m \in \mathbb{N}$ with $d \leqslant m$, $\{(f_1, x_1), \ldots, (f_m, x_m)\} \in \mathcal{T}$ and $1 \leqslant k_1 < \cdots < k_d \leqslant m$ so that the following are satisfied:
  (a) $w(f_{k_i}) = \phi(w(g_i))$ and
  (b) if $d \geqslant 3$ then $|g_i(x_{k_i})| > 10 \quad$ for $i = 2, \ldots, d-1$.

In this case, if $n \in \mathbb{N}$ with $d \leqslant n$ we call the average

$$\alpha_0 = \frac{1}{n}\sum_{i=1}^{d} g_i$$

an $\mathcal{IR}$-average of G.

Any average which is of one of the forms defined above, shall be called an α_c-average of G. Basic averages will be referred to as α_c-averages as well, where a basic average is a functional of the form $\alpha_0 = (1/n)\sum_{i=1}^{d} \varepsilon_i e_{j_i}^*$ where $d, n, j_1 < \cdots < j_d \in \mathbb{N}$ with $d \leqslant n$ and $(\varepsilon_i)_{i=1}^d$ are any signs in $\{-1, 1\}$.

Remark 2.7 The class of α_c-averages, is a much more restricted version of the one of α-averages and, with the exception of basic averages, α_c-averages are determined using the coding function σ, more precisely the tree $\mathcal{T}$.

Remark 2.8 If $(g_i)_{i=1}^d$ is a sequence in W_α which is of one of the three types described in Definition 2.6, then any subsequence of it is of the same type. Moreover, if E is an interval of $\mathbb{N}$ and $i_1 = \min\{i : E \cap \operatorname{ran} g_i \neq \varnothing\}$ and $i_2 = \max\{i : E \cap \operatorname{ran} g_i \neq \varnothing\}$, then the sequence $Eg_{i_1}, Eg_{i_1+1}, \ldots, Eg_{i_2}$ is of the same type as $(g_i)_{i=1}^d$. This last part in particular implies that whenever α_0 is an average which is of one of the three types described in Definition 2.6 and E is an interval of $\mathbb{N}$, then $E\alpha_0$ is an average of the same type.

2.6 The norming set $W_{\mathcal{T}}$ of the space $\mathfrak{X}_{\mathcal{T}}$

Definition 2.9 We define $W_{\mathcal{T}}$ to be the smallest subset of W_α which satisfies the following properties.

(i) For every $i \in \mathbb{N}$, $e_i^* \in W_{\mathcal{T}}$ and the set $W_{\mathcal{T}}$ is symmetric.
(ii) The set $W_{\mathcal{T}}$ contains all α_c-averages of $W_{\mathcal{T}}$, i.e. it contains all basic averages and all $\mathcal{IC}$, $\mathcal{CO}$and $\mathcal{IR}$-averages of $W_{\mathcal{T}}$.
(iii) For every $n \in \mathbb{N}$ and every $\mathcal{S}_n$-admissible and very fast growing sequence of α_c-averages $(\alpha_q)_{q=1}^d$ of $W_{\mathcal{T}}$, $f = (1/2^n)\sum_{q=1}^d \alpha_q$ is also in $W_{\mathcal{T}}$.

Remark 2.10 The set $W_{\mathcal{T}}$ satisfies the properties mentioned below. Note that property (ii) follows from an inductive argument using Remark 2.8 and property (iii).

(i) Every $f \in W_{\mathcal{T}}$ is either of the form $f = \pm e_i^*$, either an α_c-average of $W_{\mathcal{T}}$ or a weighted functional $f = (1/2^n)\sum_{q=1}^d \alpha_q$, where $(\alpha_q)_{q=1}^d$ is a very fast growing and $\mathcal{S}_n$-admissible sequence of α_c-averages of $W_{\mathcal{T}}$.
(ii) For every $f \in W_{\mathcal{T}}$ and interval of the natural numbers E, the functional Ef, i.e. the restriction of f onto E, is also in $W_{\mathcal{T}}$.
(iii) The set $W_{\mathcal{T}}$ can be recursively constructed to be the union of an increasing sequence of sets $(W_m)_{m=0}^\infty$, where $W_0 = \{\pm e_i^* : i \in \mathbb{N}\}$ and if W_m has been defined, then $W_{m+1}^{\alpha_c}$ is the set of all α_c-averages of W_m, W_{m+1}^w is the set of all weighted functionals constructed on very fast growing sequences of elements of $W_{m+1}^{\alpha_c}$, and $W_{m+1} = W_m \cup W_{m+1}^{\alpha_c} \cup W_{m+1}^w$.

The norm of the space $\mathfrak{X}_{\mathcal{T}}$ is the one induced by the set $W_{\mathcal{T}}$, i.e. for every $x \in c_{00}(\mathbb{N})$ we set $\|x\| = \sup\{f(x) : f \in W_{\mathcal{T}}\}$ and we define $\mathfrak{X}_{\mathcal{T}}$ to be the completion of $c_{00}(\mathbb{N})$ with respect to this norm. By Remark 2.10 the unit vector basis of $c_{00}(\mathbb{N})$ forms a bimonotone Schauder basis for $\mathfrak{X}_{\mathcal{T}}$.

Remark 2.11 The conditional structure of the space $\mathfrak{X}_{\mathcal{T}}$ is only imposed by the $\mathcal{CO}$-averages in the norming set $W_{\mathcal{T}}$, which are merely averages. In this sense, the conditionality appearing in the space $\mathfrak{X}_{\mathcal{T}}$ is not as strict as in other HI constructions.

3 Special convex combinations and evaluation of their norm

We first recall the notion of the (n, ε) special convex combinations, (see [4,5,10]) which is one of the main tools used in the sequel. We then include, without proof, some estimates from [8], which also apply to the present case.

Definition 3.1 Let $x = \sum_{k\in F} c_k e_k$ be a vector in $c_{00}(\mathbb{N})$ and $n \in \mathbb{N}$, $\varepsilon > 0$. Then x is called a (n, ε)-basic special convex combination (or a (n, ε)-basic s.c.c.) if the following are satisfied:

(i) $F \in \mathcal{S}_n$, $c_k \geqslant 0$ for $k \in F$ and $\sum_{k\in F} c_k = 1$,
(ii) for any $G \subset F$, with $G \in \mathcal{S}_{n-1}$, we have that $\sum_{k\in G} c_k < \varepsilon$.

Remark 3.2 We note for later use the following easy fact. If $x = \sum_{i\in F} c_i e_i$ is a (n, ε)-basic s.c.c. with $0 < \varepsilon < 1/2$ and for $i \in F\setminus\{\min F\}$ we set $c'_i = c_i/(\sum_{j\in F\setminus\{\min F\}} c_j)$ then $y = \sum_{i\in F\setminus\{\min F\}} c'_i e_i$ is a $(n, 2\varepsilon)$-basic s.c.c.

The next result is from [7]. For a proof see [10, Chapter 2, Proposition 2.3].

Proposition 3.3 *For every infinite subset of the natural numbers M, any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist $F \subset M$ and non-negative real numbers $(c_k)_{k\in F}$, such that the vector $x = \sum_{k\in F} c_k e_k$ is a (n, ε)-basic s.c.c.*

Definition 3.4 Let $x_1 < \cdots < x_m$ be vectors in $c_{00}(\mathbb{N})$ and $\psi(k) = \min \operatorname{supp} x_k$, for $k = 1, \ldots, m$. If the vector $\sum_{k=1}^m c_k e_{\psi(k)}$ is a (n, ε)-basic s.c.c., for some $n \in \mathbb{N}$ and $\varepsilon > 0$, then the vector $x = \sum_{k=1}^m c_k x_k$ is called a (n, ε)-special convex combination (or (n, ε)-s.c.c.).

By T we denote Tsirelson space and by $\|\cdot\|_T$ its norm, as they were defined in [13]. This space is actually the dual of Tsirelson's original Banach space defined in [25]. The proof of the following result can be found in [8, Proposition 2.5].

Proposition 3.5 *Let $n \in \mathbb{N}$, $\varepsilon > 0$, $x = \sum_{k\in F} c_k e_k$ be a (n, ε)-basic s.c.c. and $G \subset F$. Then*

$$\left\|\sum_{k\in G} c_k e_k\right\|_T \leqslant \frac{1}{2^n}\sum_{k\in G} c_k + \varepsilon.$$

The next result can also be found in [8, Corollary 2.8]. A number of steps are required in order to reach this estimate, however the arguments used there also work in the present case unchanged and therefore we omit the proof.

Proposition 3.6 *Let $(x_k)_k$ be a block sequence in $\mathfrak{X}_\mathcal{T}$ with $\|x_k\| \leqslant 1$ for all $k \in \mathbb{N}$, $(c_k)_k$ be a sequence of real numbers and $\phi(k) = \max\operatorname{supp} x_k$ for all k. Then*

$$\left\|\sum_k c_k x_k\right\| \leqslant 6\left\|\sum_k c_k e_{\phi(k)}\right\|_T.$$

The next crucial estimate follows from Propositions 3.5 and 3.6. A proof can be found in [8, Corollary 2.9].

Corollary 3.7 *Let $n \in \mathbb{N}$, $\varepsilon > 0$ and $x = \sum_{k=1}^m c_k x_k$ be a (n, ε)-s.c.c. in $\mathfrak{X}_\mathcal{T}$, such that $\|x_k\| \leqslant 1$, for $k = 1, \dots, m$. If F is subset of $\{1, \dots, m\}$ then*

$$\left\|\sum_{k\in F} c_k x_k\right\| \leqslant \frac{6}{2^n}\sum_{k\in F} c_k + 12\varepsilon.$$

In particular, $\|x\| \leqslant 6/2^n + 12\varepsilon$.

Using Propositions 3.3 and 3.7 one can easily derive the next result. For a proof see [8, Corollary 2.10].

Proposition 3.8 *The basis of $\mathfrak{X}_\mathcal{T}$ is shrinking. In particular, the dual of $\mathfrak{X}_\mathcal{T}$ is separable.*

We now give some definitions which will be crucial in the next sections, where we prove the properties of the space $\mathfrak{X}_\mathcal{T}$. Rapidly increasing sequences are defined exactly as in [8, Definition 2.13].

Definition 3.9 Let $C \geqslant 1$ and $(n_k)_k$ be a strictly increasing sequence of natural numbers. A block sequence $(x_k)_k$ is called a $(C, (n_k)_k)$-rapidly increasing sequence (or $(C, (n_k)_k)$-RIS) if $\|x_k\| \leqslant C$ for all k and the following hold:

(i) for every k and every weighted functional f in $W_\mathcal{T}$ with $w(f) = j < n_k$, we have $|f(x_k)| < C/2^j$ and
(ii) for every k, $1/2^{n_{k+1}} \max\operatorname{supp} x_k < 1/2^{n_k}$.

The notion of a (C, θ, n)-vector and a (C, θ, n)-exact vector is defined identically as in [8, Definition 2.15].

Definition 3.10 Let $n \in \mathbb{N}$, $C \geqslant 1$ and $\theta > 0$. A vector $x \in \mathfrak{X}_\mathcal{T}$ is called a (C, θ, n)-vector if there exist $0 < \varepsilon < 1/(36C2^{3n})$ and a block sequence $(x_k)_{k=1}^m$ with $\|x_k\| \leqslant C$ for $k = 1, \dots, m$ such that:

(i) $\min\operatorname{supp} x_1 \geqslant 8C2^{2n}$,
(ii) there exist non-negative real numbers $(c_k)_{k=1}^m$ so that the vector $\sum_{k=1}^m c_k x_k$ is a (n, ε)-s.c.c.,
(iii) $x = 2^n \sum_{k=1}^m c_k x_k$ and $\|x\| \geqslant \theta$.

If moreover there exists a strictly increasing sequence of natural numbers $(n_k)_{k=1}^m$ with $n_1 > 2^{2n}$ so that $(x_k)_{k=1}^m$ is a $(C, (n_k)_{k=1}^m)$-RIS, then x is called a (C, θ, n)-exact vector.

Remark 3.11 Let x be a (C, θ, n)-vector in $\mathfrak{X}_{\mathcal{T}}$. Then, using Corollary 3.7 we conclude that $\|x\| < 7C$.

Remark 3.12 Let x be a (C, θ, n)-vector in $\mathfrak{X}_{\mathcal{T}}$. By the choice of ε and $\|x_k\| \leqslant C$ for $k = 1, \dots, m$, we obtain $\|x\|_\infty < 1/(2^{2n}36)$.

4 The α-index

In all recent constructions involving saturation under constraints [3,8,9,11], the α-index has been used to help determine the spreading models admitted by block sequences. In contrast to the HI constructions [8] and [9], where the α-index is not sufficient to fully characterize the spreading models of block sequences, the present case resembles more closely the unconditional example from [3], where the α-index is the only necessary tool to study spreading models admitted by the space. This is due to the fact that α-averages, more precisely α_c-averages, are the only ingredient used to construct weighted functionals. The definition of the α-index of a block sequence given below is identical to the one from [8] and [9].

Definition 4.1 Let $(x_k)_k$ be a block sequence in $\mathfrak{X}_{\mathcal{T}}$ that satisfies the following: for every $n \in \mathbb{N}$, for every very fast growing sequence of α_c-averages of $W_{\mathcal{T}}$ $(\alpha_q)_q$, for every increasing sequence of subsets of the natural numbers $(F_m)_m$, such that $(\alpha_q)_{q\in F_m}$ is $\mathcal{S}_n$-admissible for all $m \in \mathbb{N}$ and for every subsequence $(x_{k_m})_m$ of $(x_k)_k$, we have that

$$\lim_k \sum_{q\in F_m} |\alpha_q(x_{k_m})| = 0.$$

Then we say that the α-index of $(x_k)_k$ is zero and write $\alpha((x_k)_k) = 0$. Otherwise we write $\alpha((x_k)_k) > 0$.

The next characterization, of when a block sequence has α-index zero, and its proof can be found in [8, Proposition 3.3]. Although here it is formulated slightly differently, the two versions are easily seen to be equivalent.

