STUDIA MATHEMATICA 253 (1) (2020)

Joint spreading models and uniform approximation of bounded operators

by

S. A. Argyros (Athens), A. Georgiou (Athens),
A.-R. Lagos (Athens) and Pavlos Motakis (Urbana, IL)

Abstract. We investigate the following property for Banach spaces. A Banach space X satisfies the *Uniform Approximation on Large Subspaces* (UALS) if there exists $C > 0$ with the following property: for any $A \in \mathcal{L}(X)$ and convex compact subset W of $\mathcal{L}(X)$ for which there exists $\varepsilon > 0$ such that for every $x \in X$ there exists $B \in W$ with $\|A(x) - B(x)\| \leq \varepsilon\|x\|$, there exists a subspace Y of X of finite codimension and a $B \in W$ with $\|(A-B)|_Y\|_{\mathcal{L}(Y,X)} \leq C\varepsilon$. We prove that a class of separable Banach spaces including ℓ_p for $1 \leq p < \infty$, and $C(K)$ for K countable and compact, satisfy the UALS. On the other hand, every $L_p[0,1]$, for $1 \leq p \leq \infty$ and $p \neq 2$, fails the property and the same holds for $C(K)$ where K is an uncountable metrizable compact space. Our sufficient conditions for UALS are based on joint spreading models, a multidimensional extension of the classical concept of spreading model, introduced and studied in the present paper.

Introduction. This paper is devoted to the study of the Uniform Approximation on Large Subspaces (UALS) for an infinite-dimensional Banach space X. This concept concerns a special case of the following general question. Find conditions such that the ε-pointwise approximation of a function f by the elements of a family W of functions implies that there exists a $g \in W$ which uniformly ε'-approximates f. One of the best results in this framework is the well known consequence of Hahn–Banach theorem. If X is a Banach space, it may be viewed as a subspace of X^{**} through the natural embedding. If $x_0 \in X$, a closed convex subset W of X and $\varepsilon > 0$ are such that for every $x^* \in X^*$ there exists $x \in W$ with $|x^*(x_0) - x^*(x)| \leq \varepsilon\|x^*\|$, then for every $\varepsilon' > \varepsilon$ there exists $y_0 \in W$ such that $\|x_0 - y_0\| \leq \varepsilon'$. It is natural to ask how the above can be extended to the space of bounded linear operators $\mathcal{L}(X)$. The UALS property is an attempt to provide an answer.

2010 *Mathematics Subject Classification*: Primary 46B03, 46B06, 46B25, 46B28, 46B45.
Key words and phrases: joint spreading models, plegma spreading sequences, convex sets of operators, uniform aproximation on large subspaces.
Received 26 March 2019; revised 29 October 2019.
Published online 8 January 2020.

DOI: 10.4064/sm181126-26-3

Notice that in the definition of UALS there are two differences from the above result. The first one is that the set W is norm compact, and this is necessary since the normalized operators of rank one ε-pointwise approximate the identity for every $\varepsilon > 0$. The second difference is that we expect the uniform approximation to happen on a finite-codimensional subspace. This is also necessary, since as Bill Johnson pointed out, for every X with $\dim X \geq 2$ there exist $C > 0$, $A \in \mathcal{L}(X)$ and a convex compact $W \subset \mathcal{L}(X)$ such that, for every x in the unit ball of X, there is a B in W with $\|A(x) - B(x)\| = 0$, whereas the norm distance of A from W is greater than C. There are two classes of Banach spaces satisfying the UALS. The first class is the spaces with the scalar-plus-compact property [AH], [AF$^+$], and the second is the spaces with strong asymptotic homogeneity. The latter property concerns the uniform uniqueness of l-joint spreading models, an extension of the classical spreading models [BS].

The paper is organized in five sections. In Section 1 we introduce the notion of plegma spreading sequences. These are finite collections of Schauder basic sequences in a Banach space that interact with one another in a spreading way when indexed by plegma families, a notion which first appeared in [AKT].

Section 2 is motivated by the definition of plegma spreading sequences and concerns the problem of whether or not finite collections of Schauder (unconditional) basic sequences contain subsequences that form, under a suitable order, a common Schauder (unconditional) basic sequence. For Schauder basic sequences we provide a complete characterization given in the following.

Theorem I. *Let $(x^1_n)_n, \ldots, (x^l_n)_n$ be seminormalized sequences in a Banach space X such that each one is either weakly null, equivalent to the basis of ℓ_1, or non-trivial weak-Cauchy. Let $I \subset \{1, \ldots, l\}$ be such that $(x^i_n)_n$ is a non-trivial weak-Cauchy sequence with $w^*\text{-}\lim x^i_n = x^{**}_i$ for every $i \in I$ and set $F = \operatorname{span}\{x^{**}_i\}_{i \in I}$. Then there exist infinite subsets $M_1, \ldots, M_l$ of $\mathbb{N}$ such that $\bigcup_{i=1}^l \{x^i_n\}_{n \in M_i}$ is a Schauder basic sequence, under a suitable enumeration, if and only if $X \cap F = \{0\}$.*

For unconditional sequences the following holds.

Theorem II. *Let $(e^i_n)^l_{i=1,\, n\in\mathbb{N}}$ be a plegma spreading sequence such that each $(e^i_n)_n$ is unconditional. Then $(e^i_n)^l_{i=1,\, n\in\mathbb{N}}$ is also an unconditional sequence.*

We also provide a variant of the classical B. Maurey – H. P. Rosenthal example [MR] of two unconditional sequences $(e^1_n)_n, (e^2_n)_n$ in a space X such that, for any infinite subsets M, L of $\mathbb{N}$, the sequence $(e^1_n)_{n\in M} \cup (e^2_n)_{n\in L}$ is not unconditional. This shows that the assumption of a plegma spreading sequence in the above theorem is necessary. Further, it is well known that the space generated by two unconditional sequences is not necessarily unconditionally saturated.

In Section 3 we define joint spreading models, which as already mentioned are a multidimensional extension of the classical Brunel–Sucheston spreading models. We also present some of their basic properties.

Section 4 concerns spaces that admit uniformly unique joint spreading models with respect to certain families of Schauder basic sequences. Examples of such spaces are $\ell_p(\Gamma)$ for $1 \leq p < \infty$, $c_0(\Gamma)$ and, as we show, all Asymptotic ℓ_p spaces in the sense of [MMT]. We also prove that the James Tree space admits a uniformly unique l-joint spreading model with respect to the family of all normalized weakly null Schauder basic sequences in JT. Each l-joint spreading model generated by a sequence from this family is $\sqrt{2}$-equivalent to the unit vector basis of ℓ_2, and $\sqrt{2}$ is the best constant [FG], [Be]. Our proof is a variant of the well known result due to I. Amemiya and T. Ito [AI] that every normalized weakly null sequence in JT has a subsequence equivalent to the basis of ℓ_2.

Section 5 is devoted to the study of spaces satisfying the UALS and to classical spaces where this property fails. In the first part we study the property for spaces with very few operators, namely those with the scalar-plus-compact property [AH], [AF$^+$]. We prove the following.

THEOREM III. *Every Banach space with the scalar-plus-compact property satisfies the UALS.*

The basic result for UALS concerns spaces which admit uniformly unique joint spreading models with respect to families of Schauder basic sequences that have certain stability properties. For this we first introduce the class of difference-including families (see Definition 5.9) and we prove the following.

THEOREM IV. *Let X be a Banach space and assume that for every separable subspace Z of X we have a difference-including collection $\mathscr{F}_Z$ of normalized Schauder basic sequences in Z. If there exists a uniform $K \geq 1$ such that each such Z admits a K-uniformly unique l-joint spreading model with respect to $\mathscr{F}_Z$, then X satisfies the UALS property.*

A key ingredient of the proof is Kakutani's Fixed Point Theorem for multivalued mappings [BK], [Ka]. This argument has appeared in a related work of W. T. Gowers and B. Maurey (see [GM, Lemma 9]), and was the motivation for defining the UALS property. As a consequence of the above theorem, the following spaces and all of their subspaces satisfy the UALS: the space $\ell_p(\Gamma)$ for $1 \leq p < \infty$, $c_0(\Gamma)$, the James Tree space and all Asymptotic ℓ_p spaces for $1 \leq p \leq \infty$. The UALS property behaves quite well in duality. In particular the following hold.

THEOREM V. *Let X be a reflexive Banach space with an FDD. Then X satisfies the UALS if and only if X^* does.*

THEOREM VI. *Let X be a Banach space with an FDD. Assume that there exist a uniform constant $C > 0$ and, for every separable subspace Z of X^*, a difference-including family $\mathscr{F}_Z$ of normalized sequences in X^* such that Z admits a C-uniformly unique l-joint spreading model with respect to $\mathscr{F}_Z$. Then X satisfies the UALS property.*

As a consequence of the above, $\mathscr{L}_\infty$ spaces with separable dual and their quotients with an FDD satisfy the UALS. Thus the spaces $C(K)$ for K countable compact have this property. We also provide an example of a reflexive Banach space that admits a uniformly unique spreading model and fails the UALS property. This example shows that if a space admits a uniformly unique spreading model, this does not necessarily imply that it admits a uniformly unique l-joint spreading model for every $l \in \mathbb{N}$, and that the assumption in Theorems IV and VI of a uniformly unique l-joint spreading model cannot be weakened by assuming a uniformly unique spreading model. Finally, we prove that the spaces $L_p[0,1]$ for $1 \leq p \leq \infty$ and $p \neq 2$, and $C(K)$ for any uncountable and metrizable compact space K, fail the UALS.

1. Plegma spreading sequences. We recall the notion of plegma families which first appeared in [AKT] and was used to define higher order spreading models. Interestingly, they were used in a rather different way there and we slightly modify their definition. We shall refer to the notion from [AKT] as *strict* plegma families. We use them to introduce the notion of plegma spreading sequences. These are finite collections of sequences that interact with one another in a spreading way when indexed by plegma families. We start with some notation we will use throughout the paper.

NOTATION. We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of all positive integers. We will use capital letters $L, M, N, \ldots$ (resp. lower case letters $s, t, u, \ldots$) to denote infinite subsets (resp. finite subsets) of $\mathbb{N}$. For every infinite subset L of $\mathbb{N}$, the notation $[L]^\infty$ (resp. $[L]^{<\infty}$) stands for the set of all infinite (resp. finite) subsets of L. For every $s \in [\mathbb{N}]^{<\infty}$, we denote by $|s|$ the cardinality of s. For $L \in [\mathbb{N}]^\infty$ and $k \in \mathbb{N}$, $[L]^k$ (resp. $[L]^{\leq k}$) is the set of all $s \in [L]^{<\infty}$ with $|s| = k$ (resp. $|s| \leq k$). For every $s, t \in [\mathbb{N}]^{<\infty}$, we write $s < t$ if either at least one of them is the empty set, or $\max s < \min t$. Also, for $\emptyset \neq s \in [\mathbb{N}]^\infty$ and $n \in \mathbb{N}$ we write $n < s$ if $n < \min s$.

We shall identify strictly increasing sequences in $\mathbb{N}$ with their corresponding range, i.e. we view every strictly increasing sequence in $\mathbb{N}$ as a subset of $\mathbb{N}$ and conversely every subset of $\mathbb{N}$ as the sequence resulting from the increasing order of its elements. Thus, for an infinite subset $L = \{l_1 < l_2 < \cdots\}$ of $\mathbb{N}$

and $i \in \mathbb{N}$, we set $L_i = l_i$, and similarly, for a finite subset $s = \{n_1, \ldots, n_k\}$ of $\mathbb{N}$ and for $1 \le i \le k$, we set $s(i) = n_i$.

Given a Banach space X with a Schauder basis $(e_n)_n$, for every $x \in X$ with $x = \sum_n a_n e_n$ we write $\mathrm{supp}(x)$ to denote the support of x, i.e. $\mathrm{supp}(x) = \{n \in \mathbb{N} : a_n \neq 0\}$. Generally, we follow [LT] for standard notation and terminology concerning Banach space theory.

Definition 1.1. Let $M \in [\mathbb{N}]^\infty$ and $\mathcal{F}$ be either $[M]^k$ for some $k \in \mathbb{N}$ or $[M]^\infty$. A *plegma* (resp. *strict plegma*) family in $\mathcal{F}$ is a finite sequence $(s_i)_{i=1}^l$ in $\mathcal{F}$ satisfying the following properties:

(i) $s_{i_1}(j_1) < s_{i_2}(j_2)$ for every $1 \le j_1 < j_2 \le k$ or $j_1 < j_2 \in \mathbb{N}$ and $1 \le i_1, i_2 \le l$.
(ii) $s_{i_1}(j) \le s_{i_2}(j)$ (resp. $s_{i_1}(j) < s_{i_2}(j)$) for every $1 \le i_1 < i_2 \le l$ and every $1 \le j \le k$ or $j \in \mathbb{N}$.

For each $l \in \mathbb{N}$, the set of all sequences $(s_i)_{i=1}^l$ which are plegma families in $\mathcal{F}$ will be denoted by $\mathrm{Plm}_l(\mathcal{F})$ and that of strict plegma ones by $\text{S-Plm}_l(\mathcal{F})$.

The following is a consequence of Ramsey's theorem [Ra].

Theorem 1.2 ([AKT]). *Let M be an infinite subset of $\mathbb{N}$ and $k, l \in \mathbb{N}$. Then for every finite partition $\text{S-Plm}_l([M]^k) = \bigcup_{i=1}^n P_i$, there exist $L \in [M]^\infty$ and $1 \le i_0 \le n$ such that $\text{S-Plm}_l([L]^k) \subset P_{i_0}$.*

Definition 1.3. Let $\pi = \{1, \ldots, l\} \times \{1, \ldots, k\}$, $s = (s_i)_{i=1}^l$ be a plegma family in $[\mathbb{N}]^k$ and $(e_n^i)_{i=1, n\in\mathbb{N}}^l$ be a sequence in a linear space E.

(i) The *plegma shift* of π with respect to the plegma family s is the set $s(\pi) = \{(i, s_i(j)) : (i, j) \in \pi\}$, and for a subset A of π, the *plegma shift* of A with respect to s is the set $s(A) = \{(i, s_i(j)) : (i, j) \in A\}$.
(ii) Let $x \in E$ with $x = \sum_{(i,j)\in F} a_{ij} e_j^i$ and $F \subset \pi$. The *plegma shift* of x with respect to s is the vector $s(x) = \sum_{(i,j)\in F} a_{ij} e_{s_i(j)}^i$.

Recall that a sequence $(e_n)_n$ in a seminormed space E is called *spreading* if $\|\sum_{i=1}^n a_i e_i\| = \|\sum_{i=1}^n a_i e_{k_i}\|$ for all $n \in \mathbb{N}$, $k_1 < \cdots < k_n$ and $a_1, \ldots, a_n \in \mathbb{R}$. Then, under Definition 1.3, we have the following reformulation: $(e_n)_n$ is spreading if $\|\sum_{i=1}^n a_i e_i\| = \|s(\sum_{i=1}^n a_i e_i)\|$ for all $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$ and every plegma family $s \in \mathrm{Plm}_1([\mathbb{N}]^n)$. Next we introduce the notion of plegma spreading sequences, which are an extension of the above.

Definition 1.4. A sequence $(e_n^i)_{i=1, n\in\mathbb{N}}^l$ in a Banach space E will be called *plegma spreading* if each $(e_n^i)_n$ is a normalized Schauder basic sequence and, for every $x \in \mathrm{span}\{e_n^i\}_{i=1, n\in\mathbb{N}}^l$, we have $\|x\| = \|s(x)\|$ for all plegma shifts $s(x)$ of x.

Remark 1.5. Let $(e_n^i)_{i=1, n\in\mathbb{N}}^l$ be a plegma spreading sequence.

(i) For every $I \subset \{1, \ldots, l\}$ the sequence $(e_n^i)_{i\in I, n\in\mathbb{N}}$ is also plegma spreading and in particular the sequence $(e_n^i)_n$ is spreading for every $1 \le i \le l$.

(ii) The set $\{e_n^i\}_{i=1,\,n\in\mathbb{N}}^l$ is linearly independent.
(iii) For every $(s_i)_{i=1}^l \in \mathrm{Plm}_l([\mathbb{N}]^\infty)$, the sequence $(e_{s_i(n)}^i)_{i=1,\,n\in\mathbb{N}}^l$ is isometric to $(e_n^i)_{i=1,\,n\in\mathbb{N}}^l$ under the natural mapping $T(e_n^i) = e_{s_i(n)}^i$.
(iv) For every $k \in \mathbb{N}$, $x \in \mathrm{span}\{e_n^i\}_{i=1,\,n=1}^{l,k}$ and $s_n = (s_i^n)_{i=1}^l \in \mathrm{Plm}_l([\mathbb{N}]^k)$ such that $s_l^m(k) < s_1^n(1)$ for every $m < n$, the sequence $(x_n)_n$ with $x_n = s_n(x)$ is spreading.

2. Finite families of sequences in Banach spaces. In this section we study in which cases l-tuples of Schauder (unconditional) basic sequences in a given Banach space have subsequences indexed by plegma families that form a common Schauder (unconditional) basic sequence with a natural order. This turns out to be related to the w^*-limits of these sequences in the second dual. The case where some of the sequences are equivalent to the unit vector basis of ℓ_1 is of interest and we use ultrafilters to deduce the desired conclusion.

2.1. Finite families of Schauder basic sequences. We first treat non-trivial weak-Cauchy sequences and sequences equivalent to the unit vector basis of ℓ_1, and eventually we consider weakly null sequences as well. We include a proof of the following well known lemma for completeness.

LEMMA 2.1. *Let $(x_n)_n$ be a normalized sequence in a Banach space X and $x_1^*, \dots, x_k^* \in X^*$ with $\lim x_i^*(x_n) = 0$ for all $1 \le i \le k$. For every $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, \bigcap_{i=1}^k \ker x_i^*) < \delta$ for every $n \ge n_0$.*

Proof. Let $Y = \mathrm{span}\{x_1^*, \dots, x_k^*\}$ and F be a finite $\delta/4$-net of S_Y. Then there exists $n_0 \in \mathbb{N}$ such that $f(x_n) < \delta/4$ for every $f \in F$ and $n \ge n_0$. Pick any $n \ge n_0$. Then, if $d(x_n, \bigcap_{i=1}^k \ker x_i^*) \ge \delta$, we may find $x^* \in X^*$ with $\|x^*\| = 1$ such that $x^*(x_n) \ge \delta$ and $\bigcap_{i=1}^k \ker x_i^* \subset \ker x^*$. Hence $x^* \in Y$ and there exists $f \in F$ with $\|x^* - f\| < \delta/4$, which is a contradiction to $x^*(x_n) \ge \delta$ since $f(x_n) < \delta/4$. ∎

The following is a variation of Mazur's method [LT, Theorem 1.a.5] for finding Schauder basic sequences in infinite-dimensional Banach spaces.

PROPOSITION 2.2. *Let $(e_n^1)_n, \dots, (e_n^l)_n$ be seminormalized sequences in a Banach space X and let E denote the closed linear span of $\{e_n^i\}_{i=1,\,n\in\mathbb{N}}^l$ and S_E the unit sphere of E. Assume that there exist $\varepsilon > 0$ and a collection $\{K_F : F \subset S_E \text{ finite}\}$ of finite subsets of X^* such that*

(i) *for every finite $F \subset S_E$ and $x \in F$ there exists $x^* \in K_F$ with $\|x^*\| = 1$ and $x^*(x) \ge \varepsilon$,*
(ii) *for every finite $F \subset S_E$ and $1 \le i \le l$, there exists $L \in [\mathbb{N}]^\infty$ such that $\lim_{n\in L} x^*(e_n^i) = 0$ for all $x^* \in K_F$, and*
(iii) *for every finite $F' \subset F \subset S_E$, we have $K_{F'} \subset K_F$.*

Then there exist $M_1, \dots, M_l \in [\mathbb{N}]^\infty$ and a suitable enumeration under which $\bigcup_{i=1}^l \{e_n^i\}_{n\in M_i}$ is a Schauder basic sequence.

Proof. We may assume that the sequences $(e_n^i)_n$, $1 \le i \le l$, are normalized. Indeed, if we normalize the given sequences then conditions (i)–(iii) will not be affected. If we obtain the result for the normalized versions of the given sequences, then we can revert to subsequences of the initial ones. Let $(\varepsilon_n)_n$ be a sequence in $(0, 1/2)$ such that $\sum_{n=1}^\infty \varepsilon_n < \varepsilon/5$. We will construct, by induction on $\mathbb{N}$, a Schauder basic sequence $(x_k)_k$ with $x_k = e_{n_k}^{i_k}$, where $i_k = (k-1 \bmod l) + 1$ and $n_{k+1} > n_k$. Hence the sets $M_i = \{n_k : i_k = i\}$ for $1 \le i \le l$, and the lexicographic order on $\mathbb{N} \times \{1, \dots, l\}$, yield the desired result.

We set $x_1 = y_1 = e_1^1$ and $F_1 = \{x_1/\|x_1\|, -x_1/\|x_1\|\}$. Assume that $x_1, \dots, x_k$, $y_1, \dots, y_k$, and $F_1 \subset \dots \subset F_k$ have been chosen, for some $k \in \mathbb{N}$, such that the following are satisfied: for $1 \le m \le k$ each x_m is of the form $e_{n_m}^{i_m}$, each F_m is an $\varepsilon_m/2$-net of the unit sphere of $X_m = \operatorname{span}\{x_1, y_1, \dots, x_m, y_m\}$, and for $m > 1$ each y_m is in $Y_m = \bigcap\{E \cap \ker x^* : x^* \in K_{F_{m-1}}\}$ with $\|x_m - y_m\| < \varepsilon_m$. We describe the next inductive step. By property (ii) there is $L \in [\mathbb{N}]^\infty$ such that for all $x^* \in K_{F_k}$ we have $\lim_{n\in L} x^*(e_n^{i_{k+1}}) = 0$. Apply Lemma 2.1 to the sequence $(e_n^{i_{k+1}})_n$ and the subset $\{x^*|_E : x^* \in F_k\}$ of E^* to find $x_{k+1} = e_{n_{k+1}}^{i_{k+1}}$ such that

$$d\Big(x_{k+1}, \bigcap\{Y \cap \ker x^* : x^* \in K_{F_k}\}\Big) < \varepsilon_{k+1}/2.$$

Then we may choose a $y_{k+1} \in \bigcap\{Y \cap \ker x^* : x^* \in K_{F_k}\}$ with $\|x_{k+1} - y_{k+1}\| < \varepsilon_{k+1}$. Finally, pick an ε_{k+1}-net F_{k+1} of the unit sphere of $X_{k+1} = \operatorname{span}\{x_1, y_1, \dots, x_{k+1}, y_{m+1}\}$.

Note that for each $k \in \mathbb{N}$ and $x \in X_m$ there exists $x^* \in K_{F_m}$ with $\|x^*\| = 1$ and $x^*(x) \ge (\varepsilon - \varepsilon_m/2)\|x\| \ge (9\varepsilon/10)\|x\|$. Also, because for $m \le n$ we have $K_{F_m} \subset K_{F_n}$, we also have $y_{n+1} \in Y_{n+1} \subset Y_m$ and hence for $x^* \in K_{F_m}$ we deduce $x^*(y_{n+1}) = 0$. We use these facts to first observe that $(y_k)_k$ is K-Schauder basic for $K = 10/(9\varepsilon)$. Indeed, if $m \le n$ and $a_1, \dots, a_n \in \mathbb{R}$ then $x = \sum_{i=1}^m a_i y_i \in X_m$ and for some $x^* \in F_m$ with $\|x^*\| = 1$ we have

$$\Big\|\sum_{i=1}^m a_i y_i\Big\| \le \frac{6}{5\varepsilon} x^*\Big(\sum_{i=1}^m a_i y_i\Big) = \frac{10}{9\varepsilon} x^*\Big(\sum_{i=1}^n a_i y_i\Big) \le \frac{10}{9\varepsilon}\Big\|\sum_{i=1}^n a_i y_i\Big\|.$$

By the principle of small perturbations, for $(x_k)_k$ to be Schauder basic it suffices to show that $2K\sum_k \|x_k - y_k\|/\|y_k\| < 1$. But this follows from $\sum_k \|x_k - y_k\|/\|y_k\| < 2\sum_n \varepsilon_k < 2\varepsilon/5$. ■

REMARK 2.3. Let us observe that $(M_i)_{i=1}^l$, as constructed in the previous proof, is a plegma family in $[\mathbb{N}]^\infty$. In general, for every $M_1, \dots, M_l \in [\mathbb{N}]^\infty$, there exists a plegma family $(s_i)_{i=1}^l$ in $[\mathbb{N}]^\infty$ with $s_i \subset M_i$.

Moreover, with any plegma family $(s_i)_{i=1}^l \in \mathrm{Plm}_l([\mathbb{N}]^\infty)$ we associate a natural order on $\{s_i(n)\}_{i=1,\,n\in\mathbb{N}}^l$ and that is the lexicographic order on $[\mathbb{N}] \times \{1,\dots,l\}$.

The following lemma is an immediate consequence of the principle of local reflexivity [LR], but we also give the following easy proof.