Proposition 4.2 *Let $(x_k)_k$ be a block sequence in $\mathfrak{X}_{\mathcal{T}}$. The following assertions are equivalent.*

(i) *The α-index of $(x_k)_k$ is zero.*
(ii) *For every $\varepsilon > 0$ there exists $j \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that for every $k \geqslant k_n$ and for every very fast growing and $\mathcal{S}_n$-admissible sequence of α_c-averages $(\alpha_q)_{q=1}^d$, with $s(\alpha_q) \geqslant j$ for $q = 1, \dots, d$, we have that $\sum_{q=1}^d |\alpha_q(x_k)| < \varepsilon$.*

The next result is proved in [8, Proposition 3.5].

Proposition 4.3 *Let $(x_k)_k$ be a seminormalized block sequence in $\mathfrak{X}_{\mathcal{T}}$ with $\alpha((x_k)_k) > 0$. Then there exist $\theta > 0$ and a subsequence $(x_{k_m})_m$ of $(x_k)_k$ that generates an ℓ_1^n spreading model with a lower constant $\theta/2^n$, for all $n \in \mathbb{N}$. More precisely, for every*

$n \in \mathbb{N}$, subset of the natural numbers F, so that $(x_{k_m})_{m\in F}$ is $\mathcal{S}_n$-admissible, and real numbers $(c_m)_{m\in F}$ we have that

$$\left\| \sum_{m\in F} c_m x_{k_m} \right\| \geqslant \frac{\theta}{2^n} \sum_{m\in F} |c_m|.$$

In particular, for all k_0, $n \in \mathbb{N}$, there exists a finite subset of the natural numbers F with $\min F \geqslant k_0$ and non-negative real numbers $(c_m)_{m\in F}$, such that the vector $x = 2^n \sum_{m\in F} c_m x_{k_m}$ is a (C, θ, n)-vector, where $C = \sup\{\|x_k\| : k \in \mathbb{N}\}$.

We now prove that block sequences with α-index zero admit only c_0 as a spreading model and that Schreier sums of them define rapidly increasing sequences.

Proposition 4.4 *Let $(x_k)_k$ be a normalized block sequence in $\mathfrak{X}_{\mathcal{T}}$ with $\alpha((x_k)_k) = 0$. Then $(x_k)_k$ has a subsequence, which we also denote by $(x_k)_k$, that generates a spreading model which is isometric to the unit vector basis of c_0. Moreover, there exists a strictly increasing sequence of natural numbers $(j_k)_k$ so that for every natural numbers $n \leqslant k_1 < \cdots < k_n$, real numbers $(c_i)_{i=1}^n$ and weighted functional f of $W_{\mathcal{T}}$ with $w(f) = j < j_n$, we have*

$$\left| f\left(\sum_{i=1}^n c_i x_{k_i}\right)\right| < \frac{9/8}{2^j} \max_{1\leqslant i\leqslant n} |c_i|.$$

Proof Using Proposition 4.2, we pass to a subsequence of $(x_k)_k$, again denoted by $(x_k)_k$, and choose a strictly increasing sequence of natural numbers so that the following are satisfied:

(i) for every $k \in \mathbb{N}$, $1/2^{j_{k+1}} \max\operatorname{supp} x_k < 1/2^k$ and
(ii) for every $k_0, k \in \mathbb{N}$ with $k \geqslant k_0$ and every very fast growing and $\mathcal{S}_{j_{k_0}}$-admissible sequence of α_c-averages $(\alpha_q)_{q=1}^n$ with $s(\alpha_q) \geqslant \max\operatorname{supp} x_{k_0}$ we have

$$\sum_{q=1}^d |\alpha_q(x_k)| < 1/(k_0 2^{k_0}).$$

We claim that $(x_k)_k$ generates a spreading model isometric to c_0. Using the third assertion of Remark 2.10 we shall inductively prove the following: for every $f \in W_m$, natural numbers $n \leqslant k_1 < \cdots < k_n$ and real numbers $c_1, \ldots, c_n$ in $[-1, 1]$ we have

$$\left| f\left(\sum_{i=1}^n c_i x_{k_i}\right)\right| < 1 + \frac{3}{2^n}. \tag{3}$$

If moreover f is a weighted functional with $w(f) = j < j_n$, then

$$\left| f\left(\sum_{i=1}^n c_i x_{k_i}\right)\right| < \frac{1 + 4/2^n}{2^j}. \tag{4}$$

The desired conclusion clearly follows from the above and the fact that the basis of $\mathfrak{X}_{\mathcal{T}}$ is bimonotone, omitting if necessary a finite number of terms of the sequence $(x_k)_k$.

We now proceed to the proof of the inductive step. The case $m = 0$ is an immediate consequence of the fact that the sequence $(x_k)_k$ is normalized and $W_0 = \{\pm e_i^* : i \in \mathbb{N}\}$. Assume now that m is such that the conclusion holds for every functional in W_m and let $f \in W_{m+1}$. If f is an α_c-average of W_m, then by the inductive assumption we conclude that (3) holds. Otherwise, f is a weighted functional of weight $w(f) = j$, i.e. there is a very fast growing and $\mathcal{S}_j$ admissible sequence of α_c-averages of W_m $(\alpha_q)_{q=1}^d$ so that $f = (1/2)^j \sum_{q=1}^d \alpha_q$. Assuming that $f\left(\sum_{i=1}^n c_i x_{k_i}\right) \neq 0$, set $q_0 = \min\{q : \max\operatorname{supp}\alpha_q \geqslant \min\operatorname{supp} x_{k_1}\}$. Omitting, if it is necessary, the first $q_0 - 1$ averages, we may assume that $q_0 = 1$. the We distinguish three cases concerning weight of f.

Case 1: $j < j_{k_1}$. Since the sequence $(\alpha_q)_{q=1}^d$ is very fast growing, for $q > 1$ we have $s(\alpha_q) > \max\operatorname{supp}\alpha_1 \geqslant \min\operatorname{supp} x_{k_1}$. Also, since $(\alpha_q)_{q=2}^d$ is $\mathcal{S}_j$ admissible with $j < j_{k_1}$, by (ii) we conclude:

$$\sum_{q=2}^d \left|\alpha_q\left(\sum_{i=1}^n c_i x_{k_i}\right)\right| < n\frac{1}{k_1 2^{k_1}} \leqslant \frac{1}{2^n}. \tag{5}$$

Moreover, by the inductive assumption we obtain $\left|\alpha_1\left(\sum_{i=1}^n c_i x_{k_i}\right)\right| < 1 + 3/2^n$. Combining this with (5):

$$\left|f\left(\sum_{i=1}^n c_i x_{k_i}\right)\right| < \frac{1 + 4/2^n}{2^j}. \tag{6}$$

This concludes the proof of the first case and also (4) of the inductive assumption.

Case 2: there is $1 \leqslant i_0 < n$ so that $j_{k_{i_0}} \leqslant j < j_{k_{i_0+1}}$. Arguing in an identical manner as in the previous case, we obtain

$$\left|f\left(\sum_{i>i_0} c_i x_{k_i}\right)\right| < \frac{1 + 4/2^{k_{i_0+1}}}{2^{j_{k_{i_0}}}} \leqslant \frac{2}{2^n}. \tag{7}$$

Also, if $i_0 > 1$, by (i) we have that $1/2^j \max\operatorname{supp} x_{k_{i_0-1}} < 1/2^{k_{i_0-1}}$ and hence:

$$\left|f\left(\sum_{i<i_0} c_i x_{k_i}\right)\right| \leqslant \|f\|_\infty \max\operatorname{supp} x_{k_{i_0-1}} < \frac{1}{2^{k_{i_0-1}}} \leqslant \frac{1}{2^n}. \tag{8}$$

Combining (7) and (8) with the fact that $|f(x_{k_{i_0}})| \leqslant 1$ we conclude

$$\left|f\left(\sum_{i=1}^n c_i x_{k_i}\right)\right| < 1 + \frac{3}{2^n}. \tag{9}$$

Case 3: $j \geqslant j_{k_n}$. Using that $|f(x_{k_n})| \leqslant 1$ and arguing as in (8) we obtain $\left|f\left(\sum_{i=1}^n c_i x_{k_i}\right)\right| < 1 + 1/2^n$ and this concludes the proof. □

Propositions 4.3 and 4.4 yield the following result, which characterizes the spreading models admitted by a given block sequence.

Corollary 4.5 *Let $(x_k)_k$ be a normalized block sequence in $\mathfrak{X}_\mathcal{T}$. Then $(x_k)_k$ has a subsequence that generates either an isometric c_0 spreading model or an ℓ_1^n spreading model for every $n \in \mathbb{N}$. More precisely, the assertions stated below hold.*

(i) *The sequence $(x_k)_k$ admits only c_0 as a spreading model if and only if $\alpha((x_k)_k) = 0$.*
(ii) *The sequence $(x_k)_k$ has a subsequence that generates an ℓ_1^n spreading model for every $n \in \mathbb{N}$ if and only if $\alpha((x_k)_k) > 0$.*

5 Estimations on exact vectors

In this section we provide estimations on exact vectors whose sums define non-trivial weakly Cauchy sequences in $\mathfrak{X}_\mathcal{U}$ and in the general case provide the fact that the space $\mathfrak{X}_\mathcal{T}$ is hereditarily indecomposable. We give the definitions of exact vectors and exact sequences and several technical intermediate steps are presented in order to achieve the main estimate.

The next estimate uses Proposition 3.6 and the properties of special convex combinations. It is proved in [8, Lemma 3.8] and identical arguments also apply in this case.

Lemma 5.1 *Let x be a (C, θ, n)-vector in $\mathfrak{X}_\mathcal{T}$. Let also $(a_q)_{q=1}^d$ be a very fast growing and $\mathcal{S}_j$-admissible sequence of α_c-averages, with $j < n$. Then*

$$\sum_{q=1}^d |\alpha_q(x)| < \frac{6C}{s(\alpha_1)} + \frac{1}{2^n}.$$

These next two results follows readily form Lemma 5.1 and Proposition 4.4. Their proof can also be found in [8, Propositions 3.9 and 3.10]

Proposition 5.2 *Let $C \geqslant 1$ and $\theta > 0$. If $(x_k)_k$ is a block sequence in $\mathfrak{X}_\mathcal{T}$ so that each x_k is a (C, θ, n_k)-vector, with $(n_k)_k$ a strictly increasing sequence of natural numbers, then $\alpha((x_k)_k) = 0$ and hence, every spreading model admitted by $(x_k)_k$ is isometric, up to scaling, to the unit vector basis of c_0.*

Proposition 5.3 *Let x be a (C, θ, n)-vector in $\mathfrak{X}_\mathcal{T}$. Then for any weighted functional f in $W_\mathcal{T}$ such that $w(f) = j < n$ we have*

$$|f(x)| < \frac{7C}{2^j}.$$

We now give the definition of an exact pair and a dependent sequence.

Definition 5.4 A pair (f, x) where x is a (C, θ, n)-exact vector in $\mathfrak{X}_{\mathcal{T}}$ and f is a weighted functional in $W_{\mathcal{T}}$ with $w(f) = n$, $\operatorname{ran} f \subset \operatorname{ran} x$ and $f(x) = \theta$ is called a (C, θ, n)-exact pair.

Definition 5.5 Let $C \geqslant 1$ and $\theta > 0$. A sequence of pairs $\{(f_k, x_k)\}_{k=1}^{\ell}$, where $f_k \in W_{\mathcal{T}}$ and x_k is a vector with rational coefficients in $\mathfrak{X}_{\mathcal{T}}$ for $k = 1, \dots, \ell$, is called a (C, θ)-dependent sequence if the following are satisfied:

(i) (f_k, x_k) is a $(C, \theta, w(f_k))$-exact pair for $k = 1, \dots, \ell$ and
(ii) $\{(f_k, x_k)\}_{k=1}^{\ell}$ is in $\mathcal{T}$,

We introduce some notation baring similarities to the one used in [8, Subsection 3.2] and [9].

Notation Let $x = 2^n \sum_{k=1}^m c_k x_k$ be a (C, θ, n)-exact vector, with $(x_k)_{k=1}^m$ a $(C, (n_k)_{k=1}^m)$-RIS. Let also $g_1 < \cdots < g_d$ be weighted functionals in $W_{\mathcal{T}}$, all of which have weight greater than or equal to $\min L$ satisfying $\phi(w(g_1)) < \cdots < \phi(w(g_d))$ (see Definition 2.4). We define the following subsets of $\mathbb{N}$:

$$
\begin{aligned}
I_0(x, (g_i)_{i=1}^d) &= \{j : n \leqslant w(g_j) < 2^{2n}\},\\
I_1(x, (g_i)_{i=1}^d) &= \{j : w(g_j) < n\} \text{ and}\\
I_2(x, (g_i)_{i=1}^d) &= \{j : 2^{2n} \leqslant w(g_j)\}.
\end{aligned}
$$

Remark 5.6 Let x be a (C, θ, n)-exact vector and $g_1 < \cdots < g_d$ be weighted functionals in $W_{\mathcal{T}}$, all of which have weight greater than or equal to $\min L$ satisfying $\phi(w(g_1)) < \cdots < \phi(w(g_d))$.

(i) If $n \in L$, then the set $I_0(x, (g_i)_{i=1}^d)$ is either empty or a singleton. Indeed, by the choice of L', the fact that $L \subset L'$ and the definition of ϕ it is straightforward to check that if $j \in I_0(x, (g_i)_{i=1}^d)$, then $\phi(w(g_j)) = n$ and clearly at most one j can satisfy this condition.
(ii) Also, the sets $I_1(x, (g_i)_{i=1}^d)$, $I_2(x, (g_i)_{i=1}^d)$ are successive intervals of $\{1, \dots, d\}$, which clearly follows from the fact that ϕ is non-decreasing.