LEMMA 2.4. *Let X be a Banach space and F be a linear subspace of X^{**} with finite dimension and $X \cap F = \{0\}$. Then, for every $\delta > 0$ and $x \in X$ with $\|x\| = 1$, there exists $x^* \in X^*$ with $\|x^*\| \le 1+\delta$ such that $x^*(x) \ge \varepsilon = d(S_X, F)$ and $x^{**}(x^*) = 0$ for every $x^{**} \in F$.*

Proof. Let $x \in X$ with $\|x\| = 1$ and $Y = \mathrm{span}\{F \cup \{x\}\}$. Then there exists $x^{***} \in S_{Y^*}$ such that $x^{***}(x) \ge \varepsilon$ and $x^{***}(x^*) = 0$ for every $x^{**} \in F$. Then we consider the identity map $I : Y \to X^{**}$ and recall that its conjugate $I^* : X^{***} \to Y^*$ is w^*-w^*-continuous. Since B_{X^*} is a w^*-dense subset of $B_{X^{***}}$ and Y is of finite dimension, it follows that $I^*(B_{X^*})$ is norm dense in B_{Y^*} and hence, for every $\delta > 0$, we see that $B_{Y^*} \subset I^*((1+\delta)B_{X^*})$, which yields the desired result. ∎

Recall that a sequence $(x_n)_n$ in a Banach space X is called *non-trivial weak-Cauchy* if there exists $x^{**} \in X^{**} \setminus X$ such that $w^*\text{-}\lim x_n = x^{**}$. The next proposition provides a complete characterization for the aforementioned problem for finite collections of such sequences.

PROPOSITION 2.5. *Let $(e_n^1)_n, \dots, (e_n^l)_n$ be seminormalized non-trivial weak-Cauchy sequences in a Banach space X and $F = \mathrm{span}\{e_i^{**}\}_{i=1}^l$, where $w^*\text{-}\lim e_n^i = e_i^{**}$. Then there exists an $(s_i)_{i=1}^l \in \mathrm{Plm}_l([N]^\infty)$ such that $\{e_{s_i(n)}^i\}_{i=1,\,n\in\mathbb{N}}^l$ is a Schauder basic sequence, enumerated according to the natural plegma order, if and only if $X \cap F = \{0\}$.*

Proof. Let $X \cap F \ne \{0\}$. Then there exists $x = \sum_{i=1}^l a_i e_i^{**} \in X$ with $x \ne 0$. If there exists a plegma family $(s_i)_{i=1}^l \in \mathrm{Plm}_l([\mathbb{N}]^\infty)$ such that $\bigcup_{i=1}^l \{e_n^i\}_{n\in M_i}$ is a Schauder basic sequence, we consider the sequence $(x_n)_n$ with $x_n = \sum_{i=1}^l a_i e_{s_i(n)}^i$. Notice that $(x_n)_n$ is a Schauder basic sequence with $w\text{-}\lim x_k = x$, which is a contradiction since $x \ne 0$.

Suppose now that $X \cap F = \{0\}$ and let $\varepsilon = d(S_E, F)$. Then for every $x \in S_E$, by Lemma 2.4, there exists an $f_x \in E^*$ with $\|f_x\| = 1$ such that $f_x(x) \ge \varepsilon/2$ and $x^{**}(f_x) = 0$ for every $x^{**} \in F$ and hence $\lim x^*(e_n^i) = 0$ for every $1 \le i \le l$. Finally, applying Proposition 2.2 (with $K_F = \{f_x : x \in F\}$) and Remark 2.3 completes the proof. ∎

Next we give an example of a plegma spreading sequence, formed by two non-trivial weak-Cauchy sequences, that is not Schauder basic.

DEFINITION 2.6 ([J1]). On the space $c_{00}(\mathbb{N})$ we define the following norm:

$$\|x\|_J = \sup\Big(\sum_{i=1}^{n}\Big(\sum_{k\in I_i} x(k)\Big)^2\Big)^{1/2},$$

where the supremum is taken over all finite collections $I_1, \ldots, I_n$ of disjoint intervals of natural numbers. The *James space*, denoted by J, is the completion of $c_{00}(\mathbb{N})$ with respect to $\|\cdot\|_J$.

EXAMPLE 2.7. Let $(e_n)_n$ denote the standard basis of the James space and recall that it is a non-trivial weak-Cauchy sequence. We consider the sequences $(e_n^1)_n$ and $(e_n^2)_n$ in J with $e_n^1 = e_{2n} + e_1$ and $e_n^2 = e_{2n+1} - e_1$, which are also non-trivial weak-Cauchy, and denote by e_1^{**}, e_2^{**} their w^*-limits. Notice that $(e_n^i)_{i=1,\, n\in\mathbb{N}}^2$ is a plegma spreading sequence in J. Moreover, since $e_1 \in J \cap \operatorname{span}\{e_1^{**}, e_2^{**}\}$ and $T(e_n^i) = e_{s_i(n)}^i$ is an isometry for every $(s_i)_{i=1}^2 \in \mathrm{Plm}_l([\mathbb{N}]^\infty)$, the same arguments as in Proposition 2.5 show that $(e_n^i)_{i=1,\, n\in\mathbb{N}}^2$ is not Schauder basic.

We now pass to study the case of finite families of ℓ_1 sequences in a Banach space. As is well known, $\beta\mathbb{N}$ denotes the Stone–Čech compactification of $\mathbb{N}$ and therefore $\ell_\infty(\mathbb{N})$ is isometric to $C(\beta\mathbb{N})$. It is also known that the elements of $\beta\mathbb{N}$ are the ultrafilters on $\mathbb{N}$. The identification of $\ell_\infty(\mathbb{N})$ with $C(\beta\mathbb{N})$ implies that the conjugate space of $\ell_\infty(\mathbb{N})$ is isometric to $\mathcal{M}(\beta\mathbb{N})$, the set of all regular measures on $\beta\mathbb{N}$.

For $f \in \ell_\infty(\mathbb{N})$ and p an ultrafilter on $\mathbb{N}$, the evaluation of the Dirac measure δ_p on the function f is given as $\delta_p(f) = \lim_p f(n)$, where $\lim_p f(n)$ is the unique limit of $(f(n))_n$ with respect to the ultrafilter p. Let us also observe that if $T : \ell_1 \to X$ is an isomorphic embedding, then $T^{**} : \mathcal{M}(\beta\mathbb{N}) \to X^{**}$ and for any $p \in \beta\mathbb{N}$ and $x^* \in X^*$ we have $T^{**}\delta_p(x^*) = \lim_p x^*(Te_n)$. For further information on ultrafilters we refer to [CN].

LEMMA 2.8. *Let X be a Banach space and $T : \ell_1 \to X$ be an isomorphic embedding. Let $\alpha \in \mathbb{R}$, $x_1^*, \ldots, x_k^* \in X^*$ and p be a non-principal ultrafilter on $\mathbb{N}$ such that $T^{**}\delta_p(x_i^*) = \alpha$ for all $1 \le i \le k$. Then there exists $M \in [\mathbb{N}]^\infty$ such that $\lim_{n\in M} x_i^*(Te_n) = \alpha$ for every $1 \le i \le k$.*

Proof. Notice that $T^{**}\delta_p(x_i^*) = \lim_p x_i^*(Te_n)$ and also that, for any $n \in \mathbb{N}$, the set $M_n = \{m \in \mathbb{N} : |x_i^*(Te_m) - \alpha| < 1/n \text{ for all } 1 \le i \le l\}$ is in p and is not finite, since p is a non-principal ultrafilter. Let M be a diagonalization of $(M_n)_n$, i.e. $M(n) \in M_n$ for all $n \in \mathbb{N}$. Then $(Te_n)_{n\in M}$ is the desired subsequence. ■

LEMMA 2.9. *Let X be a separable Banach space and F be a finite-dimensional subspace of X^{**} with $X \cap F = \{0\}$. Let also $(e_n^1)_n, \ldots, (e_n^l)_n$ be sequences in X such that each one is equivalent to the basis of ℓ_1 and* *denote by T_i the corresponding embedding. Then there exist non-principal ultrafilters $p_1, \dots, p_l$ on $\mathbb{N}$ such that*

(i) *the set $F \cup \{T_i^{**}\delta_{p_i}\}_{i=1}^l$ is linearly independent,*
(ii) $X \cap \operatorname{span}\{F \cup \{T_i^{**}\delta_{p_i}\}_{i=1}^l\} = \{0\}$.

Proof. Let us observe that the cardinality of $\beta\mathbb{N}$ is $2^{\mathfrak{c}}$ whereas that of any separable Banach space is less than or equal to $\mathfrak{c}$. Also recall that the family $\{\delta_p : p \in \beta\mathbb{N}\}$ is equivalent to the basis of $\ell_1(2^{\mathfrak{c}})$ and hence linearly independent. The same remains valid for a fixed $1 \le i \le l$ and the family $\{T_i^{**}\delta_p : p \in \beta\mathbb{N}\}$, since T_i^{**} is an isomorphism. We consider the linear space X^{**}/X and we denote by Q the natural quotient map $Q : X^{**} \to X^{**}/X$.

Claim. *For every $1 \le i \le l$, there exists an uncountable subset A_i of $\beta\mathbb{N}$ such that the family $\{QT_i^{**}\delta_p : p \in A_i\}$ is linearly independent.*

Proof of Claim. If not, there would exist a countable subset A_i of $\beta\mathbb{N}$ such that $\{QT_i^{**}\delta_p : p \in A_i\}$ is a maximal independent subfamily of $\{QT_i^{**}\delta_p : p \in \beta\mathbb{N}\}$ for some $1 \le i \le l$. Then $\{T_i^{**}\delta_p : p \in \beta\mathbb{N}\} \subset \operatorname{span}\{X \cup \{T_i^{**}\delta_p : p \in A_i\}\}$, which yields a contradiction, since the algebraic dimension of X is less than or equal to $\mathfrak{c}$. ∎

Since F satisfies $X \cap F = \{0\}$, it follows that $Q|_F$ is an isomorphism, and by induction we will choose, for every $1 \le i \le l$, an ultrafilter p_i on $\mathbb{N}$ such that $p_i \in A_i$ and $QT_i^{**}\delta_{p_i} \notin \operatorname{span}\{Q[F] \cup \{QT_j^{**}\delta_{p_j}\}_{j<i}\}$. For $i = 1$, there exists $p_1 \in A_1$ with $QT_1^{**}\delta_{p_1} \notin Q[F]$, since F has finite dimension and A_1 is uncountable. Suppose that $p_1, \dots, p_i$ have been chosen for some $i < l$. Then there exists $p_{i+1} \in A_{i+1}$ with $QT_{i+1}^{**}\delta_{p_{i+1}} \notin \operatorname{span}\{Q[F] \cup \{QT_j^{**}\delta_{p_j}\}_{j\le i}\}$, for the same reason as above, and this completes the inductive construction. Notice that the ultrafilters $p_1, \dots, p_l$ are non-principal since $T^{**}\delta_{p_i} \notin X$ for every $1 \le i \le l$. ∎

Corollary 2.10. *Let $(e_n^1)_n, \dots, (e_n^l)_n$ be sequences in a separable Banach space X such that each one is equivalent to the basis of ℓ_1 and denote by T_i the corresponding embedding. Then, for every $k \in \mathbb{N}$ and every $1 \le i \le l$ and $1 \le j \le k$, there exists a non-principal ultrafilter p_{ij} on $\mathbb{N}$ such that*

(i) *the set $\{T_i^{**}\delta_{p_{ij}}\}_{i=1,\,j=1}^{l,k}$ is linearly independent,*
(ii) $X \cap \operatorname{span}\{T_i^{**}\delta_{p_{ij}}\}_{i=1,\,j=1}^{l,k} = \{0\}$.

The following lemma is an immediate consequence of the above, and it will be used in the next subsection.

Lemma 2.11. *Let $(e_n^1)_n, \dots, (e_n^l)_n$ be sequences in a separable Banach space X such that each one is equivalent to the basis of ℓ_1 and denote by T_i the corresponding embedding. Then there exist $x_1^* \dots, x_l^* \in X^*$ such that*

(i) *for every* $1 \le i \le l$, *there exists* $M_i \in [\mathbb{N}]^\infty$ *with* $\lim_{n\in M_i} x_i^*(e_n^i) = 1$,
(ii) *for every* $1 \le i, j \le l$, *there exists* $M_j^i \in [\mathbb{N}]^\infty$ *with* $\lim_{n\in M_j^i} x_i^*(e_n^j) = 0$.

Proof. From Corollary 2.10, there exist non-principal ultrafilters $p_1, \dots, p_l$, $q_1, \dots, q_l$ on $\mathbb{N}$ such that the set $\{T_i^{**}\delta_{p_i}\}_{i=1}^l \cup \{T_i^{**}\delta_{q_i}\}_{i=1}^l$ is linearly independent. Then, for each $1 \le i \le l$, we choose $x_i^{***} \in X^{***}$ such that $x_i^{***}(T_j^{**}\delta_{p_j}) = \delta_{ij}$ and $x_i^{***}(T_j^{**}\delta_{q_j}) = 0$ for every $1 \le j \le l$. The principle of local reflexivity then yields an $x_i^* \in X^*$ such that $T_j^{**}\delta_{p_j}(x_i^*) = \delta_{ij}$ and $T_j^{**}\delta_{q_j}(x_i^*) = 0$ for every $1 \le j \le l$. Finally, applying Lemma 2.8 we obtain the desired subsequences. ∎

Next we give a characterization in the general case. Recall that by Rosenthal's ℓ_1 theorem [Ro] and the theory of Schauder bases, if $(x_n)_n$ is a Schauder basic sequence in a Banach space X, then it contains a subsequence which is either weakly null or equivalent to the basis of ℓ_1 or non-trivial weak-Cauchy.

Theorem 2.12. *Let* $(e_n^1)_n, \dots, (e_n^l)_n$ *be seminormalized sequences in a Banach space* X *such that each one is either weakly null, equivalent to the basis of* ℓ_1*, or non-trivial weak-Cauchy. Let* $I \subset \{1, \dots, l\}$ *be such that* $(e_n^i)_n$ *is a non-trivial weak-Cauchy sequence with* $w^*\text{-}\lim e_n^i = e_i^{**}$ *for every* $i \in I$ *and let* $F = \operatorname{span}\{e_i^{**}\}_{i\in I}$. *Then there exists* $(s_i)_{i=1}^l \in \operatorname{Plm}_l([N]^\infty)$ *such that* $\{e_{s_i(n)}^i\}_{i=1,\, n\in\mathbb{N}}^l$ *is a Schauder basic sequence, enumerated according to the natural plegma order, if and only if* $X \cap F = \{0\}$.

Proof. Let $J \subset \{1, \dots, l\}$ be such that $(e_n^i)_n$ is equivalent to the basis of ℓ_1 for each $i \in J$ and denote by T_i the corresponding embedding. Then Lemma 2.9 yields for every $i \in J$ a non-principal ultrafilter p_i on $\mathbb{N}$ such that $X \cap Y = \{0\}$, where $Y = \operatorname{span}\{F \cup \{T_i^{**}\delta_{p_i}\}_{i\in J}\}$. For $\varepsilon = d(S_X, Y)$ it follows from Lemma 2.4 that, for every $x \in X$ with $\|x\| = 1$, there exists $f_x \in X^*$ with $\|f_x\| = 1$ such that $f_x(x) \ge \varepsilon/2$ and $x^{**}(f_x) = 0$ for every $x^{**} \in Y$. For every finite $F \subset S_X$, we set $K_F = \{f_x : x \in F\}$. Note that $\lim x^*(e_n^i) = 0$ for every $i \in I$, every finite $F \subset S_X$ and every $x^* \in K_F$. Also, from Lemma 2.8, for each $i \in J$, there exists $M_i \in [\mathbb{N}]^\infty$ with $\lim_{n\in M_i} x^*(e_n^i) = 0$ for all $x^* \in K_f$. Applying Proposition 2.2, we derive the desired result. ∎

More specifically, for a plegma spreading sequence $(e_n^i)_{i=1}^l$, Rosenthal's theorem shows that each (e_n^i) is either weakly null, equivalent to the unit vector basis of ℓ_1, or non-trivial weak-Cauchy, since it is spreading. Taking also into account the behavior of plegma spreading sequences we give a corollary of the above theorem.

Corollary 2.13. *Let* $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ *be a plegma spreading sequence in a Banach space* X *and* $I \subset \{1, \dots, l\}$ *be such that* $(e_n^i)_n$ *is a non-trivial weak-Cauchy sequence with* $w^*\text{-}\lim e_n^i = e_i^{**}$ *for every* $i \in I$. *Set* $F = \operatorname{span}\{e_i^{**}\}_{i\in I}$.

Then $(e_n^i)_{i=1,\,n\in\mathbb{N}}^l$ *is a Schauder basic sequence, enumerated according to the lexicographic order on* $[\mathbb{N}]\times\{1,\dots,l\}$*, if and only if* $X\cap F=\{0\}$.

Proof. Theorem 2.12 yields a plegma family $(s_i)_{i=1}^l$ in $[\mathbb{N}]^\infty$ such that $(e_{s_i(n)}^i)_{i=1,\,n\in\mathbb{N}}^l$ is a Schauder basic sequence if and only if $X\cap F=\{0\}$, and since $T(e_n^i)=e_{s_i(n)}^i$ is an isometry, the same holds for $(e_n^i)_{i=1,\,n\in\mathbb{N}}^l$. ∎

2.2. Finite families of unconditional sequences. We now study the case of unconditional sequences. We start with plegma spreading sequences. Recall that every weakly null spreading sequence in a Banach space is unconditional. The following proposition extends this result to plegma spreading sequences, using similar arguments to those in the classical case.

Proposition 2.14. *Let* $(e_n^i)_{i=1,\,n\in\mathbb{N}}^l$ *be a plegma spreading sequence such that each* $(e_n^i)_n$ *is weakly null. Then* $(e_n^i)_{i=1,\,n\in\mathbb{N}}^l$ *is an unconditional sequence.*

Proof. Let $\pi=\{1,\dots,l\}\times\{1,\dots,k\}$ and $x=\sum_{(i,j)\in\pi}a_{ij}e_j^i$. Since each $(e_n^i)_n$ is weakly null, we see that for every $\varepsilon>0$ and $(i_0,j_0)\in\pi$ there exist $s_1(x),\dots,s_m(x)$ plegma shifts of x and a convex combination $\sum_{t=1}^m\lambda_t s_t(x)$ such that

(i) $s_{t_1}(e_j^i)=s_{t_2}(e_j^i)$ for every $(i,j)\in\pi'=\pi\setminus\{(i_0,j_0)\}$ and $1\le t_1,t_2,\le m$,
(ii) $\|\sum_{t=1}^n\lambda_t s_t(e_{j_0}^{i_0})\|<\varepsilon\|x\|/a_{i_0j_0}$,
(iii) $\sum_{t=1}^m\lambda_t s_t(x)=\sum_{(i,j)\in\pi'}a_{ij}s_1(e_j^i)+\sum_{t=1}^n\lambda_t a_{i_0j_0}s_t(e_{j_0}^{i_0})$.

Then since $(e_n^i)_{i=1,\,n\in\mathbb{N}}^l$ is plegma spreading, we have $\|s_t(x)\|=\|x\|$ for all $1\le t\le m$ and also $\|\sum_{(i,j)\in\pi'}a_{ij}s_1(e_j^i)\|=\|\sum_{(i,j)\in\pi'}a_{ij}e_j^i\|$ and hence

$$\Big\|\sum_{(i,j)\in\pi'}a_{ij}e_j^i\Big\|\le(1+\varepsilon)\Big\|\sum_{(i,j)\in\pi}a_{ij}e_j^i\Big\|.$$

Finally, applying iteration we show that for every $\varepsilon>0$ and every $F\subset\pi$,

$$\Big\|\sum_{(i,j)\in F}a_{ij}e_j^i\Big\|\le(1+\varepsilon)\Big\|\sum_{(i,j)\in\pi}a_{ij}e_j^i\Big\|. \ \blacksquare$$

Proposition 2.15. *Let* $(e_n^i)_{i=1,\,n\in\mathbb{N}}^l$ *be a plegma spreading sequence. If each* $(e_n^i)_n$ *is equivalent to the basis of* ℓ_1*, then the same holds for* $(e_n^i)_{i=1,\,n\in\mathbb{N}}^l$.

Proof. Let $0<\varepsilon<1$, $0<\delta<(1-\varepsilon)/2l$, and $x=\sum_{j=1}^k\sum_{i=1}^l a_{ij}e_j^i$ with $\sum_{j=1}^k\sum_{i=1}^l|a_{ij}|=1$. Then either for x or for $-x$ there exists $1\le i_0\le l$ such that $\sum_{j\in J_{i_0}^+}a_{i_0j}\ge 1/2l$, where $J_{i_0}^+=\{j:a_{i_0j}>0\}$. Moreover, from Lemma 2.11, for any $1\le i,j\le l$, there exist $x_i\in X^*$ and M_i,M_j^i in $[\mathbb{N}]^\infty$ such that $\lim_{n\in M_i}x_i^*(e_n^i)=1$ and $\lim_{n\in M_j^i}x_i^*(e_n^j)=0$. We set $M=$

$\max\{\|x_i^*\| : i = 1, \dots, l\}$ and choose a plegma family $(s_i)_{i=1}^l$ in $[\mathbb{N}]^k$ such that

(i) $s_{i_0}(j) \in M_{i_0}$ and $x^*(e^i_{s_{i_0}(j)}) > 1 - \varepsilon$ for every $j \in J^+_{i_0}$,
(ii) $s_{i_0}(j) \in M^{i_0}_{i_0}$ for every $j \in J^-_{i_0} = \{j : a_{i_0 j} < 0\}$,
(iii) $s_j \subset M^{i_0}_j$ for every $1 \le j \le l$ with $j \ne i_0$,
(iv) $x^*_{i_0}(\sum_{j=1}^k \sum_{i=1,\, i\ne i_0}^l a_{ij} e^i_{s_i(j)} + \sum_{j \in J^-_{i_0}} a_{i_0 j} e^{i_0}_{s_{i_0}(j)}) < \delta$.

Hence $x^*_{i_0}(x) \ge \frac{1-\varepsilon}{2l} - \delta$ and therefore $\|x\| \ge \big(\frac{1-\varepsilon}{2l} - \delta\big)/M$, which yields the desired result. ∎

Combining the previous two propositions we have the following final result.

THEOREM 2.16. *Let $(e^i_n)_{i=1,\, n\in\mathbb{N}}^l$ be a plegma spreading sequence such that each $(e^i_n)_n$ is unconditional. Then $(e^i_n)_{i=1,\, n\in\mathbb{N}}^l$ is also an unconditional sequence.*

Proof. Let $I \subset \{1, \dots, l\}$ be such that the sequence $(e^i_n)_n$ is weakly null for every $i \in I$ and denote its complement by J. We denote by E_0 the closed linear span of $\{e^i_n\}_{i\in I,\, n\in\mathbb{N}}$ and by E_1 that of $\{e^i_n\}_{i\in J,\, n\in\mathbb{N}}$. Then, for any $x \in E_0 + E_1$ with $x = \sum_{j=1}^k (\sum_{i\in I} a_{ij} e^i_j + \sum_{i\in J} b_{ij} e^i_j)$, using similar arguments to those in the proof of Proposition 2.14, we find that for every $\varepsilon > 0$,

$$\Big\| \sum_{j=1}^k \sum_{i\in J} b_{ij} e^i_j \Big\| \le (1+\varepsilon) \Big\| \sum_{j=1}^k \Big(\sum_{i\in I} a_{ij} e^i_j + \sum_{i\in J} b_{ij} e^i_j \Big) \Big\|.$$

Hence $E_0 + E_1 = E_0 \oplus E_1$ and since both $(e^i_n)_{i\in I,\, n\in\mathbb{N}}$ and $(e^i_n)_{i\in J,\, n\in\mathbb{N}}$ are unconditional sequences, as follows from the previous two propositions, the same holds for their union. ∎

2.3. Unconditional sequences in singular position. The following is a variant of the Maurey–Rosenthal classical example [MR]. As Theorem 2.16 asserts, the strong assumption of being plegma spreading implies that l-tuples of unconditional sequences are jointly unconditional. The purpose of this example is to demonstrate that this strong condition is in fact necessary.

PROPOSITION 2.17. *Let N_1, N_2 be a partition of $\mathbb{N}$ into two infinite sets. There exists a Banach space X with a Schauder basis $(e_n)_n$ such that*

(i) *the sequences $(e_n)_{n\in N_1}$ and $(e_n)_{n\in N_2}$ are unconditional,*
(ii) *for every $M \subset \mathbb{N}$ such that $M \cap N_1$ and $M \cap N_2$ are both infinite sets, the sequence $(e_n)_{n\in M}$ is not unconditional.*

We fix a strictly increasing sequence of natural numbers $(\mu_i)_i$ such that

$$\sum_{i=1}^{\infty}\sum_{j>i}\frac{\sqrt{\mu_i}}{\sqrt{\mu_j}}\le\frac{1}{2}.$$

Denote by $\mathcal{P}$ the collection of all finite sequences $(E_k)_{k=1}^n$ of successive non-empty finite subsets of $\mathbb{N}$. Take an injection $\sigma:\mathcal{P}\to\mathbb{N}$ such that $\sigma((E_k)_{k=1}^n)>\max\{\#E_k\}_{k=1}^n$ for every $(E_k)_{k=1}^n\in\mathcal{P}$, and finally fix a partition of $\mathbb{N}$ into two infinite subsets N_1 and N_2.