Lemma 5.7 *Let $n \geqslant 2$, x be a (C, θ, n)-exact vector in $\mathfrak{X}_{\mathcal{T}}$ and $g_1 < \cdots < g_d$ be weighted functionals in $W_{\mathcal{T}}$, all of which have weight greater than or equal to* $\min L$ *satisfying $\phi(w(g_1)) < \cdots < \phi(w(g_d))$. If we set $I_2(x) = I_2(x, (g_i)_{i=1}^d)$, then*

$$\sum_{j \in I_2(x)} |g_j(x)| < d\frac{C}{2^n}.$$

Proof We will actually show that if g is a weighted functional in $W_{\mathcal{T}}$ with $w(g) \geqslant 2^{2n}$, then $|g(x)| < C/2^n$. If $x = 2^n \sum_{k=1}^m c_k x_k$, with $(x_k)_{k=1}^m$ a $(C, (n_k)_{k=1}^m)$-RIS, recall that according to Definition 3.10 we have that $2^{2n} < n_1$. Set

$$A = \{k : \ n_k \leqslant w(g)\} \quad \text{and} \quad B = \{k : \ w(g) < n_k\}.$$

If $A \neq \varnothing$, set $k_0 = \max A$.

For $k \in B$ and $1 \leqslant k \leqslant m$, since $(x_k)_{k=1}^m$ is a $(C, (n_k)_{k=1}^m)$-RIS, we obtain $|g(x_k)| < C/2^{w(g)}$ and hence:

$$\left| g\left(2^n \sum_{k \in B} c_k x_k\right)\right| \leqslant 2^n C \sum_{k \in B} \frac{c_k}{2^{w(g)}} \leqslant 2^n C \frac{1}{2^{2^{2n}}} < C \frac{1}{2^n 6} \tag{10}$$

where we used that $w(g) \geqslant 2^{2n}$ while the last inequality holds for all $n \geqslant 2$.

If $A = \varnothing$ we are done. Otherwise we need some further calculations. Observe that

$$\left| g\left(2^n c_{k_0} x_{k_0}\right)\right| \leqslant 2^n C c_{k_0} < \frac{C}{2^{2n} 36} \tag{11}$$

where we used that, according to Definition 3.10, the vector $\sum_{k=1}^m c_k x_k$ is an (n, ε)-s.c.c. with $\varepsilon < 1/(36C2^{3n})$.

If A is a singleton, then (10) and (11) yield the desired estimate. Otherwise, if A is not a singleton:

$$\begin{aligned} \left| g\left(2^n \sum_{k<k_0} c_k x_k\right)\right| &\leqslant \|g\|_\infty \max \operatorname{supp} x_{k_0-1} \left\| 2^n \sum_{k<k_0} c_k x_k \right\|_\infty \\ &\leqslant \frac{2^{n_{k_0}}}{2^{w(g)}} \left(\frac{1}{2^{n_{k_0}}} \max \operatorname{supp} x_{k_0-1}\right) \frac{1}{2^{2n} 36} \\ &\leqslant \frac{1}{2^{n_{k_0-1}}} \frac{1}{2^{2n} 36} < \frac{1}{2^{2n} 36} \end{aligned}$$

where we used property (ii) from Definition 3.9, Remark 3.12 and that k_0 is in A, i.e. $n_{k_0} \leqslant w(g)$. The result follows from the above, (10) and (11). □

Lemma 5.8 *Let* $1 \leqslant C \leqslant 10/7$, $\theta > 0$, $\{(f_k, x_k)\}_{k=1}^\ell$ *be a* (C, θ)*-dependent sequence and* $1 \leqslant n \leqslant m \leqslant \ell$ *be natural numbers. Let also* $(g_j)_{j=1}^d$ *be a sequence of weighted functionals in* $W_{\mathcal{T}}$ *and* $(\varepsilon_j)_{j=1}^d$ *be a sequence of signs in* $\{-1, 1\}$*, so that one of the following is satisfied:*

(i) *the sequence* $(g_j)_{j=1}^d$ *is comparable and the signs* $(\varepsilon_j)_{j=1}^d$ *are alternating or*
(ii) *the sequence* $(g_j)_{j=1}^d$ *is either incomparable or irrelevant.*

If for $j = 1, \ldots, d$ *we define* $D_j = \left\{n \leqslant k \leqslant m : \ w(g_j) < w(f_k)\right\}$*, then*

$$\left| \sum_{j=1}^d \varepsilon_j g_j \left(\sum_{k=n}^m x_k\right) - \sum_{j=1}^d \varepsilon_j g_j \left(\sum_{k \in D_j} x_k\right)\right| \leqslant 22C + d \frac{2C}{2^{w(f_n)}}.$$

Proof Recall that each x_k is a $(C, \theta, w(f_k))$-exact vector and for all $1 \leqslant k \leqslant \ell$ define $A_k = I_0\left(x_k, (g_j)_{j=1}^d\right)$ and $B_k = I_1\left(x_k, (g_j)_{j=1}^d\right)$. Observe that

$$\sum_{j=1}^{d} \varepsilon_j g_j \left(\sum_{k \in D_j}^{m} x_k \right) = \sum_{k=1}^{m} \sum_{j \in B_k} \varepsilon_j g_j(x_k).$$

Therefore, if we define $C_k = I_2 \left(x_k, (g_j)_{j=1}^d \right)$ we conclude

$$\begin{aligned} &\left| \sum_{j=1}^{d} \varepsilon_j g_j \left(\sum_{k=n}^{m} x_k \right) - \sum_{j=1}^{d} \varepsilon_j g_j \left(\sum_{k \in D_j}^{m} x_k \right) \right| = \\ &\quad \left| \sum_{k=n}^{m} \left(\sum_{j \in A_k} \varepsilon_j g_j(x_k) + \sum_{j \in C_k} \varepsilon_j g_j(x_k) \right) \right| \leqslant \left| \sum_{k=n}^{m} \sum_{j \in A_k} \varepsilon_j g_j(x_k) \right| + \sum_{k=n}^{m} d \frac{C}{2^{w(f_k)}} \\ &\quad \leqslant \left| \sum_{k=n}^{m} \sum_{j \in A_k} \varepsilon_j g_j(x_k) \right| + d \frac{2C}{2^{w(f_n)}} \end{aligned}$$

where the first inequality follows from Lemma 5.7 while the second one follows from the fact that the $w(f_j)$'s are strictly increasing (see Remark 2.3).

We will show that $|\sum_{k=1}^{m} \sum_{j \in A_k} \varepsilon_j g_j(x_k)| \leqslant 22C$, which will conclude the proof. We remind that by Remark 3.11, $\|x_k\| < 7C$ for all $1 \leqslant k \leqslant \ell$. We also remind that by Remark 5.6 each set A_k is either empty or a singleton and in particular, we note the following: if $j \in A_k$ then $\phi(w(g_j)) = w(f_k)$. Moreover, the assumptions yield the $\phi(w(g_j))$'s are strictly increasing. If the sets A_k are all empty there is nothing to prove. Otherwise, let $k_1 < \cdots < k_s$ be all the k's in $\{1, \ldots, \ell\}$ satisfying $A_{k_i} \neq \varnothing$. Let also $1 \leqslant j_1 < \cdots < j_s \leqslant d$ be so that for each i, j_i is the unique element of A_{k_i}, and hence $\phi(w(g_{j_i})) = w(f_{k_i})$ for $i = 1, \ldots, s$.

If $s \leqslant 2$ then the desired estimate follows from $\|x_k\| < 7C$ for all $1 \leqslant k \leqslant \ell$. Otherwise, $s \geqslant 3$ which implies that the sequence $(g_i)_{i=1}^d$ is not incomparable, i.e. there are $1 \leqslant i < i' \leqslant d$ so that $w(g_{j_2})$ and $w(g_{j_3})$ are not incomparable in the sense of Definition 2.5. Indeed, since $\{(f_k, x_k)\}_{k=1}^m$ is in $\mathcal{T}$ we have that

$$\begin{aligned} \sigma^{-1}(\phi(w(g_{j_2}))) = \sigma^{-1}(w(f_{k_2})) = \{(f_k, x_k)\}_{k=1}^{k_2-1} \sqsubseteq \{(f_k, x_k)\}_{k=1}^{k_3-1} \\ = \sigma^{-1}(w(f_{k_3})) = \sigma^{-1}(\phi(w(g_{j_3}))) \end{aligned}$$

which means that $w(g_{j_2})$ and $w(g_{j_3})$ are comparable.

We conclude that the sequence $(g_j)_{j=1}^d$ is either comparable, or irrelevant and therefore there exists $m' \in \mathbb{N}$ with $d \leqslant m'$, natural numbers $1 \leqslant k'_1 < \cdots < k'_d \leqslant m'$ and $\{(h_k, y_k)\}_{k=1}^{m'}$ in $\mathcal{T}$, so that $\phi(w(g_j)) = w(h_{k'_j})$ for $j = 1, \ldots, d$. Observe the following:

$$\{(h_k, y_k)\}_{k=1}^{k'_{j_s}-1} = \sigma^{-1}(\phi(w(g_{j_s}))) = \sigma^{-1}(w(f_{k_s})) = \{(f_k, x_k)\}_{k=1}^{k_s-1}. \tag{12}$$

The above implies that $\{j_1, \ldots, j_s\}$ is an initial interval of $\{1, \ldots, d\}$, in particular:

(a) $j_i = i$ for $i = 1, \dots, s$ and
(b) $k'_i = k_i$ for $i = 1, \dots, s$.

Indeed, if $1 \leqslant t < j_s$ then $\phi(w(g_t)) = w(h_{k'_t}) = w(f_{k'_t})$ and hence $j \in A_{k'_t}$. This yields that there is $1 \leqslant i < s$ so that $t = j_i$ and $k_i = k'_t$. A simple cardinality argument yields that $\{j_1, \dots, j_s\} = \{1, \dots, s\}$ and for $1 \leqslant i < s$ $k'_i = k_i$. Also, since $j_s = s$, (12) clearly yields that $k_s = k'_s$.

Observe that the sequence $(g_j)_{j=1}^d$ is not irrelevant. Indeed, the opposite would imply that $10 < |g_2(y_{k'_2})| = |g_2(x_{k_2})| \leqslant 7C \leqslant 10$, a contradiction.

In the last remaining case, the sequence $(g_j)_{j=1}^d$ is comparable. Define $E = \{i : k_i \in \{n, \dots, m\}\}$, observe that E is an interval of $\{1, \dots, s\}$ and choose successive two-point intervals $E_1, \dots, E_p$ of $E \setminus \{\max E, \min E\}$, so that $E \setminus \cup_{i=1}^p E_i$ has at most three elements. The fact that the sequence $(g_j)_{j=1}^d$ is comparable and (b) yield that $|g_i(x_{k_i}) - g_j(x_{k_j})| < 1/2^i$ for all $2 \leqslant i < j \leqslant s-1$ and therefore, since the signs $(\varepsilon_i)_{i=1}^d$ are alternating, if for each i we write $E_i = \{r_i, r_i + 1\}$ then we obtain

$$\left|\sum_{j \in E_i} \varepsilon_j g_j(x_{k_j})\right| = \left|g_r(x_{k_{r_i}}) - g_{r+1}(x_{k_{r_i+1}})\right| < \frac{1}{2^{r_i}} \leqslant \frac{1}{2^i}$$

for $i = 1, \dots, p$ and hence

$$\left|\sum_{k=n}^m \sum_{j \in A_k} \varepsilon_j g_j(x_k)\right| = \left|\sum_{i \in E} \varepsilon_i g_i(x_{k_i})\right| \leqslant 21C + \sum_{i=1}^p \left|\sum_{j \in E_i} \varepsilon_i g_i(x_{k_i})\right| \leqslant 22C.$$

□

The result below is the main one of this section and it is used later to prove the main properties of the space $\mathfrak{X}_{\mathcal{T}}$ and its operators.

Proposition 5.9 *Let* $1 \leqslant C \leqslant 10/7$, $\{(f_k, x_k)\}_{k=1}^\ell$ *be a* (C, θ)*-dependent sequence and* f *be a weighted functional in* $W_{\mathcal{T}}$. *If for some natural numbers* $1 \leqslant n \leqslant m \leqslant \ell$ *we set* $D = \{k \in \{n, \dots, m\} : \ w(f) < w(f_k)\}$, *then:*

$$\left|f\left(\sum_{k \in D} x_k\right)\right| \leqslant \frac{47C}{2^{w(f)}}.$$

In particular, for every natural numbers $1 \leqslant n \leqslant m \leqslant \ell$, $\|\sum_{k=n}^m x_k\| \leqslant 24C$.

Proof We first assume that the first statement holds to prove the fact that for $1 \leqslant n \leqslant m \leqslant \ell$, $\|\sum_{k=n}^m x_k\| \leqslant 24C$. Let $f \in W_{\mathcal{T}}$. We may assume that f is either an element of the basis, or a weighted functional. In the first case, $\left|f\left(\sum_{k=n}^m x_k\right)\right| \leqslant \max\{\|x_k\|_\infty : \ n \leqslant k \leqslant m\} < 24C$ by Remark 3.12. If on the other hand f is a weighted functional, we distinguish three cases regarding the weight of f. If $w(f) < w(f_n)$, then the first statement yields that $\left|f\left(\sum_{k=n}^m x_k\right)\right| < 47C/2^{w(f)} < 24C$. If there is $n \leqslant k_0 < m$ with $w(f_{k_0}) \leqslant w(f) < w(f_{k_0+1})$, then as before we obtain that $\left|f\left(\sum_{k>k_0} x_k\right)\right| \leqslant 47C/2^{w(f)} \leqslant 47C/2^{w(f_{k_0})} < C$ (recall that $w(f_{k_0}) \in L$ and $\min L \geqslant 8$). Also, by Remark 3.7, $|f(x_{k_0})| \leqslant 7C$ while (2) and Remark 3.12 yield that $\left|f\left(\sum_{k<k_0} x_k\right)\right| < C$. We obtain that $\left|f\left(\sum_{k=n}^m x_k\right)\right| < 9C$. In the last case we have $w(f) \geqslant w(f_m)$, where by using similar arguments we obtain $\left|f\left(\sum_{k=n}^m x_k\right)\right| < 8C$.

We now proceed to prove the first statement, for which we will use the third statement of Remark 2.10. In particular, by induction on p, where $W_{\mathcal{T}} = \cup_p W_p$, we shall prove that for every weighted functional f in W_p and natural numbers $1 \leqslant n \leqslant m \leqslant \ell$, if $D = \{k \in \{n, \ldots, m\} : \ w(f) < w(f_k)\}$ then $\left|f\left(\sum_{k\in D} x_k\right)\right| \leqslant 24C/2^{w(f)}$.