Definition 2.18. A sequence $(E_k^i)_{i=1,k=1}^{2,n}$ of non-empty finite subsets of $\mathbb{N}$ is called a *special sequence* if

(i) $E_k^1\subset N_1$ and $E_k^2\subset N_2$ for every $1\le k\le n$,
(ii) the sets E_k^1 and E_k^2 are successive for every $1\le k\le n$,
(iii) the sets E_k^2 and E_{k+1}^1 are successive for every $1\le k<n$,
(iv) $\#E_1^1=\#E_1^2=\mu_{j_1}$ for some $j_1\in\mathbb{N}$,
(v) $\#E_{k_0}^1=\#E_{k_0}^2=\mu_{j_{k_0}}$, where $j_{k_0}=\sigma((E_k^i)_{i=1,\,k=1}^{2,k_0-1})$ for every $1<k_0\le n$.

Remark 2.19. Let $(E_k^i)_{i=1,\,k=1}^{2,n}$ and $(F_k^i)_{i=1,\,k=1}^{2,m}$ be special sequences and set $k_0=\min\{k:\#E_k^1\ne\#F_k^1\}$. If $k_0>1$ then since σ is an injection,

(i) if $k_0>2$ then $E_k^1=F_k^1$ and $E_k^2=F_k^2$ for every $1\le k<k_0-1$,
(ii) $\#E_{k_0-1}^1=\#F_{k_0-1}^1$ and $E_{k_0-1}^1\ne F_{k_0-1}^1$ or $E_{k_0-1}^2\ne F_{k_0-1}^2$,
(iii) $\#E_k^1\ne\#F_k^1$ for every $k_0\le k\le\min\{n,m\}$.

Let $(e_i^*)_i$ as well as $(e_i)_i$ denote the unit vector basis of $c_{00}(\mathbb{N})$ and for every $f,x\in c_{00}(\mathbb{N})$ with $f=\sum_{i=1}^n a_ie_i^*$ and $x=\sum_{i=1}^m b_ie_i$ set $f(x)=\sum_{i=1}^{\min\{n,m\}}a_ib_i$. Finally, for $f,g\in c_{00}(\mathbb{N})$ with $f=\sum_{k=1}^n a_{i_k}e_{i_k}^*$ and $g=\sum_{k=1}^n b_{j_k}e_{j_k}^*$, where $a_{i_k},b_{j_k}\ne 0$, we will say that f and g are *consistent* if $\operatorname{sgn}(a_{i_k})=\operatorname{sgn}(b_{j_k})$ for every $1\le k\le n$.

Definition 2.20. Consider the following subsets of $c_{00}(\mathbb{N})$:

$$W_0=\{0\}\cup\{\pm e_i^*:i\in\mathbb{N}\},$$

$$W_1=\left\{\frac{1}{\sqrt{\mu_j}}\sum_{i\in E}\varepsilon_ie_i^*:E\subset\mathbb{N}_1\text{ or }E\subset N_2,\ \#E=\mu_j,\ \varepsilon_i\in\{-1,1\}\text{ for }i\in E\right\},$$

$$W_2=\Big\{\sum_{i=1}^2\sum_{k=1}^n f_k^i:f_k^i\in W_0\cup W_1,\ (\operatorname{supp}(f_k^i))_{i=1,\,k=1}^{2,n}\text{ is a special sequence,}$$
$$f_k^1\text{ and }f_k^2\text{ are consistent for every }1\le k\le n\Big\},$$

and set $W=\{P_E(f):f\in W_0\cup W_1\cup W_2,\ E\text{ is an interval of }\mathbb{N}\}$. Define a norm on $c_{00}(\mathbb{N})$ by setting $\|x\|=\sup\{f(x):f\in W\}$ and let $X_{\mathrm{MR}}^{(2)}$ denote its completion with respect to this norm.

REMARK 2.21. For any $f = \sum_{i=1}^n a_i e_i^*$ in W and $1 \le k < l \le n$, $\sum_{i=k}^l a_i e_i^*$ is in W as well. That is, $(e_i)_i$ forms a normalized and bimonotone Schauder basis for $X_{\mathrm{MR}}^{(2)}$.

REMARK 2.22. For any $f = \sum_{i=1}^n a_i e_i^*$ in W and any signs $(\varepsilon_i)_{i\in N_1}$ (or $(\varepsilon_i)_{i\in N_2}$), there exist $(b_i)_{i=1}^n$ such that $b_i = \varepsilon_i a_i$ for all $i \in N_1 \cap \{1,\dots,n\}$ (or $i \in N_2 \cap \{1,\dots,n\}$) and $g = \sum_{i=1}^n b_i e_i^*$ is in W. Hence, the sequences $(e_i)_{i\in N_1}$ and $(e_i)_{i\in N_2}$ are 1-unconditional.

DEFINITION 2.23. We will call an x in $X_{MR}^{(2)}$ a *weighted vector* if $x = \frac{1}{\sqrt{\mu_\ell}} \sum_{i\in E} \varepsilon_i e_i$ with $\#E = \mu_\ell$ and $\varepsilon_i \in \{-1,1\}$. We also define the *weight* of x as $w(x) = \mu_\ell$.

Moreover, any $f = \frac{1}{\sqrt{\mu_\ell}} \sum_{i\in E} \varepsilon_i e_i^*$ in W with $\#E = \mu_\ell$ and $\varepsilon_i \in \{-1,1\}$ will be called a *weighted functional* and we define the *weight* of f as $w(f) = \mu_\ell$.

LEMMA 2.24. *Let $x_1,\dots,x_n$ be successive weighted vectors with increasing weights, and $f_1,\dots,f_m$ be successive functionals with increasing weights. If $w(x_i) \neq w(f_j)$ for all $1 \le i \le n$ and $1 \le j \le m$, then $\sum_{i=1}^m \sum_{i=1}^n |f_j(x_i)| \le \frac{1}{2}$.*

Proof. For x, f weighted with $w(x) = \mu_\ell$ and $w(f) = \mu_k$ such that $\mu_\ell \neq \mu_k$, we have $|f(x)| \le \frac{\min\{\sqrt{\mu_\ell},\sqrt{\mu_k}\}}{\max\{\sqrt{\mu_\ell},\sqrt{\mu_k}\}}$ and hence $|f(x)| \le \sqrt{\mu_\ell}/\sqrt{\mu_k}$ if $\mu_\ell < \mu_k$, and $|f(x)| \le \sqrt{\mu_k}/\sqrt{\mu_\ell}$ if $\mu_k < \mu_\ell$. Let $w(x_i) = \mu_{\ell_i}$ and $w(f_j) = \mu_{k_j}$. Then $|f_j(x_i)| \le \min\Big\{\frac{\sqrt{\mu_{\ell_i}}}{\sqrt{\mu_{k_j}}}, \frac{\sqrt{\mu_{k_j}}}{\sqrt{\mu_{\ell_i}}}\Big\}$ for each pair (i,j), and since each pair (μ_{ℓ_i},μ_{k_j}) appears only once, we have

$$\sum_{j=1}^m \sum_{i=1}^n |f_j(x_i)| \le \sum_{j=1}^\infty \sum_{j>i} \frac{\sqrt{\mu_i}}{\sqrt{\mu_j}} \le \frac{1}{2}. \ \blacksquare$$

PROPOSITION 2.25. *Let $(E_k^i)_{i=1,\,k=1}^{2,n}$ be a special sequence. Define the vector $x_k^i = (1/\sqrt{\#E_k^i}) \sum_{i\in E_k^i} e_i$ for $1 \le k \le n$ and $i = 1,2$. Then*

$$\Big\|\sum_{i=1}^2 \sum_{k=1}^n x_k^i\Big\| \ge 2n$$

whereas $\|\sum_{i=1}^2 \sum_{k=1}^n (-1)^i x_k^i\| \le 5$.

Proof. Set $\mu_{j_k} = \#E_k^1$ for $1 \le k \le n$. The first part follows easily from the fact that $f = \sum_{i=1}^2 \sum_{k=1}^n (1/\sqrt{\#E_k^i}) \sum_{j\in E_k^i} e_j^*$ is in W. For the second part set $y = \sum_{i=1}^2 \sum_{k=1}^n (-1)^i x_k^i$ and let $g = \sum_{i=1}^2 \sum_{k=1}^m \frac{1}{\sqrt{\mu_{j_k}}} \sum_{j\in F_k^i} e_j^*$ in W_2 and also $k_0 = \min\{k : \#E_k^1 \neq \#F_k^1\}$, under the convention $\min\emptyset = m+1$. If $k_0 = 1$, then the previous lemma yields $|g(y)| \le 1/2$. Otherwise, by Remark 2.19 and Lemma 2.24,

(i) if $k_0 > 2$, then $g^1_k(y) = -g^2_k(y)$ for every $1 \le k < k_0 - 1$,
(ii) $|g^i_{k_0-1}(y)| \le 1$ and $\sum_{k=k_0}^m |g^i_k(y)| \le 1/2$ for $i = 1, 2$.

Hence $|g(y)| \le 3$. Finally, in the general case of $g \in W$, using similar arguments we conclude that $|g(y)| \le 5$. ∎

Proposition 2.26. *Let $M \subset \mathbb{N}$ be such that $M \cap N_1$ and $M \cap N_2$ are both infinite sets. Then the sequence $(e_i)_{i\in M}$ is not unconditional.*

Proof. We may choose for each $n \in \mathbb{N}$ a special sequence $(E^i_k)^{2,n}_{i=1,\,k=1}$ such that $E^1_k \subset M \cap N_1$ and $E^2_k \subset M \cap N_2$ for every $1 \le k \le n$, and apply Proposition 2.25 to conclude that

$$\sup\Big\{\Big\|\sum_{i\in M} \varepsilon_i a_i e_i\Big\| : \varepsilon_i \in \{-1,1\},\ \Big\|\sum_{i\in M} \varepsilon_i a_i e_i\Big\| \le 1\Big\} \ge \frac{2n}{5}.$$

Since n is arbitrary, it follows that $(e_i)_{i\in M}$ is not unconditional. ∎

The following problem is however open.

Problem 1. *Let $(e^1_n)_n$ and $(e^2_n)_n$ be subsymmetric sequences in a Banach space, i.e. spreading and unconditional. Do there exist infinite subsets M, L of $\mathbb{N}$ such that $\{e^1_n\}_{n\in M} \cup \{e^2_n\}_{n\in L}$ is unconditional?*

Despite the fact that for $(e_n)_{n\in N_1}$ and $(e_n)_{n\in N_2}$ any subsequences fail to form a common unconditional sequence, the following more general result shows that we may find further block subsequences which satisfy this property.

Proposition 2.27. *Let $(x_n)_n$ and $(y_n)_n$ be unconditional sequences in a Banach space X. There exist block sequences $(z_n)_n$ and $(w_n)_n$ of $(x_n)_n$ and $(y_n)_n$ respectively such that $\{z_n\}_{n\in\mathbb{N}} \cup \{w_n\}_{n\in\mathbb{N}}$ is an unconditional sequence.*

Proof. Assume that there exist subsequences $(x_n)_{n\in M_1}$ and $(y_n)_{n\in M_2}$ such that $d(S_Z, S_Y) > 0$, where $Z = \operatorname{span}\{x_n\}_{n\in M_1}$ and $Y = \operatorname{span}\{y_n\}_{n\in M_2}$. Then $Y + Z$ is closed. Hence $Y + Z = Y \oplus Z$ by the Closed Graph Theorem and this shows that $\{x_n\}_{n\in M_1} \cup \{y_n\}_{n\in M_2}$ is unconditional.

Otherwise, we choose by induction normalized blocks $(z_n)_n$ and $(w_n)_n$ of $(x_n)_n$ and $(y_n)_n$ respectively with $\sum_{n=1}^\infty \|z_n - w_n\| < 1/(2C)$, where C is the basis constant of $(x_n)_n$, and hence also of $(z_n)_n$. Then, by the principle of small perturbations, $\{z_{2n}\}_{n\in\mathbb{N}} \cup \{w_{2n-1}\}_{n\in\mathbb{N}}$ is equivalent to $(z_n)_n$, which is unconditional. ∎

Remark 2.28. A natural question arising from the previous proposition is whether every space generated by two unconditional sequences is unconditionally saturated. The answer is negative and this follows from a well known more general result. Let X be a Banach space with a Schauder basis $(x_n)_n$, Y be a separable Banach space and $(d_n)_n$ be a dense subset of the unit ball of Y. Then the sequences $(x_n)_n$ and $(y_n)_n$ with $y_n = x_n + d_n/2^n$ are equivalent and generate the space $X \oplus Y$. Hence if $(x_n)_n$ is unconditional and Y contains no unconditional sequence, we obtain the desired result. We thank Bill Johnson for bringing this classical argument to our attention.

3. Joint spreading models. We introduce the notion of l-joint spreading models which is the central concept of this paper. It describes the joint asymptotic behavior of a finite collection of sequences. As is demonstrated in [AM1], in certain spaces this behavior may be radically more rich than the one of usual spreading models. It is worth pointing out that spreading models have been tied to the study of bounded linear operators [AM2] and the present paper clarifies that joint spreading models are no exception.

DEFINITION 3.1. Let $l \in \mathbb{N}$, $(x_n^1)_n, \dots, (x_n^l)_n$ be Schauder basic sequences in a Banach space $(X, \|\cdot\|)$ and $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ be a sequence in a Banach space $(E, \|\cdot\|_*)$.

Let $M \in [\mathbb{N}]^\infty$. We will say that the l-tuple $((x_n^i)_{n\in M})_{i=1}^l$ *generates* $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ *as an l-joint spreading model* if the following is satisfied. There exists a null sequence $(\delta_n)_n$ of positive reals such that for every $k \in \mathbb{N}$, $(a_{ij})_{i=1,\, j=1}^{l,k} \subset [-1, 1]$ and every strict plegma family $(s_i)_{i=1}^l \in \text{S-Plm}_l([M]^k)$ with $M(k) \le s_1(1)$, we have

$$\left| \Big\| \sum_{j=1}^{k} \sum_{i=1}^{l} a_{ij} x_{s_i(j)}^i \Big\| - \Big\| \sum_{j=1}^{k} \sum_{i=1}^{l} a_{ij} e_j^i \Big\|_* \right| < \delta_k.$$

We will also say that $((x_n^i)_n)_{i=1}^l$ *admits* $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ *as an l-joint spreading model* if there exists $M \in [\mathbb{N}]^\infty$ such that $((x_n^i)_{n\in M})_{i=1}^l$ generates $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$.

Finally, for a subset A of X, we will say that A *admits* $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ *as an l-joint spreading model* if there exists an l-tuple $((x_n^i)_n)_{i=1}^l$ of sequences in A which admits $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ as an l-joint spreading model.

Notice that for $l = 1$, the previous definition recovers the classical Brunel–Sucheston spreading models.

REMARK 3.2. Let $(x_n^1)_n, \dots, (x_n^l)_n$ be Schauder basic sequences in a Banach space $(X, \|\cdot\|)$. Let also $M \in [\mathbb{N}]^\infty$ be such that the l-tuple $((x_n^i)_{n\in M})_{i=1}^l$ generates the sequence $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ as an l-joint spreading model. Then the following hold:

(i) For every $1 \le i \le l$, the sequence $(e_n^i)_n$ is the spreading model admitted by $(x_n^i)_n$.
(ii) The sequence $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ is plegma spreading. Although l-joint spreading models are defined using strict plegma families, these sequences behave in a spreading way that involves plegma families.

(iii) For every $M' \in [M]^\infty$, $((x_n^i)_{n\in M'})_{i=1}^l$ generates $(e_n^i)_{i=1,\,n\in\mathbb{N}}^l$ as an l-joint spreading model.
(iv) For every $(\delta_n)_n$ null sequence of positive reals there exists $M' \in [M]^\infty$ such that $((x_n^i)_{n\in M'})_{i=1}^l$ generates $(e_n^i)_{i=1,\,n\in\mathbb{N}}^l$ as an l-joint spreading model with respect to $(\delta_n)_n$.
(v) If $|||\cdot|||$ is an equivalent norm on X, then every l-joint spreading model admitted by $(X, \|\cdot\|)$ is equivalent to an l-joint spreading model admitted by $(X, |||\cdot|||)$.

Next we prove a Brunel–Sucheston type result for l-joint spreading models.

THEOREM 3.3. *Let $l \in \mathbb{N}$ and X be a Banach space. Then every l-tuple of Schauder basic sequences in X admits an l-joint spreading model.*

First we present the following combinatorial lemma which will yield the theorem.

LEMMA 3.4. *Let $(x_n^1)_n, \dots, (x_n^l)_n$ be bounded sequences in a Banach space X and $(\delta_n)_n$ be a decreasing null sequence of positive real numbers. Then for every $M \in [\mathbb{N}]^\infty$, there exists $L \in [M]^\infty$ such that*

$$\left|\Big\|\sum_{j=1}^k\sum_{i=1}^l a_{ij}x^i_{s_i(j)}\Big\| - \Big\|\sum_{j=1}^k\sum_{i=1}^l a_{ij}x^i_{t_i(j)}\Big\|\right| < \delta_k$$

for every $k \in \mathbb{N}$, $(a_{ij})_{i=1,\,j=1}^{l,k} \subset [-1,1]$ and $(s_i)_{i=1}^l, (t_i)_{i=1}^l \in \text{S-Plm}_l([L]^k)$ with $s_1(1), t_1(1) \geq L(k)$.

Proof. Let $C > 0$ be such that $\|x_n^i\| < C$ for all $i = 1, \dots, l$ and $n \in \mathbb{N}$, and set $L_0 = M$. We will construct, by induction, a decreasing sequence $(L_k)_{k\geq 0}$ such that for every $k \in \mathbb{N}$, $(a_{ij})_{i=1,\,j=1}^{l,k} \subset [-1,1]$ and $(s_i)_{i=1}^l, (t_i)_{i=1}^l \in \text{S-Plm}_l([L_k]^k)$,

$$\left|\Big\|\sum_{j=1}^k\sum_{i=1}^l a_{ij}x^i_{s_i(j)}\Big\| - \Big\|\sum_{j=1}^k\sum_{i=1}^l a_{ij}x^i_{t_i(j)}\Big\|\right| < \delta_k.$$

Suppose that $L_0, \dots, L_{k-1}$ have been chosen for some $k \in \mathbb{N}$. Let A be a finite $\frac{\delta_k}{4klC}$-net of $[-1,1]$ and B be a partition of $[0, lC]$ consisting of disjoint intervals with length less than $\delta_k/4$. We set $\mathcal{F} = \{f : A^{kl} \to B\}$, and for $f \in \mathcal{F}$,

$$P_f = \Big\{(s_i)_{i=1}^l \in \text{S-Plm}_l([L_{k-1}]^k) : \Big\|\sum_{j=1}^k\sum_{i=1}^l a_{ij}x^i_{s_i(j)}\Big\| \in f(a)$$
$$\text{for all } a = ((a_{ij})_{j=1}^k)_{i=1}^l \in A^{kl}\Big\}.$$

Then $\text{S-Plm}_l([L_{k-1}]^k) = \bigcup_{f\in\mathcal{F}} P_f$ and by Theorem 1.2 there exist $L_k \in [\mathbb{N}]^\infty$ and $f \in \mathcal{F}$ such that $\text{S-Plm}_l([L_k]^k) \subset P_f$. Hence for all $(a_{ij})_{i=1,\,j=1}^{l,k} \subset A$ and $(s_i)_{i=1}^l, (t_i)_{i=1}^l \in \text{S-Plm}_l([L_k]^k)$, we have

$$\Big|\Big\|\sum_{j=1}^{n}\sum_{i=1}^{l} a_{ij}x^i_{s_i(j)}\Big\| - \Big\|\sum_{j=1}^{n}\sum_{i=1}^{l} a_{ij}x^i_{t_i(j)}\Big\|\Big| < \frac{\delta_k}{4}.$$

Since A is a net of $[-1,1]$, it is easy to see that L_k is as desired. Finally, choosing L to be a diagonalization of $(L_k)_k$ completes the proof. ∎

Proof of Theorem 3.3. Let $(x^1_n)_n, \ldots, (x^l_n)_n$ be Schauder basic sequences. Observe that Lemma 3.4 yields an infinite subset L of $\mathbb{N}$ such that for every $k \in \mathbb{N}$ and $(a_{ij})_{i=1,\,j=1}^{l,k} \subset [-1,1]$ and every sequence $((s^n_i)_{i=1}^l)_n$ of strict plegma families in $[L]^k$ with $\lim s^n_1(1) = \infty$, the sequence $(\|\sum_{j=1,\,i=1}^{k,l} a_{ij}x^i_{s^n_i(j)}\|)_n$ is Cauchy, with the limit independent of the choice of $((s^n_i)_{i=1}^l)_n$.

Denote by $(e_n)_n$ the usual basis of $c_{00}(\mathbb{N})$, and for every $i = 1,\ldots,l$ and $n \in \mathbb{N}$, set $e^i_n = e_{k(i,n)}$, where $k(i,n) = (n-1)l + in$. Using the above observation, we define a seminorm $\|\cdot\|_*$ on $c_{00}(\mathbb{N})$ as follows:

$$\Big\|\sum_{j=1}^{k}\sum_{i=1}^{l} a_{ij}e^i_j\Big\|_* = \lim_n \Big\|\sum_{j=1}^{k}\sum_{i=1}^{l} a_{ij}x^i_{s^n_i(j)}\Big\|$$

where $((s^n_i)_{i=1}^l)_n \subset \text{Plm}_l([L]^k)$ with $s^n_1(1) \to \infty$ and $(a_{ij})_{i=1,\,j=1}^{l,k} \in [-1,1]$. Since each $(x^i_n)_n$ is a Schauder basic sequence and hence does not contain any norm convergent subsequences, a modification of [BL, Proposition 1.B.2] shows that $\|\cdot\|_*$ is a norm. Denote by E the completion of $c_{00}(\mathbb{N})$ with respect to this norm and notice that the l-tuple $((x^i_n)_{n\in L})_{i=1}^l$ generates the sequence $(e^i_n)_{i=1,\,n\in\mathbb{N}}^l$ in E as an l-joint spreading model. ∎

The following proposition is an immediate consequence of the definition of l-joint spreading models and Theorem 2.12.

PROPOSITION 3.5. *Let $(x^1_n)_n, \ldots, (x^l_n)_n$ be Schauder basic sequences in a Banach space X such that each one is either weakly null, equivalent to the basis of ℓ_1, or non-trivial weak-Cauchy, and the l-tuple $(x^i_n)_{i=1,\,n\in\mathbb{N}}^l$ generates the sequence $(e^i_n)_{i=1,\,n\in\mathbb{N}}^l$ as an l-joint spreading model. Let $I \subset \{1,\ldots,l\}$ be such that $(x^i_n)_n$ is a non-trivial weak-Cauchy sequence with $w^*\text{-}\lim x^i_n = x^{**}_i$ for every $i \in I$ and let $F = \operatorname{span}\{x^{**}_i\}_{i\in I}$. If $X \cap F = \{0\}$, then $(e^i_n)_{i=1,\,n\in\mathbb{N}}^l$ is a Schauder basic sequence.*

The next example demonstrates that the opposite statement to the above is not always true, that is, $(e^i_n)_{i=1,\,n\in\mathbb{N}}^l$ may be Schauder basic whereas $X \cap F \neq \{0\}$.

EXAMPLE 3.6. Define a norm on $c_{00}(\mathbb{N})$ by $\|x\| = \sup \sum_{i=1}^{n} |\sum_{k\in I_i} x(k)|$, where the supremum is taken over all finite collections $I_1, \ldots, I_n$ of successive intervals of natural numbers with $n \le \min I_1$. Denote by X the completion of $c_{00}(\mathbb{N})$ with respect to this norm. Then the usual basis $(e_n)_n$ is a non-trivial weak-Cauchy sequence that generates a spreading model equivalent to the unit vector basis of ℓ_1. Consider the sequences $e_n^1 = e_{2n-1} - e_1$ and $e_n^2 = e_{2n} + e_1$. As follows from Proposition 2.5, none of their subsequences is a common Schauder basic sequence whereas the l-joint spreading model admitted by $(e_n^i)_{i=1,\, n\in\mathbb{N}}^2$ is equivalent to the unit vector basis of ℓ_1.

Recall that spreading models generated by weakly null sequences are unconditional. This is extended to joint spreading models by an easy modification of the classical case [BL, Proposition 5.1].

PROPOSITION 3.7. *Let $(x_n^1)_n, \ldots, (x_n^l)_n$ be weakly null Schauder basic sequences in a Banach space X that admit $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ as an l-joint spreading model. Then $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ is 1-suppression unconditional and hence for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every $(s_i)_{i=1}^l \in \text{S-Plm}_l([\mathbb{N}]^k)$ with $n \le s_1(1)$ the sequence $(x_{s_i(j)}^i)_{i=1,\, j=1}^{l,k}$ is $(1+\varepsilon)$-suppression unconditional.*

REMARK 3.8. The notion of l-joint spreading models can be naturally extended, by a diagonalization argument, to ω-joint spreading models which are ω-plegma spreading sequences and are generated by countably many Schauder basic sequences.

4. Spaces with a unique joint spreading model. In this section we study spaces that admit a uniformly unique joint spreading model with respect to certain families of sequences. In the first part we prove the uniform uniqueness of l-joint spreading models for the classical ℓ_p and c_0 spaces. Then we pass to Asymptotic ℓ_p spaces [MMT] and in the last part we study this problem for the James Tree space.

DEFINITION 4.1. Let $\mathscr{F}$ be a family of normalized sequences in a Banach space X. We will say that X admits a *uniformly unique l-joint spreading model* with respect to $\mathscr{F}$ if there exists $K > 0$ such that, for every $l \in \mathbb{N}$, any two l-joint spreading models generated by sequences from $\mathscr{F}$ are K-equivalent.