The set $W_0 = \{\pm e_i^* : \ i \in \mathbb{N}\}$ does not contain any weighted functionals and so the statement for $p = 0$ trivially holds. Let $p \in \mathbb{N}$ such that every weighted functional in W_p satisfies the conclusion. Before showing that this property is satisfied by functionals in W_{p+1}, we remark the following: let α_0 be an α_c-average of W_p and $n \leqslant m$, then

$$\left|\alpha_0\left(\sum_{k=n}^m x_k\right)\right| \leqslant \frac{23C}{s(\alpha_0)} + \frac{2C}{2^{w(f_n)}}. \tag{13}$$

Indeed, if α_0 is a basic average, then

$$\left|\alpha_0\left(\sum_{k=n}^m x_k\right)\right| \leqslant \max_{n\leqslant k\leqslant m} \|x_k\|_\infty \leqslant \frac{1}{2^{w(f_n)}}$$

where the last inequality follows from Remark 3.12. If α_0 is not a basic average, then there are natural numbers $s \leqslant d$ and weighted functionals $g_1 < \cdots < g_s$ in W_p, so that $\alpha_0 = (1/d)\sum_{i=1}^s g_i$ (or $\alpha_0 = (1/d)\sum_{i=1}^s \varepsilon_i g_i$ with the ε_i's being alternating signs). We define $D_j = \{k \in \{n, \ldots, m\} : w(g_j) < w(f_k)$ and by Lemma 5.8 we obtain:

$$\left|\alpha_0\left(\sum_{k=n}^m x_k\right)\right| \leqslant \frac{1}{d}\sum_{j=1}^d \left|g_j\left(\sum_{k\in D_j} x_k\right)\right| + \frac{22C}{d} + \frac{2C}{2^{w(f_n)}} \tag{14}$$

The inductive assumption yields

$$\sum_{j=1}^d \left|g_j\left(\sum_{k\in D_j} x_k\right)\right| \leqslant \sum_{j=1}^d \frac{47C}{2^{w(g_j)}} \leqslant \sum_{j=1}^d \frac{47C}{2^{\phi(w(g_j))}} \leqslant C$$

where we used the fact that, in order to define an α_c-average, the $\phi(w(g_j))$'s must be strictly increasing elements of L and $\min L \geqslant 8$. Combining (14) with the above, (13) follows.

Let now $f = (1/2^j)\sum_{q=1}^d \alpha_q$ be a weighted functional in W_{p+1}, with $(\alpha_q)_{q=1}^d$ a very fast growing and $\mathcal{S}_j$-admissible sequence of α_c-averages of W_p, and let also $1 \leqslant n \leqslant m \leqslant \ell$ be natural numbers. Define $D = \{k \in \{n, \ldots, m\} : \ j < w(f_k)\}$ and also for $k \in D$ set

$$M_k = \left\{q : \operatorname{ran}\alpha_q \cap \operatorname{ran} x_k \neq \varnothing\right\} \quad \text{and}$$
$$N_k = \left\{q \in M_k : s(\alpha_q) > 8C2^{2w(f_k)}\right\}.$$

Lemma 5.1 yields that for $k \in D$,

$$\sum_{q \in N_k} |\alpha_q(x_k)| < \frac{2}{2^{w(f_k)}}$$

and therefore:

$$\begin{aligned} \left|\sum_{q=1}^{d} \alpha_q\left(\sum_{k \in D} x_k\right)\right| &= \left|\sum_{k \in D}\sum_{q \in M_k \setminus N_k} \alpha_q(x_k) + \sum_{k \in D}\sum_{q \in N_k} \alpha_q(x_k)\right| \\ &\leqslant \left|\sum_{k \in D}\sum_{q \in M_k \setminus N_k} \alpha_q(x_k)\right| + \sum_{k \in D} \frac{2}{2^{w(f_k)}} \\ &\leqslant \left|\sum_{k \in D}\sum_{q \in M_k \setminus N_k} \alpha_q(x_k)\right| + \frac{4}{2^{w(f_n)}} \end{aligned} \tag{15}$$

where we used that, according to Remark 2.3, the $w(f_k)$'s are strictly increasing.

Define $A = \cup_{k \in D} M_k \setminus N_k$, for $q \in A$ set $D_q = \{k \in D : q \in M_k \setminus N_k\}$ and observe the following:

$$\left|\sum_{k \in D}\sum_{q \in M_k \setminus N_k} \alpha_q(x_k)\right| = \left|\sum_{q \in A} \alpha_q\left(\sum_{k \in D_q} x_k\right)\right|. \tag{16}$$

We will show that the D_q's are disjoint intervals of $\{n, \dots, m\}$. Indeed, let $q \in A$ and $k_1, k_2 \in D_q$. If $k_1 < k < k_2$, we will show that $k \in D_q$. The fact that $q \in M_{k_1} \cap M_{k_2}$ means that $\operatorname{ran}\alpha_q \cap \operatorname{ran} x_{k_1} \neq \varnothing$ and $\operatorname{ran}\alpha_q \cap \operatorname{ran} x_{k_2} \neq \varnothing$ which, of course, yields that $\operatorname{ran}\alpha_q \cap \operatorname{ran} x_k \neq \varnothing$, i.e. $q \in M_k$. Also, $q \in M_{k_1} \setminus N_{k_1}$ means that $s(\alpha_q) \leqslant 8C2^{2w(f_{k_1})} < 8C2^{2w(f_k)}$, in other words $q \notin N_k$ and hence $k \in D_q$. We now show that the D_q's are pairwise disjoint. Let $q_1 < q_2$ be in A and assume that $k \in D_{q_1} \cap D_{q_2}$. By the fact that $\operatorname{ran}\alpha_{q_1} \cap \operatorname{ran} x_k \neq \varnothing$ and Definition 3.10 we obtain

$$8C2^{w(f_k)} \leqslant \min\operatorname{supp} x_k \leqslant \max\operatorname{supp}\alpha_{q_1}$$

and since the sequence $(\alpha_q)_{q=1}^{d}$ is very fast growing, we obtain that

$$s(\alpha_{q_2}) > \max\operatorname{supp}\alpha_{q_1} \geqslant 8C2^{w(f_k)}$$

which means that $q_2 \in N_k$, which contradicts $k \in D_{q_2}$.

If we set $n_q = \min D_q$, then the n_q's are strictly increasing and since the D_q's are intervals, by (13)

$$\left|\alpha_q\left(\sum_{k\in D_q} x_k\right)\right| \leqslant \frac{23C}{s(\alpha_q)} + \frac{2C}{2^{w(f_{n_q})}} \tag{17}$$

for all $q \in A$. Combining (15), (16) and (17):

$$\left|\sum_{q=1}^{d}\alpha_q\left(\sum_{k\in D} x_k\right)\right| \leqslant \sum_{q\in A}\frac{23C}{s(\alpha_q)} + \sum_{q\in A}\frac{2C}{2^{w(f_{n_q})}} + \frac{4}{2^{w(f_n)}} < 47C$$

where we used that, as implied by the definition of very fast growing sequences, $\sum_q(1/s(\alpha_q)) < 2$, that the $w(f_{n_q})$'s are strictly increasing elements of L and that $\min L \geqslant 8$. Finally, we conclude that $|f(\sum_{k\in D})x_k| < 47C/2^j$. □

6 Non-trivial weakly Cauchy sequences and the HI property of the space $\mathfrak{X}_\mathcal{T}$

In this section we prove that in every block subspace of $\mathfrak{X}_\mathcal{T}$ one can find a seminormalized block sequence $(x_k)_k$ and a sequence of weighted functionals $(f_k)_k$ so that $\{(f_k, x_k)\}_k$ forms a maximal chain in $\mathcal{T}$. We conclude that $\mathfrak{X}_\mathcal{T}$ is hereditarily indecomposable. We also show that in the case $\mathcal{T}$ is well founded, then the space $\mathfrak{X}_\mathcal{T}$ reflexive. On the other hand, if $\mathcal{T} = \mathcal{U}$, then we show that $\mathfrak{X}_\mathcal{U}$ contains no reflexive subspace.

Lemma 6.1 *Let $(f_k)_k$ be an infinite sequence of non-averages in $W_\mathcal{T}$ so that for each $n \in \mathbb{N}$ the set $\{k : f_k$ is a weighted functional with $w(f_k) = n\}$ is finite. Then there exists a subsequence of $(f_k)_k$, again denoted by $(f_k)_k$, so that for every natural numbers $k_1 < \cdots < k_d$ and alternating signs $(\varepsilon_i)_{i=1}^d$ in $\{-1, 1\}$, the functional $\alpha_0 = (1/d)\sum_{i=1}^d \varepsilon_i f_{k_i}$ is an α_c-average in $W_\mathcal{T}$.*

Proof By passing to a subsequence, either all f_k's are weighted functionals, or they are all of the form $f_k = \varepsilon_k e^*_{i_k}$ where $\varepsilon_k \in \{-1, 1\}$. If the second case holds, the result follows easily.

Assume now that the f_k's are all weighted functionals. Then $\lim_k w(f_k) = \infty$ and so we may pass to a subsequence so that the sequence $\phi(w(f_k))$ is strictly increasing. By Ramsey's theorem [23, Theorem A], by passing to a further subsequence, the $\phi(w(f_k))$'s are either all pairwise incomparable, or all pairwise comparable, in the sense of Definition 2.5. If the first one holds, then for any natural numbers $d \leqslant n$, $k_1 < \cdots < k_d$ and for any choice of signs ε_j, $j = 1, \ldots, d$ the sequence of functionals $(\varepsilon_j f_{k_j})_{j=1}^d$ is incomparable, which easily implies the desired result.

We assume now that the $\phi(w(f_k))$'s are pairwise comparable in the sense of Definition 2.5. Observe first that for at most one $k \in \mathbb{N}$ $\phi(w(f_k)) \in L_0$ and hence we may assume that $\phi(w(f_k)) \in L_1$ for all $k \in \mathbb{N}$. This further implies that $(\sigma^{-1}(\phi(w(f_k))))_k$ is a chain in $\mathcal{T}$ and hence, there exist sequences $(h_i)_i$ in W_α and $(y_i)_i$ in $c_{00}(\mathbb{N}, \mathbb{Q})$, so that $\{(h_i, y_i)\}_{i=1}^n$ is in $\mathcal{T}$ for all $n \in \mathbb{N}$ and there is a strictly increasing sequence of natural numbers $(m_k)_k$, so that $w(h_{m_k}) = \phi(w(f_k))$ for all $k \in \mathbb{N}$. By passing once more to a subsequence, we may assume that either $|f_k(y_{m_k})| > 10$ for all $k \in \mathbb{N}$, or $|f_k(y_{m_k})| \leqslant 10$ for all $k \in \mathbb{N}$. If the first one holds, then for any natural numbers $d \leqslant n$, $k_1 < \cdots < k_d$ and for any choice of signs ε_j, $j = 1, \dots, d$ the sequence of functionals $(\varepsilon_j f_{k_j})_{j=1}^d$ is irrelevant, which implies the desired result. Otherwise, we pass to an even further subsequence so that for every natural numbers $k < n$ we have that $|f_k(y_{m_k}) - f_n(y_{m_n})| < 1/2^k$. This means that for any natural numbers $d \leqslant n$, $k_1 < \cdots < k_d$ sequence of functionals $(f_{k_j})_{j=1}^d$ is comparable and therefore for alternating signs $(\varepsilon_j)_{j=1}^d$, $\alpha_0 = (1/n)\sum_{j=1}^d \varepsilon_j f_{k_j}$ is a $\mathcal{CO}$-average. □

If we assume that the tree $\mathcal{T}$ is well founded, then there does not exist a strictly increasing sequence of natural numbers which are pairwise comparable in the sense of Definition 2.5. In this case, the proof of Lemma 6.1 yields the following.

Lemma 6.2 *Assume that the tree $\mathcal{T}$ is well founded and let $(f_k)_k$ be an infinite sequence of non-averages in $W_{\mathcal{T}}$ so that for each $n \in \mathbb{N}$ the set $\{k : f_k$ is a weighted functional with $w(f_k) = n\}$ is finite. Then there exists a subsequence of $(f_k)_k$, again denoted by $(f_k)_k$, so that for every natural numbers $k_1 < \cdots < k_d$ the functional $\alpha_0 = (1/d)\sum_{i=1}^d f_{k_i}$ is an α_c-average in $W_{\mathcal{T}}$.*

Lemma 6.3 *Let $(x_k)_k$ be a block sequence in $\mathfrak{X}_{\mathcal{T}}$ and assume that there is a constant $C > 0$ so that $\|\sum_{k=1}^{\ell} x_k\| \leqslant C$ for all $\ell \in \mathbb{N}$. Then $\alpha((x_k)_k) = 0$.*

Proof Assume that this is not the case. Then there exist $\varepsilon > 0$, $m \in \mathbb{N}$, a very fast growing sequence of α_c-averages $(\alpha_q)_q$, a sequence of successive subsets $(F_n)_n$ of $\mathbb{N}$, with $(\alpha_q)_{q\in F_n}$ $\mathcal{S}_m$-admissible for all $n \in \mathbb{N}$ and a subsequence $(x_{k_n})_n$ of $(x_k)_k$ so that for all $n \in \mathbb{N}$ $\sum_{q\in F_n} \alpha_q(x_{k_n}) > \varepsilon$. We may also assume that $\operatorname{ran}\alpha_q \subset \operatorname{ran} x_{k_n}$ for all $q \in F_n$ and $n \in \mathbb{N}$, hence $\sum_{q\in F_n} \alpha_q(x_{k'}) = 0$ for $k' \neq k_n$. Choose $n_0 > 2^{m+1}C/\varepsilon$ and observe that the functional

$$f = \frac{1}{2^{m+1}} \sum_{n=n_0}^{2n_0-1} \sum_{q\in F_n} \alpha_q$$

is a weighted functional in $W_{\mathcal{T}}$ of weight $w(f) = m + 1$. We conclude

$$C \geqslant \left\| \sum_{k=1}^{k_{2n_0-1}} x_k \right\| \geqslant f\left(\sum_{k=1}^{k_{2n_0-1}} x_k\right) > \frac{1}{2^{m+1}} n_0\varepsilon > C$$

which is absurd. □

Lemma 6.4 *Let $(x_k)_k$ be a seminormalized block sequence in $\mathfrak{X}_{\mathcal{T}}$ with $\alpha((x_k)_k) = 0$. Let also $(f_k)_k$ be a sequence of non-zero functionals in $W_{\mathcal{T}}$, so that $f_k(x_k) \geqslant (3/4)\|x_k\|$ for all $k \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ the set $\{k : f_k$ is a weighted functional with $w(f_k) = n\}$ is finite.*

Proof Assume that, passing to a subsequence, there is $m \in \mathbb{N}$ so that f_k is a weighted functional with $w(f_k) = m$ for all $k \in \mathbb{N}$. Proposition 4.4 yields that, passing to a further subsequence, there is $k_0 \in \mathbb{N}$ so that $f_k(x_k) \leqslant (1/2^m)(8/9)\|x_k\| < (3/4)\|x_k\|$ for all $k \geqslant k_0$, which is absurd. □

Lemmas 6.3 and 6.4 immediately yield the following.

Lemma 6.5 *Let $(x_k)_k$ be a seminormalized block sequence in $\mathfrak{X}_{\mathcal{T}}$ and assume that there is a constant $C > 0$ so that $\|\sum_{k=1}^{\ell} x_k\| \leqslant C$ for all $\ell \in \mathbb{N}$. Let also $(f_k)_k$ be a sequence of non-averages in $W_{\mathcal{T}}$, so that $f_k(x_k) > (3/4)\|x_k\|$ for all $k \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ the set $\{k :\ f_k$ is a weighted functional with $w(f_k) = n\}$ is finite.*

We obtain the first result that depends on the properties of the tree $\mathcal{T}$.

Proposition 6.6 *If the tree $\mathcal{T}$ is well founded, then the space $\mathfrak{X}_{\mathcal{T}}$ is reflexive.*

Proof We will show that the basis of $\mathfrak{X}_{\mathcal{T}}$ is boundedly complete, which in conjunction with Proposition 3.8 and James' well known theorem [17, Theorem 1] yield the desired result. Let us assume that this is not the case, i.e. there is a seminormalized block sequence $(x_k)_k$ and a constant C with $\|\sum_{k=1}^{m} x_k\| \leqslant C$ for all $m \in \mathbb{N}$. For each $k \in \mathbb{N}$ choose a functional in $W_{\mathcal{T}}$, which is not an average, so that $\operatorname{ran} f_k \subset \operatorname{ran} x_k$ and $f_k(x_k) > (3/4)\|x_k\|$. Lemmas 6.2 and 6.5 yield that there is an infinite subset of the natural numbers M, so that for every finite subset F of M the functional $\alpha_F = (1/\#F)\sum_{k\in F} f_k$ is an α_c-average of $W_{\mathcal{T}}$. Note that for $m \geqslant \max F$ we have

$$\alpha_F\left(\sum_{k=1}^{m} x_k\right) = \frac{1}{\#F}\sum_{k\in F} f_k(x_k) > \frac{3}{4}\inf_k \|x_k\|$$

Choose a natural number $d > 6C/(4\inf\|x_k\|)$ and $F_1 < \cdots < F_d$ so that the sequence $(\alpha_{F_q})_{q=1}^{d}$ is $\mathcal{S}_1$-admissible and very fast growing. Then $f = (1/2)\sum_{q=1}^{d} \alpha_{F_q}$ is in $W_{\mathcal{T}}$ and if $m = \max F_d$ we obtain $f(\sum_{k=1}^{m} x_k) > C$, which is absurd. □

The next result is one of the main features of saturation under constraints and it plays an important role in deducing the properties of the space.

Proposition 6.7 *Every block subspace X of $\mathfrak{X}_{\mathcal{T}}$ contains a block sequence generating an ℓ_1 spreading model, as well as a block sequence generating a c_0 spreading model.*

Proof By Corollary 4.5, it suffices to find, given a block sequence generating an ℓ_1 spreading model, a further block sequence with α-index zero and, given a block sequence generating a c_0 spreading model, a further block sequence with α-index positive. Assume that $(x_k)_k$ is a block sequence generating an ℓ_1 spreading model, i.e. $\alpha((x_k)_k) > 0$. By Proposition 4.3 we may find $C \geqslant 1$, $\theta > 0$ and a further block sequence $(y_k)_k$ so that each y_k is a (C, θ, n_k)-vector, with $(n_k)_k$ strictly increasing. Proposition 5.2 yields the desired result. Assume now that $(x_k)_k$ is a normalized block generating a c_0 spreading model, i.e. $\alpha((x_k)_k) = 0$. Choose a sequence $(f_k)_k$ of non-averages in $W_{\mathcal{T}}$ so that for each k, $\operatorname{ran} f_k \subset \operatorname{ran} x_k$ and $f_k(x_k) > 3/4$. By Lemmas 6.1 and 6.4 we may pass to a further subsequence so that for every $k_1 < \cdots < k_d$

and alternating signs $(\varepsilon_i)_{i=1}^d$, the functional $(1/d)\sum_{i=1}^d \varepsilon_i f_{k_i}$ is an α_c-average of $W_{\mathcal{T}}$. Choose a sequence $(F_n)_n$ of successive subsets of $\mathbb{N}$ with $\#F_n \leqslant \min F_n$ for all $n \in \mathbb{N}$ and $\lim_n \#F_n = \infty$. Also choose sequences of alternating signs $(\varepsilon_i)_{i\in F_n}$ and set $y_n = \sum_{i\in F_n} \varepsilon_i x_i$, $\alpha_n = (1/\#F_n)\sum_{\varepsilon_i} f_i$ for all $n \in \mathbb{N}$. Since $(x_k)_k$ generates a c_0 spreading model we conclude that $(y_n)_n$ is bounded. Furthermore for each n, α_n is an α_c-average of size $\#F_n$ so that $\alpha_n(y_n) > 3/4$. It easily follows that $\alpha((y_n)_n) > 0$. ◻

Lemma 6.8 *Let $(x_k)_k$ be a block sequence in the unit ball of $\mathfrak{X}_{\mathcal{T}}$ generating a c_0 spreading model and $(f_k)_k$ be a sequence of functionals in $W_{\mathcal{T}}$ so that the following are satisfied:*

(a) *f_k is not an α_c-average,* $\operatorname{ran} f_k \subset \operatorname{ran} x_k$ *for all $k \in \mathbb{N}$ and*
(b) *there is a $\theta > 0$, so that $(3/4)\|x_k\| < f_k(x_k) = \theta$ for all $k \in \mathbb{N}$.*

Then for every $n \in \mathbb{N}$ there are successive finite subsets of the natural numbers $(F_k)_{k=1}^m$, sequences of signs $(\varepsilon_i)_{i\in F_k}$, $k = 1, \dots, m$ and a sequence of non-negative real numbers $(c_k)_{k=1}^m$ so that the following are satisfied:

(i) *the vector $x = 2^n \sum_{k=1}^m c_k(\sum_{i\in F_k} \varepsilon_i x_i)$ is a $(9/8, \theta, n)$-exact vector,*
(ii) *the functional $\alpha_k = (1/\#F_k)\sum_{i\in F_k} \varepsilon_i f_i$ is an α_c-average of $W_{\mathcal{T}}$ for $k = 1, \dots, m$ and*
(iii) *the sequence $(\alpha_k)_{k=1}^d$ is $\mathcal{S}_n$-admissible and very fast growing. In particular, $f = (1/2^n)\sum_{k=1}^m \alpha_k$ is a weighted functional in $W_{\mathcal{T}}$ with,* $\operatorname{ran} f \subset \operatorname{ran} x$*, $w(f) = n$ and $f(x) = \theta$.*

Proof By Corollary 4.5 $\alpha((x_k)_k) = 0$ and by Lemma 6.4 we obtain that, passing to a subsequence, $(f_k)_k$ satisfies the conclusion of Lemma 6.1, i.e.

(c) for every natural numbers $k_1 < \cdots < k_d$ and alternating signs $(\varepsilon_i)_{i=1}^d$, the functional $\alpha_0 = (1/d)\sum_{i=1}^d \varepsilon_i f_{k_i}$ is an α_c-average of $W_{\mathcal{T}}$.

Corollary 4.5 yields that $\alpha((x_k)_k) = 0$ and so we pass once more to a subsequence and find a strictly increasing sequence of natural numbers $(j_k)_k$, so that the conclusion of Proposition 4.4 holds, i.e. for every natural numbers $d \leqslant k_1 < \cdots < k_d$, scalars $(\lambda_i)_{i=1}^d$ we have

$$\left\|\sum_{i=1}^d \lambda_i x_{k_i}\right\| \leqslant (9/8) \max_{1\leqslant i\leqslant d} |\lambda_i| \tag{18}$$

and for every weighted functional in $W_{\mathcal{T}}$ f with $w(f) = j < j_d$, we have

$$\left|f\left(\sum_{i=1}^d \lambda_i x_{k_i}\right)\right| < \frac{9/8}{2^j} \max_{1\leqslant i\leqslant d} |\lambda_i|. \tag{19}$$

Inductively choose a sequence of successive intervals of $\mathbb{N}$ $(I_q)_q$ so that the following are satisfied:

(d) $\min I_q \leqslant \#I_q$ for all $q \in \mathbb{N}$,
(e) $\#I_{q+1} > 2^{\max \operatorname{supp} x_{\max I_q}}$ for all $q \in \mathbb{N}$ and
(f) $1/2^{j_{\min I_{q+1}}} \max \operatorname{supp} x_{\max I_q} < 1/2^{j_{\max I_q}}$ for all $q \in \mathbb{N}$.

For each q choose alternating signs $(\varepsilon_i)_{i\in I_q}$ and define

$$w_q = \sum_{i\in I_q} \varepsilon_i x_i \quad \text{and} \quad \alpha_q = \frac{1}{\#I_q} \sum_{i\in I_q} \varepsilon_i f_i .$$

Then (18) and (d) yield $\|w_q\| \leqslant 9/8$ for all $q \in \mathbb{N}$ while by (c) and (e) $(\alpha_q)_q$ is a very fast growing sequence of α_c-averages of $W_{\mathcal{T}}$. Since ran $\alpha_q \subset$ ran w_q for all q we easily obtain the following:

(g) whenever $F \subset \mathbb{N}$ is such that $(w_q)_{q\in F}$ is $\mathcal{S}_n$-admissible, then $f = (1/2^n)\sum_{q\in F}\alpha_q$ is in $W_{\mathcal{T}}$ and hence, if $(\lambda_q)_{q\in F}$ are non-negative scalars with $\sum_{q\in F}\lambda_q = 1$, then $f\left(2^n \sum_{q\in F}\lambda_q w_q\right) = \theta$.

Furthermore, by (19) and (f), the sequence $(w_q)_q$ is $(9/8, (j'_q)_q)$-RIS, where $j'_q = j_{\min I_q}$ for all $q \in \mathbb{N}$.

By Proposition 3.3 we may choose $q_1 < \cdots < q_m$ and non-negative real numbers $(c_k)_{k=1}^m$ so that the vector $x = 2^n \sum_{k=1}^m c_k w_{q_k}$ satisfies all assumptions of the definition of a $(9/8, \theta, n)$-exact vector (see Definition 3.10). Therefore, the $(I_{q_k})_{k=1}^m$, $(\varepsilon_i)_{i\in I_{q_k}}$ for $k = 1, \ldots, m$ and $(c_k)_{k=1}^m$ satisfy the desired conclusion. □

Remark 6.9 Let $(x_k)_k, (f_k)_k$ satisfy the assumptions of Lemma 6.8. Assume moreover that $(g_k)_k$ is a sequence of successive functionals in $W_{\mathcal{T}}$ such that for each $n \in \mathbb{N}$, the set $\{k : \ g_k$ is a weighted functional with $w(g_k) = n\}$ is finite. The same method of proof, and an argument involving Proposition 3.3, Remark 3.2 and the spreading property of the Schreier families, yields that we may find $(F_k)_{k=1}^m$, $(\varepsilon_i)_{i\in F_k}$, $k = 1, \ldots, m$ and $(c_k)_{k=1}^m$ satisfying the conclusion of Proposition 6.8 so that moreover the functional $g = (1/2^n)\sum_{k=1}^m \left((1/(\#F_k))\sum_{i\in F_k}\varepsilon_i g_i\right)$ is a weighted functional of weight $w(g) = n$ in $W_{\mathcal{T}}$.

Lemma 6.10 *Let X be a block subspace of $\mathfrak{X}_{\mathcal{T}}$ and $n \in \mathbb{N}$. Then there exists a $(9/8, 8/9, n)$-exact pair (f, x) so that x is in X.*

Proof By Proposition 6.7 there exists a normalized block sequence $(x_k)_k$ in X generating a c_0 spreading model. Choose a sequence of functionals f_k in $W_{\mathcal{T}}$ so that for each k, f_k is not an average, $f_k(x)_k > 8/9$ and ran $f_k \subset$ ran x_k. Define $x'_k = (8/(9 f_k(x_k)))x_k$ and observe that the assumptions of Lemma 6.8 are satisfied for $(x'_k)_k$, $(f_k)_k$ and $\theta = 8/9$. The first and third assertions of the conclusion of that proposition yield the desired result. □

Lemma 6.11 *Let X and Y be block subspaces of $\mathfrak{X}_{\mathcal{T}}$, both generated by vectors with rational coefficients. Then there exists an initial interval E of $\mathbb{N}$ (finite or infinite) and a sequence of exact pairs $\{(f_k, x_k)\}_{k\in E}$ so that the following are satisfied:*

(i) *for k odd x_k is in X while for k even x_k is in Y,*
(ii) *$\{(f_k, x_k)\}_{k=1}^m$ is a $(9/8, 8/9)$-dependent sequence for all $m \in E$ and*
(iii) *$\{\{(f_k, x_k)\}_{k=1}^m : \ m \in E\}$ is a maximal chain of $\mathcal{T}$.*

Proof Using an inductive argument and Lemma 6.10, we choose a sequence of $(9/8, 8/9, n_k)$-exact pairs $\{(f_k, x_k)\}_{k=1}^{\infty}$ so that (i) of the conclusion holds and $\{(f_k, x_k)\}_{k=1}^{m}$ is in $\mathcal{U}$ for all $m \in \mathbb{N}$. By property (i) of $\mathcal{T}$ from Sect. 2.4 we obtain that $\{(f_1, x_1)\}$ is in $\mathcal{T}$. If for all $m \in \mathbb{N}$ we have that $\{(f_k, x_k)\}_{k=1}^{m}$ is in $\mathcal{T}$, then we obtain that for $E = \mathbb{N}$ the conclusion is satisfied. Otherwise, set $m_0 = \max\{m \in \mathbb{N} : \{(f_k, x_k)\}_{k=1}^{m} \in \mathcal{T}\}$ and by property (ii) of $\mathcal{T}$ from Sect. 2.4 we obtain that, setting $E = \{1, \dots, m_0\}$, the conclusion holds. □

Recall that $\mathcal{T}$ is a subtree of the universal tree $\mathcal{U}$ associated with the coding function σ. If we take $\mathcal{T}$ to be all of $\mathcal{U}$, we obtain the result below.