REMARK 4.2. Let $\mathscr{F}$ be a family of normalized sequences in a Banach space X such that, for some $l \in \mathbb{N}$, there exists $K_l > 0$ such that any two l-joint spreading models generated by sequences from $\mathscr{F}$ are K_l-equivalent.

(i) For every $l' < l$, there exists $K_{l'} \le K_l$ such that any two l'-joint spreading models generated by sequences from $\mathscr{F}$ are $K_{l'}$-equivalent.

(ii) The space X may fail to admit a uniformly unique l-joint spreading model with respect to $\mathscr{F}$. For examples of such spaces see [AM1] and Definition 5.28 below.

NOTATION 4.3. For a Banach space X, we will denote by $\mathscr{F}(X)$ the set of all normalized Schauder basic sequences in X, by $\mathscr{F}_0(X)$ its subset consisting of the sequences that are weakly null, and by $\mathscr{F}_C(X)$ the set of all normalized C-Schauder basic sequences in X. Finally, if X has a Schauder basis, we shall denote by $\mathscr{F}_b(X)$ the set of all normalized block sequences in X.

Next we present some examples of spaces admitting a uniformly unique l-joint spreading model. We start with the classical sequence spaces ℓ_p and c_0.

PROPOSITION 4.4. *Each of the following spaces admits a uniformly unique l-joint spreading model which is in fact equivalent to its unit vector basis:*

(i) *the spaces ℓ_p, for $1 < p < \infty$, with respect to $\mathscr{F}(\ell_p)$,*
(ii) *the space ℓ_1 with respect to $\mathscr{F}_b(\ell_1)$, but not with respect to $\mathscr{F}(\ell_1)$,*
(iii) *the space c_0 with respect to $\mathscr{F}_0(c_0)$, but not with respect to $\mathscr{F}(c_0)$.*

It is immediate to see that the above remain valid for the spaces $\ell_p(\Gamma)$ for $1 \leq p < \infty$, and $c_0(\Gamma)$ for any infinite set Γ.

REMARK 4.5. For $C \geq 1$, the space ℓ_1 admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_C(\ell_1)$. To see this, let $(x_n)_n$ be an arbitrary normalized C-Schauder basic sequence in ℓ_1. Passing, if necessary, to a subsequence, we see that $(x_n)_n$ has a pointwise limit x_0 in ℓ_1. That is, $\lim_n e_i^*(x_n) = e_i^*(x_0)$ for all $i \in \mathbb{N}$. Also, if we set $z_n = x_n - x_0$, then $\lim_n \|z_n\| = \lambda$ may be assumed to exist. It follows that $\|x_0\| + \lambda = 1$ and that $0 < \lambda \leq 1$. We can also assume that $(\lambda^{-1} z_n)_{n \geq n_0}$ is $(1 + 1/n_0)$-equivalent to the usual basis of ℓ_1. We conclude that for any $M \in \mathbb{N}$,

$$\begin{aligned} M(1-\lambda) = M\|x_0\| \leq \lim_k \Big\| \sum_{n=1}^{M} x_{n+k} \Big\| &\leq C \lim_k \Big\| \sum_{n=1}^{M} x_{n+k} - \sum_{n=M+1}^{2M} x_{n+k} \Big\| \\ &= C \lim_k \Big\| \sum_{n=1}^{M} z_{n+k} - \sum_{n=M+1}^{2M} z_{n+k} \Big\| = C2M\lambda. \end{aligned}$$

Therefore, $\lambda \geq 1/(2C+1)$. Hence if $(x_n^i)_n$, $1 \leq i \leq l$, is an l-tuple of C-Schauder basic sequences, we may pass to subsequences such that $(x_n^i)_n$ converges pointwise to some x_0^i for $1 \leq i \leq l$ and if we set $(z_n^i)_n = (x_n^i - x_0^i)$ for $1 \leq i \leq l$ then these sequences are pointwise null and they are all bounded below by $1/(2C+1)$. We may then conclude that for any $\varepsilon > 0$, passing to appropriate subsequences, $(x_n^i)_n$, $1 \leq i \leq m$, is jointly $(2C+1+\varepsilon)$-equivalent to the unit vector basis of ℓ_1.

REMARK 4.6. Although for any $C \geq 1$ the space ℓ_1 admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_C(\ell_1)$, this is no longer true for spaces with the Schur property. For example, define for each $n \in \mathbb{N}$ the norm $\|\cdot\|_n$ on ℓ_1 by $\|x\|_n = \max\{\|x\|_{\ell_2}, n^{-1}\|x\|_{\ell_1}\}$. Set $X = (\sum_n \oplus X_n)_{\ell_1}$, where $X_n = (\ell_1, \|\cdot\|_n)$, which has the Schur property. Although every spreading model of this space is equivalent to the unit vector basis of ℓ_1, this does not happen for a uniform constant.

Another example of spaces admitting a uniformly unique joint spreading model are asymptotic ℓ_p spaces. We start with their definition.

DEFINITION 4.7 ([MT]). A Banach space X with a normalized Schauder basis is *asymptotic* ℓ_p (resp. *asymptotic* c_0) if there exists $C > 0$ such that any finite sequence $(x_i)_{i=1}^n$ of normalized vectors in X with $n < \operatorname{supp}(x_1) < \cdots < \operatorname{supp}(x_n)$ is C-equivalent to the standard basis of ℓ_p^n for $1 \leq p < \infty$ (resp. of c_0^n).

The classical examples of asymptotic ℓ_p spaces are Tsirelson's original space [T] and its p-convexifications [FJ]. The next proposition follows easily from the above definition and the fact that an asymptotic ℓ_p space, for $1 < p < \infty$, is reflexive.

PROPOSITION 4.8. *Every asymptotic ℓ_p or asymptotic c_0 space X admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_b(X)$. Moreover, every asymptotic ℓ_p space, for $1 < p < \infty$, admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}(X)$.*

The following proposition concerns spaces with uniformly unique joint spreading models with respect to families that have certain stability properties. The joint spreading models of such spaces are unconditional and sometimes even equivalent to some ℓ_p or to c_0. Families with such properties play an important role in the study of the UALS in the next section.

PROPOSITION 4.9. *Let X be a Banach space that admits a K-uniformly unique l-joint spreading model with respect to a family $\mathscr{F}$ of normalized Schauder basic sequences. Assume that $\mathscr{F}$ has the following properties:*

(a) *If $(x_j)_j$ is in $\mathscr{F}$ then any subsequence of $(x_j)_j$ is in $\mathscr{F}$.*
(b) *If $(x_j)_j$ is in $\mathscr{F}$ then there exists an infinite subset $L = \{l_i : i \in \mathbb{N}\}$ of $\mathbb{N}$ such that if $z_i = \|x_{l_{2i-1}} - x_{l_{2i}}\|^{-1}(x_{l_{2i-1}} - x_{l_{2i}})$ for $i \in \mathbb{N}$, then the sequence $(z_i)_i$ is in $\mathscr{F}$.*
(c) *If $(x_j)_j$ is in $\mathscr{F}$ and $(\lambda_i)_{i=1}^N$ is a finite sequence of scalars, not all zero, then there exists an infinite subset $L = \{l_i : i \in \mathbb{N}\}$ of $\mathbb{N}$ such that if*

$$z_n = \Big\| \sum_{i=1}^{N} \lambda_i x_{l_{N(n-1)+i}} \Big\|^{-1} \Big(\sum_{i=1}^{N} \lambda_i x_{l_{N(n-1)+i}} \Big) \quad \text{for } n \in \mathbb{N},$$

then the sequence $(z_i)_i$ is in $\mathscr{F}$.

Then the following statements hold:

(i) *If* $\mathscr{F}$ *satisfies* (a) *then every* l*-joint spreading model admitted by an* l*-tuple of sequences in* $\mathscr{F}$ *is spreading when enumerated with the natural plegma order and it is* K*-equivalent to the spreading model generated by any sequence in* $\mathscr{F}$.

(ii) *If* $\mathscr{F}$ *satisfies* (a) *and* (b) *then every* l*-joint spreading model admitted by sequences in* $\mathscr{F}$ *is* K*-suppression unconditional.*

(iii) *If* $\mathscr{F}$ *satisfies* (a) *and* (c) *then every* l*-joint spreading model admitted by sequences in* $\mathscr{F}$ *is* K*-equivalent to the unit vector basis of* ℓ_p (*for* $p = \infty$ *we mean the unit vector basis of* c_0).

Proof. (i) follows by taking an arbitrary sequence $(x_j)_j$ in $\mathscr{F}$, passing to a subsequence that generates some spreading model $(e_i)_i$, and then taking disjointly indexed subsequences $(x_i^1)_i, \dots, (x_i^l)_i$, which by assumption are all in $\mathscr{F}$. Clearly, they generate an l-joint spreading model that is isometrically equivalent to $(e_i)_i$. We conclude that any l-joint spreading model generated by an l-tuple of sequences in $\mathscr{F}$ is K-equivalent to $(e_i)_i$, when endowed with the natural plegma order.

For (ii) it is sufficient, by (i), to show that any spreading model admitted by a sequence in $\mathscr{F}$ has the desired property. Pick an arbitrary sequence $(x_j)_j$ in $\mathscr{F}$ which by (a) may be chosen to generate some spreading model $(e_j)_j$. Applying (b) to $(x_j)_j$ we can deduce that there is a sequence in $\mathscr{F}$ that generates as a spreading model the sequence $(\|e_{2j-1} - e_{2j}\|^{-1}(e_{2j-1} - e_{2j}))_j$, which by [BL, Proposition 4.3] is 1-suppression unconditional. Observe that any sequence that is K-equivalent to a 1-suppression unconditional sequence is K-suppression unconditional.

Assume now that (a) and (c) hold. Clearly, (a) and (c) together imply (b) so we may pick up where we left off, namely having at hand a sequence $(x_j)_j$ in $\mathscr{F}$ that generates a spreading model that is 1-suppression unconditional. By [MMT, 1.6.3], as a direct application of Krivine's theorem [Kr], [L], for any $m \in \mathbb{N}$ and $\varepsilon > 0$, we may choose scalars $\lambda_1, \dots, \lambda_N$ such that any m terms of the resulting sequence $(z_n)_n$ are $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^m for some $1 \le p \le \infty$. This means that there exists a constant K such that, for any $m \in \mathbb{N}$, there exists $1 \le p_m \le \infty$ such that the first m terms of any spreading model generated by a sequence from $\mathscr{F}$ are K-equivalent to the unit vector basis of ℓ_{p_m}. Taking a limit point of $(p_m)_m$ yields the conclusion. ■

4.1. Coordinate-free asymptotic ℓ_p spaces (Asymptotic ℓ_p spaces). Notice that Definition 4.7 of an asymptotic ℓ_p space from [MT] depends on the Schauder basis of X and not only on X. A coordinate-free version of this definition can be found in [MMT, Subsection 1.7] and it is based on a game of two players (S) and (V). In each turn of the game player (S) chooses a closed finite-codimensional subspace Y of X and player (V) chooses a normalized vector $y \in Y$. A Banach space X is called *Asymptotic* ℓ_p if there exists a constant C such that, for every $n \in \mathbb{N}$, player (S) has a winning strategy in the game $G(p, n, C)$, that is, to force in n steps player (V) to choose a sequence $(y_i)_{i=1}^n$ that is C-equivalent to the unit vector basis of ℓ_p^n (or c_0^n for $p = \infty$). We point out that the original formulation of this property is different. The equivalence of the original definition with this more convenient version follows from [MMT, Subsection 1.5].

Next we show that for a separable Asymptotic ℓ_p space X for $1 \leq p \leq \infty$, there exists a certain family of sequences in X, described in Proposition 4.11, with respect to which X admits a uniformly unique l-joint spreading model. This family has certain properties that $\mathscr{F}_0(X)$ fails when X contains ℓ_1 and this result will be used in the next section to prove that an Asymptotic ℓ_1 space satisfies the UALS. We start with the following lemma.

LEMMA 4.10. *Let X be a separable C-Asymptotic ℓ_p space for $1 \leq p \leq \infty$. Then there exists a countable collection $\mathscr{Y}$ of finite-codimensional subspaces of X such that, for every $\varepsilon > 0$ and $n \in \mathbb{N}$, player (S) has a winning strategy in the game $G(p, n, C+\varepsilon)$ when choosing finite-codimensional subspaces from $\mathscr{Y}$.*

Proof. If X is C-asymptotic ℓ_p in the sense described above, we shall, for fixed $n \in \mathbb{N}$, assume the role of player (V) and let player (S) follow a winning strategy during a multitude of outcomes in a game of $G(p, n, C)$. More accurately, we will describe how to define a collection of vectors of X of the form $\{x_F^n : \emptyset \neq F \in [\mathbb{N}]^{\leq n}\}$ and a collection of closed finite-codimensional subspaces of X of the form $\{Y_F^n : F \in [\mathbb{N}]^{\leq n-1}\}$ that satisfy:

(i) for all $F \in [\mathbb{N}]^{\leq n-1}$, the norm-closure of $\{x_{F\cup\{i\}}^n : i > \max F\}$ is the unit sphere of Y_F^n (here, $\max\emptyset = 0$), and
(ii) for every $\{k_1, \ldots, k_m\}$ in $[\mathbb{N}]^{\leq n}$,
$$(Y_\emptyset^n, x_{\{k_1\}}^n), (Y_{\{k_1\}}^n, x_{\{k_1,k_2\}}^n), \ldots, (Y_{\{k_1,\ldots,k_{m-1}\}}^n, x_{\{k_1,\ldots,k_m\}}^n)$$
is the outcome of a game of $G(p, n, C)$ after m rounds in which player (S) has followed a winning strategy.