Theorem 6.12 *The space $\mathfrak{X}_{\mathcal{U}}$ contains no reflexive subspace.*

Proof It is enough to show that any block sequence with rational coefficients is not boundedly complete. Indeed, let $(z_k)_k$ be such a block sequence and apply Lemma 6.11, for $X = Y = [(z_k)_k]$ to find a sequence of exact pairs $\{(f_k, x_k)\}_{k\in E}$ satisfying the conclusion of that lemma. Recall that every maximal chain in $\mathcal{U}$ is infinite and hence $E = \mathbb{N}$. Finally, $\|x_k\| \geqslant 8/9$ for all $k \in \mathbb{N}$ while by Proposition 5.9 we have that $\|\sum_{k=1}^{n} x_k\| \leqslant 27$ for all $n \in \mathbb{N}$. □

Theorem 6.13 *The space $\mathfrak{X}_{\mathcal{T}}$ is hereditarily indecomposable.*

Proof We will show that for every block subspaces X and Y of $\mathfrak{X}_{\mathcal{T}}$, both generated by vectors with rational coefficients, and for every $n \in \mathbb{N}$ there exists $x \in X$ and $y \in Y$ so that $\|x + y\| \leqslant 53$ and $\|x - y\| \geqslant (4/9)n$. by passing to further block subspaces, we may assume the X and Y are generated by block sequences $(z_k)_k$ and $(w_k)_k$ respectively, so that

(i) $\min \operatorname{supp} z_1 \geqslant n$,
(ii) $\min \operatorname{supp} z_k > 2^{\max \operatorname{supp} w_{k-1}}$ and $\min \operatorname{supp} w_k > 2^{\max \operatorname{supp} z_{k-1}}$ for all s$k \in \mathbb{N}$.

Apply Lemma 6.11 to find sequences $(x_k)_{k\in E}$ and $(f_k)_{k\in E}$ satisfying the conclusion of that Lemma. The maximality property of that conclusion in conjunction with property (iii) of $\mathcal{T}$ from Sect. 2.4 yield that there is an initial interval G of E so that the set $\{\min \operatorname{supp} f_k : k \in G\}$ is a maximal $\mathcal{S}_2$-set. By the definition of $\mathcal{S}_2$ choose a partition of G into successive intervals $G_1, \dots, G_d$ so that:

(a) $\{\min \operatorname{supp} f_{\min G_q} : q = 1, \dots, d\}$ is an $\mathcal{S}_1$-set and
(b) $\{\min \operatorname{supp} f_k : k \in G_q\}$ is an $\mathcal{S}_1$-set for $q = 1, \dots, d$.

Then (i) implies that $n \leqslant d$ while the maximality of $\{\min \operatorname{supp} f_k : k \in G\}$ implies that each $\{\min \operatorname{supp} f_k : k \in G_q\}$ is a maximal $\mathcal{S}_1$-set, i.e. $\#G_q = \min \operatorname{supp} f_{\min G_q}$, for $q = 1, \dots, d$.

Define $G_o = \{k \in G : k \text{ odd}\}$ and $G_e = \{k \in G : k \text{ even}\}$. Set $x = \sum_{k\in G_o} x_k$ and $y = \sum_{k\in G_e} x_k$. Then $x \in X$, $y \in Y$ and $\|x + y\| \leqslant 47(9/8) < 53$ by Proposition 5.9.

The sequence $(f_k)_{k\in G_q}$ can easily seen to be comparable and hence, the functional $\alpha_q = (1/\#G_q)\sum_{k\in G_q}(-1)^k f_k$ is an α_c-average for each $q = 1, \dots, n$ with $\alpha_q\left(\sum_{k\in G_q}(-1)^k x_k\right) = 8/9$. Also the sequence $(\alpha_q)_{q=1}^{d}$ is $\mathcal{S}_1$-admissible by (a).

Also by (ii), $s(\alpha_{q+1}) = \min \operatorname{supp} f_{\min G_{q+1}} > 2^{\max \operatorname{supp} \alpha_q}$ and hence the sequence $(\alpha_q)_{q=1}^d$ is very fast growing. We conclude that $f = (1/2)\sum_{q=1}^d \alpha_q$ is in $W_{\mathcal{T}}$ and $f(x-y) = (1/2)\sum_{q=1}^d \alpha_q \left(\sum_{k\in G_q}(-1)^k x_k\right) = (1/2)d(8/9) \geqslant (4/9)n$ which yields the desired result. □

7 The spreading models of non-trivial weakly Cauchy sequences in $\mathfrak{X}_{\mathcal{T}}$

In the case $\mathcal{T}$ is well founded, i.e. the space $\mathfrak{X}_{\mathcal{T}}$ is reflexive, Propositions 4.5 and 6.7 clarify all types of spreading models admitted by Schauder basic sequences in subspaces of $\mathfrak{X}_{\mathcal{T}}$. In the case of the space $\mathfrak{X}_{\mathcal{U}}$ non-trivial weakly Cauchy sequences exist in every subspace of the space and this section is devoted to determining what types of spreading models these sequences admit. We start the section by presenting some simple general facts about spreading sequences, i.e. sequences which are equivalent to their subsequences. Both Lemma 7.1 as well as Proposition 7.2 are showed using classical techniques and we omit their proofs.

Lemma 7.1 *Let $(e_k)_k$ be a conditional and spreading Schauder basic sequence so that $(e_{2k-1} - e_{2k})_k$ is equivalent to the unit vector basis of c_0. Then $(e_k)_k$ is equivalent to the summing basis of c_0.*

Proposition 7.2 *Let X be a Banach space and $(x_k)_k$, $(y_k)_k$ be Schauder basic sequences in X. If $(x_k)_k$ admits an ℓ_1 spreading model while $(y_k)_k$ does not, then $(x_k - y_k)_k$ admits an ℓ_1 spreading model.*

as well, which is the desired result. Indeed, let $(c_n)_{n=1}^m$ be a sequence

Proposition 7.3 *Let $(\lambda_i)_i$ be a sequence of scalars so that if $(e_i)_i$ is the basis of $\mathfrak{X}_{\mathcal{T}}$ and $x_k = \sum_{i=1}^k \lambda_i e_i$ for all $k \in \mathbb{N}$, then $(x_k)_k$ is bounded and non-convergent in the norm topology. Then $(x_k)_k$ admits only the summing basis of c_0 as a spreading model.*

Proof Pass to a subsequence of $(x_k)_k$ that generates a spreading model $(z_k)_k$. The fact that $(x_k)_k$ is non-trivial weakly Caushy easily implies that $(z_k)_k$ is either equivalent to the unit vector basis of ℓ_1, or a conditional spreading sequence. Then, if $y_k = x_{2k-1} - x_{2k}$ and $u_k = z_{2k-1} - z_{2k}$ for all $k \in \mathbb{N}$, the sequence $(y_k)_k$ generates $(u_k)_k$ as a spreading model. Lemma 6.3 implies that $\alpha((y_k)_k) = 0$ and hence by Proposition 4.4, $(u_k)_k$ is equivalent to the unit vector basis of c_0. Therefore, $(z_k)_k$ is conditional and spreading and by Lemma 7.1 we deduce the desired result. □

Remark 7.4 Note that the summing basis norm is the minimum conditional spreading norm, in terms of domination. An easy argument yields the following: if $(x_k)_k$ is a sequence generating the summing basis of c_0 as a spreading model, then every convex block sequence of $(x_k)_k$ admits only the summing basis of c_0 as a spreading model as well.

The next will be useful in the sequel.

Lemma 7.5 *Let $(x_k)_k$ be a non-trivial weakly Cauchy sequence in $\mathfrak{X}_{\mathcal{T}}$. Then there is a convex block sequence $(y_k)_k$ of $(x_k)_k$ that generates the summing basis of c_0 as a spreading model.*

Proof Let x^{**} be the w^*-limit of $(x_k)_k$ and $y_k = \sum_{i=1}^{k} x^{**}(e_i^*)e_i$. Then by Proposition 3.8 $(y_k)_k$, w^*-converges to x^{**}. By Lemma 7.3, passing to a subsequence, $(y_k)_k$ generates the summing basis of c_0 as a spreading model. As $(x_k - y_k)_k$ is weakly null, by Mazur's theorem there is a convex block sequence of $(x_k)_k$ that is equivalent to a convex block sequence of $(y_k)_k$. By Remark 7.4 we deduce the desired result. □

Proposition 7.6 *Every non-trivial weakly Cauchy sequence in $\mathfrak{X}_{\mathcal{T}}$ admits a spreading model which is either equivalent to the summing basis of c_0 or equivalent to the unit vector basis of ℓ_1. If moreover $\mathcal{T} = \mathcal{U}$, then every infinite dimensional subspace of $\mathfrak{X}_{\mathcal{U}}$ contains non-trivial weakly Cauchy sequences admitting both of these types of spreading models.*

Proof Let $(x_k)_k$ be a non-trivial weakly Cauchy sequence in $\mathfrak{X}_{\mathcal{T}}$ and x^{**} be its w^*-limit. If for $k \in \mathbb{N}$ we set $y_k = \sum_{i=1}^{k} x^{**}(e_i^*)e_i$, By proposition 3.8 we obtain that $(y_k)_k$ w^*-converges to x^{**} and hence, setting $z_k = y_k - x_k$, the sequence $(z_k)_k$ is weakly null. By Proposition 7.3 $(y_k)_k$ admits only the summing basis of c_0 as a spreading model, while $(z_k)_k$ is either norm null, or it is not. If it is not norm it follows from Proposition 4.5 that $(z_k)_k$ either admits only the unit vector basis of c_0 as a spreading model, or it admits the unit vector basis of ℓ_1 as a spreading model. If the first one holds, we conclude that any spreading model admitted by $(x_k)_k$ must be equivalent to the unit vector basis of c_0 and if the second one holds, Proposition 7.2 yields that $(x_k)_k$ admits an ℓ_1 spreading model.

The second assertion is proved as follows: by Theorem 6.12, and Proposition 7.3 we obtain that every subspace of $\mathfrak{X}_{\mathcal{U}}$ admits the summing basis of c_0 as a spreading model. Combining this with Propositions 6.7 and 7.2 we deduce that there is a non-trivial weakly Cauchy sequence in every subspace generating an ℓ_1 spreading model. □

Remark 7.7 We comment that using the α-index it can be shown that every non-trivial weakly Cauchy sequence in $\mathfrak{X}_{\mathcal{T}}$ admitting an ℓ_1 spreading model, has a subsequence that generates an ℓ_1^n spreading model with lower constant $\theta/2^n$, for all $n \in \mathbb{N}$ and some $\theta > 0$.

8 Operators on the space $\mathfrak{X}_{\mathcal{T}}$

In this final section we prove the properties of the operators defined on subspaces of $\mathfrak{X}_{\mathcal{T}}$. We characterize strictly singular operators with respect to their action on sequences generating certain types of spreading models. We conclude that the composition of any pair of singular operators is a compact one. This ought to be compared to [8, Theorem 5.19 and Remark 5.20]. We also show that all operators defined on block subspaces of $\mathfrak{X}_{\mathcal{T}}$ have non-trivial closed invariant subspaces and that operators defined on $\mathfrak{X}_{\mathcal{U}}$ are strictly singular if and only if they are weakly compact.

Lemma 8.1 *Let x, y be non-zero vectors in $\mathfrak{X}_{\mathcal{T}}$. Then there exist non-averages f, g in $W_{\mathcal{T}}$ so that the following hold:*

(i) $\operatorname{ran} f \subset \operatorname{ran} x$ *and* $\operatorname{ran} g \subset \operatorname{ran} y$,
(ii) $f(x) > (8/9)\|x\|$ *and* $g(y) > (8/9)\|y\|$,
(iii) $\left|g\left(\frac{8}{9f(x)}x\right)\right| \leqslant 8/9$.

Proof Choose a non-average g in $W_{\mathcal{T}}$ with $g(y) > (8/9)\|y\|$. If $|g(x)| > (8/9)\|x\|$ define $f = \operatorname{sgn}(g(x))g|_{\operatorname{ran} x}$ and observe that f, g satisfy the conclusion. Otherwise $g(x) \leqslant (8/9)\|x\|$ and choose any non-average f in $W_{\mathcal{T}}$ with $f(x) > (8/9)\|x\|$ and $\operatorname{ran} f \subset \operatorname{ran} x$. A simple calculation yields that f, g satisfy the conclusion. □

Lemma 8.2 *Let (f, x) be an $(9/8, 8/9, n)$-exact pair in $\mathfrak{X}_{\mathcal{T}}$ and let also ρ in $[-8/9, 8/9]$. Then there is a weighted functional g in $W_{\mathcal{T}}$ of weight $w(g) = n$, so that* $\operatorname{ran} g \subset \operatorname{ran} x$ *and* $|g(x) - \rho| < 1/2^{n+1}$.

Proof By Remark 3.12, we have that $\|x\|_\infty < 1/(2^{2n}36) < 1/2^{n+1}$. The fact that $f(x) = 8/9$ easily implies that there is an initial interval E of $\operatorname{ran} f$ and $\varepsilon \in \{-1, 1\}$, so that $g = \varepsilon E f$ is the desired functional. □

The following result characterizes strictly singular operators, defined on subspaces of $\mathfrak{X}_{\mathcal{T}}$, in the following manner: an operator is strictly singular if and only if it does not preserve any type of spreading model. It is worth mentioning that we could neither prove nor disprove the same result in [8]. The reason for this difference is the presence of β-averages in that paper and their absence in the present one.

Proposition 8.3 *Let X be an infinite dimensional closed subspace of $\mathfrak{X}_{\mathcal{T}}$ and $T : X \to \mathfrak{X}_{\mathcal{T}}$ be a bounded linear operator. The following assertions are equivalent.*

(i) *The operator T is strictly singular.*
(ii) *There exists a normalized weakly null sequence $(y_k)_k$ in X so that $(Ty_k)_k$ converges to zero in norm.*
(iii) *For every sequence $(x_k)_k$ in X generating a c_0 spreading model, $(Tx_k)_k$ converges to zero in norm.*
(iv) *For every sequence $(x_k)_k$ in X generating an ℓ_1 spreading model, $(Tx_k)_k$ does not admit an ℓ_1 spreading model.*

Proof That (i) implies (ii) follows from the fact that ℓ_1 does not embed into $\mathfrak{X}_{\mathcal{T}}$ and that (iv) implies (i) follows from Proposition 6.7. We shall first demonstrate that (iii) implies (iv) and then that (ii) implies (iii).

We assume that (ii) is true and towards a contradiction assume that there is a sequence in $(x_k)_k$ in X, so that both $(x_k)_k$ and $(Tx_k)_k$ generate an ℓ_1 spreading model. By taking differences, we may assume that both $(x_k)_k$ and $(Tx_k)_k$ are block sequences with α-index positive. By Proposition 4.3 we may assume that there is $\theta > 0$ so that both sequences generate an ℓ_1^n spreading model with a lower constant $\theta/2^n$ for all $n \in \mathbb{N}$. Using the same Proposition, construct a block sequence $(y_k)_k$ of $(x_k)_k$, so that each y_k is a (C, θ, n_k)-vector and $\|Ty_k\| \geqslant \theta$ for all $k \in \mathbb{N}$ with a $(n_k)_k$ a strictly increasing sequence of natural numbers. Proposition 5.2 yields that $(y_k)_k$ admits only c_0 as a spreading model, which contradicts (ii).

We shall now prove that (ii) implies (iii). Toward a contradiction assume that there is normalized weakly null sequence $(y_k)_k$ in X with $\lim_k Ty_k = 0$ in norm, as well as a sequence $(x_k)_k$ in X generating a c_0 spreading model, so that $(Tx_k)_k$ does not converge to zero in norm. By perturbing the operator T we may assume that the following are satisfied:

(A) $(y_k)_k$, $(x_k)_k$ and $(Tx_k)_k$ are all seminormalized block sequences with rational coefficients and
(B) $Ty_k = 0$ for all $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, choose f_k and g_k so that the conclusion of Lemma 8.1 is satisfied, i.e. $\operatorname{ran} f_k \subset \operatorname{ran} x_k$, $\operatorname{ran} g_k \subset \operatorname{ran} Tx_k$, $f_k(x_k) > (8/9)\|x_k\|$, $g_k(Tx_k) > (8/9)\|Tx_k\|$ and $|g_k((8/9 f_k(x_k))x_k)| \leqslant 8/9$. Hence, if for all k we set $x'_k = (8/9 f_k(x_k))x_k$ and $\theta = (8/9)^2 \inf_k \|Tx_k\| / \sup_k \|x_k\| > 0$, then for all $k \in \mathbb{N}$:

(C) $\operatorname{ran} f_k \subset \operatorname{ran} x'_k$, $\operatorname{ran} g_k \subset \operatorname{ran} Tx'_k$,
(D) $f_k(x'_k) = 8/9$, $g_k(Tx'_k) \geqslant \theta$ and
(E) $|g_k(x'_k)| \leqslant 8/9$.

We note that the boundedness of T yields that $(Tx_k)_k$ admits only c_0 as a spreading model, combining this with $g_k(Tx_k) > (8/9)\|Tx_k\|$ for all $k \in \mathbb{N}$ and Lemma 6.4 we obtain that

(F) for each $n \in \mathbb{N}$, the set of k's so that g_k is a weighted functional of weight $w(g_k) = n$ is finite.

We pass to a subsequence, so that there is ρ in $[-8/9, 8/9]$ so that

(G) $|g_k(x'_k) - \rho| < 1/2^{k+1}$ for all $k \in \mathbb{N}$.

Let now $n \in \mathbb{N}$ with $n > 162\|T\|/\theta$. We construct a $(9/8, 8/9)$-dependent sequence $\{(h_k, z_k)\}_{k=1}^m$ with the following properties:

(H) $\min\operatorname{supp} h_1 \geqslant n$ and $(h_k)_{k=1}^m$ is $\mathcal{S}_2$-admissible,
(I) There is a partition of $\mathbb{N}$ into successive intervals $(G_k)_k$ and successive subsets of the natural numbers $(F_j)_j$ as well as a sequence of signs $(\varepsilon_i)_i$ so that for k odd:

$$z_k = 2^{w(h_k)} \sum_{j \in G_k} c_j \left(\sum_{i \in F_j} \varepsilon_i x'_i \right)$$

$$h_k = \frac{1}{2^{w(h_k)}} \sum_{j \in G_k} \frac{1}{\# F_j} \sum_{i \in F_j} \varepsilon_i f_i,$$

(J) for k odd the functional

$$\phi_k = \frac{1}{2^{w(h_k)}} \sum_{j \in G_k} \frac{1}{\# F_j} \sum_{i \in F_j} \varepsilon_i g_i$$

is a weighted functional in $W_{\mathcal{T}}$ of weight $w(\phi_k) = w(h_k)$ and

(K) for k even, $\operatorname{ran}\phi_{k-1} < \operatorname{ran} z_k < \operatorname{ran}\phi_{k+1}$ and z_k is a linear combination of the $(y_k)_k$.

Note that in the construction for k odd we use Lemma 6.8 (F) and Remark 6.9. For k even we just use Lemma 6.10 while the fact that we continue this process until $(h_k)_{k=1}^m$ is $\mathcal{S}_2$-admissible follows from properties (ii) and (iii) from Sect. 2.4.

Proposition 5.9 yields $\|\sum_{k=1}^m z_k\| \leqslant 27$. We will finish the proof by showing that $\|T(\sum_{k=1}^m z_k)\| > 27\|T\|$, which is absurd.

For k even, by Lemma 8.2, we may choose ϕ_k in $W_{\mathcal{T}}$ with $\operatorname{ran}\phi_k \subset \operatorname{ran} z_k$ and $|\phi_k(z_k) - \rho| < 1/2^{w(h_k)+1} \leqslant 1/2^{k+1}$. Moreover, (G), (I) and (J) yield that for k odd, $|\phi_k(z_k) - \rho| < 1/2^{k+1}$ as well. We conclude:

(L) $|\phi_k(z_k) - \phi_{k'}(z_{k'})| < 1/2^k$ for $1 \leqslant k \leqslant k' \leqslant m$.

Since $\{(h_k, z_k)\}_{k=1}^m$ is in $\mathcal{T}$ and ϕ_k is a functional of weight $w(h_k)$ for $k = 1, \dots, m$ by (l) and (m) we conclude that the sequence $(\phi_k)_{k=1}^m$ is compatible, in the sense of Definition 2.6. Arguing identically as in the proof of Theorem 6.13, for the already fixed n we may choose a partition of $\{1, \dots, m\}$ into successive intervals $(E_q)_{q=1}^n$ so that if $\alpha_q = (1/\#E_q)\sum_{k\in E_q}(-1)^{k+1}\phi_k$, then the sequence $(\alpha_q)_{q=1}^d$ is a very fast growing and $\mathcal{S}_1$-admissible of α_c-averages of $W_{\mathcal{T}}$. Define $\psi = (1/2)\sum_{q=1}^n \alpha_q$ which is in $W_{\mathcal{T}}$. Then, by (B) and (K) $T\left(\sum_{k=1}^m z_k\right) = \sum_{k \text{ odd}} T z_k$. By (D), (I) and (J) we obtain:

$$\left\| T\left(\sum_{k=1}^m z_k\right)\right\| = \left\|\sum_{k \text{ odd}} T z_k\right\| \geqslant \psi\left(\sum_{k \text{ odd}} T z_k\right)$$
$$= \frac{1}{2}\sum_{q=1}^n \frac{1}{\#E_q}\sum_{\text{odd } k\in E_q} \phi_k(T z_k) \geqslant \frac{\theta}{2}\frac{n}{3} > 27\|T\|.$$

□

We remind that in [8, Theorem 5.19] it is proved that the composition of any triple of strictly singular operators, defined on a subspace of $\mathfrak{X}_{\text{ISP}}$, is a compact one. We were unable to determine whether that result is optimal or if it could be stated for couples of strictly singular operators. As we commented before Proposition 8.3, the construction of the space $\mathfrak{X}_{\text{ISP}}$ form [8] uses β-averages while the present one does not. A direct consequence of this difference is that in the case of the space $\mathfrak{X}_{\mathcal{T}}$ we can prove the following.

Theorem 8.4 *Let X be a closed subspace of $\mathfrak{X}_{\mathcal{T}}$ and $S, T : X \to X$ be strictly singular operators. Then the composition TS is a compact operator.*

Proof Since ℓ_1 does not embed into $\mathfrak{X}_{\mathcal{T}}$, it suffices to show that TS maps weakly null sequences to norm null ones and $(x_k)_k$ be a weakly null sequence in X. If it is norm null then there is nothing more to prove. Otherwise, it either admits a c_0 or an ℓ_1 spreading model. If the first one holds, then by Proposition 8.3 $(Sx_k)_k$ has a subsequence which is norm null. If on the other hand $(x_k)_k$ admits an ℓ_1 spreading model then, passing to subsequence, $(Sx_k)_k$ is either norm null, or it generates a c_0 spreading model and hence, arguing as above, we obtain that $(TSx_k)_k$ is norm null. □

Corollary 8.5 *Let X be an infinite dimensional closed subspace of $\mathfrak{X}_{\mathcal{T}}$ and $S : X \to X$ be a non-zero strictly singular operator. Then S admits a non-trivial closed hyperinvariant subspace.*

Proof Assume first that $S^2 = 0$. Then it is straightforward to check that $\ker S$ is a non-trivial closed hyperinvariant subspace of S. Otherwise, if $S^2 \neq 0$, then Theorem 8.4 yields that S^2 is compact and non-zero. Since S commutes with its square, by [24, Theorem 2.1], it is sufficient to check that for any $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, we have $(\alpha I - S)^2 + \beta^2 I \neq 0$ (see also [16, Theorem 2]). The fact that S is strictly singular, easily implies that this condition is satisfied. □

Lemma 8.6 *Let $(x_k)_k$ be a seminormalized block sequence in $\mathfrak{X}_{\mathcal{T}}$ with $\alpha((x_k)_k) = 0$ and $X = [(x_k)_k]$. Let $T : X \to \mathfrak{X}_{\mathcal{T}}$ be a linear operator and assume that there exist $\varepsilon > 0$ and a sequence of successive non-averages $(g_k)_k$ in $W_{\mathcal{T}}$ satisfying the following:*

(i) $g_k(Tx_k) > \varepsilon$ *and* $g_k(x_k) = 0$ *for all* $k \in \mathbb{N}$ *and*
(ii) *for all* $n \in \mathbb{N}$ *the set* $\{k : g_k$ *is a weighted functional with* $w(g_k) = n\}$ *is finite.*

Then T is unbounded.

Proof Towards a contradiction we assume that T is bounded. We may assume that the x_k's have rational coefficients. Choose a sequence of non averages in $W_{\mathcal{T}}$ so that $\operatorname{ran} f_k \subset \operatorname{ran} x_k$ and $f_k(x_k) > (8/9)\|x_k\|$ for all $k \in \mathbb{N}$. For all $k \in \mathbb{N}$ define $x'_k = (8/(9 f_k(x_k)))x_k$ and set $\theta = (8\varepsilon)/(9 \sup \|x_k\|) > 0$ and observe the following:

(a) $g_k(x'_k) = 0$ for all $k \in \mathbb{N}$ and
(b) $g_k(Tx'_k) \geqslant \varepsilon$ for all $k \in \mathbb{N}$.

Let now $n \in \mathbb{N}$ with $n > 54\|T\|/\theta$. We construct a $(9/8, 8/9)$-dependent sequence $\{(h_k, z_k)\}_{k=1}^m$ so that $\min \operatorname{supp} h_1 \geqslant n$, $(h_k)_{k=1}^m$ is $\mathcal{S}_2$-admissible, there is a partition of $\mathbb{N}$ into successive intervals $(G_k)_k$ and successive subsets of the natural numbers $(F_j)_j$ as well as a sequence of signs $(\varepsilon_i)_i$ so that for $k = 1, \ldots, m$:

$$z_k = 2^{w(h_k)} \sum_{j \in G_k} c_j \left(\sum_{i \in F_j} \varepsilon_i x'_i \right)$$

$$h_k = \frac{1}{2^{w(h_k)}} \sum_{j \in G_k} \frac{1}{\# F_j} \sum_{i \in F_j} \varepsilon_i f_i,$$

and the functional

$$\phi_k = \frac{1}{2^{w(h_k)}} \sum_{j \in G_k} \frac{1}{\# F_j} \sum_{i \in F_j} \varepsilon_i g_i$$

is a weighted functional in $W_{\mathcal{T}}$ of weight $w(\phi_k) = w(h_k)$. Note that by (a)

(c) $\phi_k(z_k) = 0$ for $k = 1, \ldots, m$.

Proposition 5.9 yields that $\|\sum_{k=1}^m z_k\| \leqslant 27$. We will show that also $\|T(\sum_{k=1}^m z_k)\| > 27\|T\|$, which will complete the proof.

Since $\{(h_k, z_k)\}_{k=1}^m$ is in $\mathcal{T}$ and ϕ_k is a functional of weight $w(h_k)$ for $k = 1, \ldots, m$ by (c) we easily conclude that the sequence $((-1)^k \phi_k)_{k=1}^m$ is compatible, in the sense of Definition 2.6. Arguing in the proof of Theorem 6.13 we choose a partition of $\{1, \ldots, m\}$ into successive intervals $(E_q)_{q=1}^n$ so that if $\alpha_q = (1/\#E_q)\sum_{k\in E_q} \phi_k$, then the sequence $(\alpha_q)_{q=1}^d$ is a very fast growing and $\mathcal{S}_1$-admissible of α_c-averages of $W_{\mathcal{T}}$. An argument similar to that used in the end of the proof of Proposition 8.3 yields $\|\sum_{k=1}^m Tz_k\| > n\theta/2 > 27\|T\|$. □

Remark 8.7 If E is an interval of $\mathbb{N}$, we denote by P_E the projection onto E, associated with the Schauder basis $(e_i)_i$ of $\mathfrak{X}_{\mathcal{T}}$. It easily follows that if $(x_k)_k$, $(y_k)_k$ are block sequences in $\mathfrak{X}_{\mathcal{T}}$, then

(i) if $\alpha((x_k)_k) = 0$ and $(E_k)_k$ is a sequence of successive intervals of the natural numbers, then $\alpha((P_{E_k}x_k)_k) = 0$.
(ii) if $\alpha((x_k)_k) = 0$ and $\alpha((y_k)_k) = 0$, then $\alpha((x_k + y_k)_k) = 0$.

Lemma 8.8 *Let $(x_k)_k$ be a seminormalized block sequence in $\mathfrak{X}_{\mathcal{T}}$ and $X = [(x_k)_k]$. Let $T : X \to \mathfrak{X}_{\mathcal{T}}$ be a bounded linear operator and for each $k \in \mathbb{N}$ set $y_k = P_{\operatorname{ran} x_k} T x_k$. If the sequence $(y_k)_k$ is norm null, then T is strictly singular.*

Proof By Proposition 8.3 it suffices to find a seminormalized weakly null sequence $(u_k)_k$ in X so that $(Tu_k)_k$ is norm null. For all k define $z_k = P_{[1,\min \operatorname{ran} x_k - 1]} T x_k$ and $w_k = P_{[\max \operatorname{ran} x_k + 1, \infty)]} T x_k$. By perturbing T and passing to a subsequence, we may assume that $Tx_k = z_k + w_k$ and $z_k < x_k < w_k$ for all $k \in \mathbb{N}$. We distinguish three cases.

Case 1: $(x_k)_k$ admits a c_0 spreading model. We will show that $(Tx_k)_k$ is norm null. If this is not the case then, passing to a subsequence, either $(z_k)_k$ or $(w_k)_k$ is bounded below. We assume that the first one holds, set $\varepsilon = (3/4)\inf \|z_k\|$ and for each k choose $(g_k)_k$ with $\operatorname{ran} g_k \subset \operatorname{ran} z_k$ and $g_k(x_k) > (3/4)\|x_k\|$. By Remark 8.7 we obtain that $\alpha((z_k)_k) = 0$ and by Lemma 6.4 we conclude that the assumptions of Lemma 8.6 are satisfied, i.e. T is unbounded, which is absurd.

Case 2: $(x_k)_k$ admits an ℓ_1 spreading model and $(Tx_k)_k$ does not, i.e. it is either norm null, or passing to a subsequence it generates a c_0 spreading model. In the first case we are done, in the second case choose a sequence of successive $\mathcal{S}_1$ sets $(F_k)_k$ with $\lim_k \#F_k = 0$ and for all k define $u_k = (1/\#F_k)\sum_{i\in F_k} x_i$. Then $(u_k)_k$ is the desired sequence.

Case 3: by passing to a subsequence, both $(x_k)_k$ and $(Tx_k)_k$ generate an ℓ_1 spreading model. Remark 8.7 yields that either $\alpha((z_k)_k) > 0$ or $\alpha((w_k)_k) > 0$ and we shall assume that the first one holds. Passing to a subsequence, there are $n \in \mathbb{N}$, $\delta > 0$, a very fast growing sequence of α_c-averages $(\alpha_q)_q$ of $W_{\mathcal{T}}$ and a sequence of successive subsets $(F_k)_k$ of $\mathbb{N}$, so that

(a) $(\alpha_q)_{q\in F_k}$ is $\mathcal{S}_n$ admissible for all $k \in \mathbb{N}$,
(b) $\operatorname{ran} \alpha_q \subset \operatorname{ran} z_k$ for all $q \in F_k$, $k \in \mathbb{N}$ and
(c) $\sum_{q\in F_k} \alpha_q(z_k) > \delta$ for all $k \in \mathbb{N}$.

By Proposition 4.3, there are $C \geqslant 1$, $\theta > 0$ and a block sequence $(u_k)_k$ so that for each k, $u_k = 2^{n_k} \sum_{j \in G_k} c_k x_k$ is a (C, θ, n_k)-vector with (n_k) strictly increasing. Using an argument involving Proposition 3.3, Remark 3.2 and the spreading properties of the Schreier families, we may also chose the sets G_k so that $(\alpha_q)_{q \in \cup_{j \in G_k} F_j}$ is $\mathcal{S}_{n+n_k}$ -admissible and hence, $g_k = (1/2^{n+n_k}) \sum_{q \in \cup_{j \in G_k} F_j} \alpha_q$ is a weighted functional of weight $w(g_k) = n + n_k$ for all $k \in \mathbb{N}$. By (b) we obtain $g_k(u_k) = 0$ and by(c) $g_k(Tu_k) > \delta/2^n$ for all $k \in \mathbb{N}$. Finally, combining these facts with Proposition 5.2 we conclude that $(u_k)_k$ admits a c_0 spreading model, i.e. the assumptions of Lemma 8.6 are satisfied. This means that T is unbounded, which is absurd. □

Theorem 8.9 *Let X be a block subspace of $\mathfrak{X}_{\mathcal{T}}$. Then for every bounded linear operator $T : X \to X$ there is a $\lambda \in \mathbb{R}$ so that $T - \lambda I$ is strictly singular.*

Proof Let $(x_k)_k$ be the normalized block sequence so that $X = [(x_k)_k]$. We may, of course, assume that $(x_k)_k$ is normalized and let $Q_{\{n\}}$ denote the projections associated with the basis $(x_n)_n$ of X, i.e. $Q_{\{n\}} x_m = \delta_{n,m}$. Then for each $k \in \mathbb{N}$, $Q_{\{k\}} T x_k = \lambda_k x_k$ for some $\lambda_k \in \mathbb{R}$. Choose an accumulation point λ of $(\lambda_k)_k$ and by Lemma 8.8 it easily follows that $T - \lambda I$ is strictly singular.

Remark 8.10 The reason the above result cannot be stated for every closed subspace of $\mathfrak{X}_{\mathcal{T}}$, is that in the definition of the norming set $W_{\mathcal{T}}$ it is not allowed to take α-averages of convex combinations of elements of $W_{\mathcal{T}}$. We note that the construction presented in this paper can also be used to obtain a space $\mathfrak{X}_{\mathcal{T}}^{\mathbb{C}}$ defined over the field of complex numbers. In that case, as it was proved in [15, Theorem 18], every subspace of $\mathfrak{X}_{\mathcal{T}}^{\mathbb{C}}$ satisfies the scalar plus strictly singular property. Therefore, compared to Theorem 8.11 which is stated for block subspaces of $\mathfrak{X}_{\mathcal{T}}$, every closed subspace of $\mathfrak{X}_{\mathcal{T}}^{\mathbb{C}}$ satisfied the invariant subspace property.

Theorem 8.11 *Let X be a block subspace of $\mathfrak{X}_{\mathcal{T}}$ and $T : X \to X$ be a non-scalar bounded linear operator. Then T admits a non-trivial closed hyperinvariant subspace.*

Proof By Theorem 8.9 there is a $\lambda \in \mathbb{R}$ so that the operator $S = T - \lambda I$ is strictly singular. Note that $S \neq 0$, otherwise T would be a scalar operator. Corollary 8.5 yields that S admits a non-trivial closed hyperinvariant subspace Y. It is straightforward to check that Y is a hyperinvariant subspace for T. □

We note that the following property of the strictly singular operators on $\mathfrak{X}_{\mathcal{U}}$, was also proved for an HI space which appeared in [2].

Theorem 8.12 *Let X be a closed subspace of $\mathfrak{X}_{\mathcal{U}}$ and $T : X \to \mathfrak{X}_{\mathcal{U}}$ be a bounded linear operator. The following assertions are equivalent.*

(i) *The operator T is strictly singular.*
(ii) *The operator T is weakly compact.*

Proof The implication (ii)⇒(i) immediately follows from Theorem 6.12. Assume now that T is strictly singular and not weakly compact, which implies that there is a sequence $(x_k)_k$ in X so that both $(x_k)_k$ and $(Tx_k)_k$ are non-trivial weakly Cauchy.

By Lemma 7.5 we may assume that $(x_k)_k$ generates the summing basis of c_0 as a spreading model. Recall that the norm of the summing basis is the minimum conditional spreading norm and thus, we may assume that $(Tx_k)_k$ generates the summing basis of c_0 as a spreading model as well. We conclude that if $y_k = x_{2k-1} - x_{2k}$ for all k, then both $(y_k)_k$ and $(Ty_k)_k$ generate the unit vector basis of c_0 spreading model. Proposition 8.3 yields a contradiction. □

Remark 8.13 A proof identical to the one of [8, Proposition 5.23] yields that every infinite dimensional closed subspace X of $\mathfrak{X}_{\mathcal{T}}$ admits non-compact strictly singular operators, in fact all such operators define a non-separable subset of $\mathcal{L}(X)$.

References

1. Alspach, D.E., Argyros, S.A.: Complexity of weakly null sequences. Diss. Math. (Rozprawy Mat.) **321**, 44 (1992)
2. Argyros, S.A., Arvanitakis, A.D., Tolias, A.G.: Saturated extensions, the attractors method and hereditarily James tree spaces. In: Methods in Banach Space Theory, London Math. Soc. Lecture Note Ser., vol. 337, pp. 1–90. Cambridge Univ. Press, Cambridge (2006)
3. Argyros, S.A., Beanland, K., Motakis, P.: The strictly singular operators in Tsirelson like reflexive spaces. Ill. J. Math. **57**(4), 1173–1217 (2013)
4. Argyros, S.A., Deliyanni, I.: Examples of asymptotic ℓ_1 Banach spaces. Trans. Am. Math. Soc. **349**(3), 973–995 (1997)
5. Argyros, S.A., Godefroy, G., Rosenthal, H.P.: Descriptive set theory and Banach spaces. In: Handbook of the geometry of Banach spaces, vol. 2, pp. 1007–1069. North-Holland, Amsterdam (2003)
6. Argyros, S.A., Haydon, R.G.: A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar-plus-compact problem. Acta Math. **206**(1), 1–54 (2011)
7. Argyros, S.A., Mercourakis, S., Tsarpalias, A.: Convex unconditionality and summability of weakly null sequences. Isr. J. Math. **107**(1), 157–193 (1998)
8. Argyros, S.A., Motakis, P.: A reflexive hereditarily indecomposable space with the hereditary invariant subspace property. Proc. London Math. Soc. **108**(6), 1381–1416 (2014)
9. Argyros, S.A., Motakis, P.: A hereditarily indecomposable Banach space with rich spreading model structure. Isr. J. Math. **203**(1), 341–387 (2014)
10. Argyros, S.A., Tolias, A.: Methods in the theory of hereditarily indecomposable Banach spaces. Mem. Am. Math. Soc. **170**, vi+114 (2004)
11. Beanland, K., Freeman, D., Motakis, P.: The stabilized set of p's in Krivine's theorem can be disconnected. Adv. Math **281**, 553–577 (2015)
12. Bourgain, J., Delbaen, F.: A class of special $\mathcal{L}_\infty$ spaces. Acta Math. **145**(3–4), 155–176 (1980)
13. Figiel, T., Johnson, W.B.: A uniformly convex Banach space which contains no ℓ_p. Compos. Math. **29**, 179–190 (1974)
14. Gowers, W.T.: A Banach space not containing c_0, ℓ_1 or a reflexive subspace. Trans. Am. Math. Soc. **344**(1), 407–420 (1994)
15. Gowers, W.T., Maurey, B.: The unconditional basic sequence problem. J. A.M.S. **6**, 851–874 (1993)
16. Hooker, N.D.: Lomonosov's hyperinvariant subspace theorem for real spaces. Math. Proc. Camb. Philos. Soc. **89**, 129–133 (1981)
17. James, R.C.: Bases and refelexivity of Banach spaces. Ann. Math. (2) **52**, 518–527 (1950)
18. Johnson, W.B., Rosenthal, H.P.: On w^*-basic sequences and their applications to the study of Banach spaces. Stud. Math. **43**, 77–92 (1972)
19. Lewis, D., Stegall, C.: Banach spaces whose duals are isomorphic to $\ell_1(\Gamma)$. J. Funct. Anal. **12**, 177–187 (1973)
20. Odell, E., Schlumprecht, Th: On the richness of the set of p's in Krivine's theorem. Oper. Theory Adv. Appl. **77**, 177–198 (1995)
21. Odell, E., Schlumprecht, Th: A Banach space block finitely universal for monotone bases. Trans. Am. Math. Soc. **352**(4), 1859–1888 (2000)
22. Pełczyński, A.: On Banach spaces containing $L_1(\mu)$. Stud. Math. **30**, 231–246 (1968)

23. Ramsey, F.P.: On a problem of formal logic. Proc. London Math. Soc. Ser. **2**(30), 264–286 (1930)
24. Sirotkin, G.: A version of the Lomonosov invariant subspace theorem for real Banach spaces Indiana Univ. Math. J. **54**(1), 257–262 (2005)
25. Tsirelson, B.S.: Not every Banach space contains ℓ_p or c_0. Funct. Anal. Appl. **8**, 138–141 (1974)