Player (S) initiates and he chooses a finite-codimensional subspace $Y_\emptyset^n$. As player (V), we choose a dense subset $\{x_{\{i\}}^n : i \in \mathbb{N}\}$ of the unit sphere of $Y_\emptyset^n$. If for some $1 \leq m < n$ we have chosen $\{x_F^n : \emptyset \neq F \in [\mathbb{N}]^{\leq m}\}$ and $\{Y_F^n : F \in [\mathbb{N}]^{\leq m-1}\}$, we complete the inductive step as follows: for every $F = \{k_1 < \cdots < k_m\}$, by assumption (ii), player (S) may continue following a winning strategy and choose a closed finite-codimensional subspace Y such that for every unit vector $y \in Y$ the sequence $(x_{k_i}^n)_{i=1}^m{}^\frown(y)$ is C-equivalent to the unit vector basis of ℓ_p^{m+1}. Set $Y_F^n = Y$ and then, for all $i > \max F$, choose a unit vector $x^n_{F\cup\{i\}}$ in Y_F such that the set $\{x^n_{F\cup\{i\}} : i > \max F\}$ is dense in the unit sphere of Y^n_F.

Set $\mathscr{Y} = \{Y^n_F : n \in \mathbb{N}, F \in [\mathbb{N}]^{\leq n-1}\}$ and fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Let us also take $\tilde{\varepsilon} > 0$ to be determined later. We will describe a winning strategy for player (S) in the game $G(p, n, C + \varepsilon)$, choosing finite-codimensional subspaces from $\mathscr{Y}$. Player (S) initiates the game and chooses the subspace $Y_1 = Y^n_\emptyset$, and player (V) chooses an arbitrary normalized vector y_1 from Y_1. Before the next turn, player (S) also chooses $k_1 \in \mathbb{N}$ such that $\|y_1 - x^n_{k_1}\| < \tilde{\varepsilon}/n$. Let $(Y_1, y_1), \ldots, (Y_m, y_m)$ be the outcome of the first m turns of the game for $1 \leq m < n$, while player (S) has also chosen $k_1, \ldots, k_m \in \mathbb{N}$ with $\|y_i - x^n_{k_i}\| < \tilde{\varepsilon}/n$ for $1 \leq i \leq m$. In the next turn, player (S) chooses the subspace $Y_{m+1} = Y^n_{\{k_1,\ldots,k_m\}}$ and $k_{m+1} \in \mathbb{N}$ such that $\|y_{m+1} - x^n_{k_{m+1}}\| < \tilde{\varepsilon}/n$, where y_{m+1} is the vector player (V) chose from Y_{m+1}. Hence if $(Y_1, y_1), \ldots, (Y_n, y_n)$ is the final outcome of the game, notice that the sequence $(x^n_{k_i})^n_{i=1}$ is C-equivalent to the unit vector basis of ℓ^n_p and $\|y_i - x^n_{k_i}\| < \tilde{\varepsilon}/n$ for all $1 \leq i \leq n$. If we take $1 \leq A, B$ with $AB \leq C$ such that $1/A \leq (\sum^n_{i=1} |a_i|^p)^{1/p} \leq \|\sum^n_{i=1} a_i x^n_{k_i}\| \leq B(\sum^n_{i=1} |a_i|^p)^{1/p}$ then we conclude that $(y_i)^n_{i=1}$ is $C(1 + \tilde{\varepsilon})/(1 - \tilde{\varepsilon}C)$-equivalent to the unit vector basis of ℓ^n_p. For $\tilde{\varepsilon}$ sufficiently small we deduce the conclusion. ∎

PROPOSITION 4.11. *Let X be a separable C-Asymptotic ℓ_p space for $1 \leq p \leq \infty$. There exists a countable subset $\mathscr{A}$ of X^* such that if*

$$\mathscr{F}_{0,\mathscr{A}} = \Big\{(x_n)_n : (x_n)_n \textit{ is normalized and } \lim_n f(x_n) = 0 \textit{ for all } f \in \mathscr{A}\Big\},$$

then X admits ℓ_p as a C^2-uniformly unique l-joint spreading model with respect to the family $\mathscr{F}_{0,\mathscr{A}}$.

Proof. Let $\mathscr{Y}$ be as in Lemma 4.10 and, for each $Y \in \mathscr{Y}$, choose a finite subset $f^Y_1, \ldots, f^Y_{k_Y}$ of X^* such that $Y = \bigcap^{k_Y}_{i=1} \ker f^Y_i$ and set $\mathscr{A} = \bigcup_{Y\in\mathscr{Y}} \{f^Y_1, \ldots, f^Y_{k_Y}\}$, which is a countable set. We will show that it is as desired. To that end, let $l \in \mathbb{N}$ and $(x^1_n)_n, \ldots, (x^l_n)_n$ be sequences in $\mathscr{F}_{0,\mathscr{A}}$ generating an l-joint spreading model $(e^i_n)^l_{i=1,n}$. Let $k \in \mathbb{N}$; we will show that $(e^i_j)^{l,k}_{i=1,j=1}$ is C-equivalent to the unit vector basis of ℓ^{lk}_p. Set $m = lk$, fix $\varepsilon > 0$ and, using Lemma 2.1, choose by induction normalized vectors $y_1, \ldots, y_m$ and $(s_i)^l_{i=1}$ in S-Plm$_l([\mathbb{N}]^k)$ such that

(i) $(Y_1, y_1), \ldots, (Y_m, y_m)$ is the outcome of the game $G(m, p, C + \varepsilon)$,
(ii) $Y_1, \ldots, Y_m \in \mathscr{Y}$,
(iii) for $1 \leq i \leq l$ and $1 \leq j \leq k$, if we take $n(i, j) = (i - 1)k + j$ then $\|y_{n(i,j)} - x^i_{s_i(j)}\| \leq \varepsilon/m$.

It follows that for any scalars $(a_{ij})_{i=1,\,j=1}^{l,k}$ with $|a_{ij}| \leq 1$, we have

$$\left|\Big\|\sum_{j=1}^{k}\sum_{i=1}^{l} a_{ij}x^i_{s_i(j)}\Big\| - \Big\|\sum_{j=1}^{k}\sum_{i=1}^{l} a_{ij}y_{n(i,j)}\Big\|\right| < \varepsilon.$$

As $(y_i)_{i=1}^m$ is $(C+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^m, the conclusion follows. ∎

The following is an immediate corollary of the above. In the general case, all Asymptotic ℓ_p spaces admit a uniformly unique joint spreading model with respect to the family of normalized weakly null Schauder basic sequences.

COROLLARY 4.12. *Every Asymptotic ℓ_p space X, for $1 \leq p \leq \infty$, admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_0(X)$. Moreover, every l-joint spreading model generated by a sequence from this family is equivalent to the unit vector basis of ℓ_p (or of c_0 if $p = \infty$).*

4.2. James Tree space. We show that the James Tree space JT admits a uniformly unique joint spreading model with respect to $\mathscr{F}_0(JT)$. This is however not true for joint spreading models with respect to $\mathscr{F}(JT)$ or $\mathscr{F}_b(JT)$.

NOTATION 4.13. We denote by $\mathscr{D}$ the *dyadic tree*, i.e. $\mathscr{D} = \{0,1\}^{<\infty}$, ordered by the initial part order. We will use S to denote segments of $\mathscr{D}$, and B to denote branches. For $m < n$, the *band* $Q_{[m,n]}$ is the set $\{s \in \mathscr{D} : m \leq |s| \leq n\}$. We set $c_{00}(\mathscr{D})$ to be the linear space of all eventually zero sequences $x : \mathscr{D} \to \mathbb{R}$. For a segment S of $\mathscr{D}$ we denote by S^* the linear functional on $c_{00}(\mathscr{D})$ defined as $S^*(x) = \sum_{s\in S} x(s)$.

DEFINITION 4.14 ([J2]). On $c_{00}(\mathscr{D})$ we define the norm

$$\|x\|_{JT} = \sup\Big(\sum_{i=1}^{n}\Big(\sum_{s\in S_i} x(s)\Big)^2\Big)^{1/2}$$

where the supremum is taken over all finite collections $S_1, \ldots, S_n$ of pairwise disjoint segments. The *James Tree space*, denoted by JT, is the completion of $c_{00}(\mathscr{D})$ with respect to the above norm.

REMARKS 4.15. (i) The following set is norming for JT:

$$W = \Big\{\sum_{i=1}^{n} b_i S_i^* : n \in \mathbb{N},\ \sum_{i=1}^{n} b_i^2 \leq 1,\ \{S_i\}_{i=1}^n \text{ pairwise disjoint segments}\Big\}.$$

(ii) Let $\varepsilon > 0$, $x \in JT$ with $\|x\| = 1$ and $\{S_i\}_{i\in I}$ be pairwise disjoint segments with the property that $|S_i^*(x)| \geq \varepsilon$ for every $i \in I$. Then $\#I \leq 1/\varepsilon^2$.

(iii) Let $S_1, \dots, S_n$ be pairwise disjoint segments and $b_1, \dots, b_n \in \mathbb{R}$. Then

$$\Big\| \sum_{i=1}^{n} b_i S_i^* \Big\|^2 \leq \sum_{i=1}^{n} b_i^2.$$

We will prove the following theorem.

THEOREM 4.16. *The space JT admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_0(JT)$, and every l-joint spreading model generated by sequences from this family is $\sqrt{2}$-equivalent to the unit vector basis of ℓ_2.*

This is a variant of the well known result due to I. Amemiya and T. Ito [AI] that every normalized weakly null sequence in JT contains a subsequence which is 2-equivalent to the usual basis of ℓ_2. From this it follows that every spreading model generated by a normalized weakly null sequence is 2-equivalent to the unit vector basis of ℓ_2. Our approach implies that every l-joint spreading model generated by sequences from $\mathscr{F}_0(JT)$ is equivalent to the unit vector basis of ℓ_2 with equivalence constant $\sqrt{2}$, which as mentioned in [FG], [Be] is the best possible.

As a consequence of the fact that the James space (see Definition 2.6) is isometric to a subspace of JT, it follows that J also admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_0(J)$.

We break up the proof of the theorem into several lemmas and we start with the following Ramsey type result.

DEFINITION 4.17. Let $(Q_{[p_n,q_n]})_n$ be successive bands in $\mathscr{D}$ and let $(F_n)_n$ be a sequence of finite subsets of JT. We will say that $(F_n)_n$ is a *weakly null level block family* with respect to $(Q_{[p_n,q_n]})_n$ if

(i) $\operatorname{supp}(x) \subset Q_{[p_n,q_n]}$ and $\|x\| = 1$ for every $n \in \mathbb{N}$ and $x \in F_n$,
(ii) the sequence $(x_n)_n$ is weakly null for any choice of $x_n \in F_n$.

LEMMA 4.18. *Let $(F_n)_n$ be a weakly null level block family with $\sup_n \#F_n < \infty$. Then, for every $\varepsilon > 0$, there exists an $L \in [\mathbb{N}]^\infty$ such that for every initial segment S there exists at most one $n \in L$ with $|S^*(x)| \geq \varepsilon$ for some $x \in F_n$.*

Proof. If the conclusion is false, then using Ramsey's theorem [Ra] we may assume that there exists $L \in [\mathbb{N}]^\infty$ such that, for any $m < n$ in L, there exist an initial segment $S_{m,n}$ and $x \in F_m$, $y \in F_n$ such that $|S_{m,n}^*(x)| \geq \varepsilon$ and $|S_{m,n}^*(y)| \geq \varepsilon$.

CLAIM. *Set $\mu = \max_n \#F_n/\varepsilon^2$. Then $\#\{S_{m,n}|_{[0,p_n]} : m \in L,\, m < n\} \leq \mu$ for every $n \in L$, where for a segment S and $p, q \in \mathbb{N}$ we denote $S|_{[p,q]} = S \cap Q_{[p,q]}$.*

Proof of Claim. If $\#\{S_{m,n}|_{[0,p_n]} : m \in L,\, m < n\} > \mu$ for some $n \in L$, then using the pigeon hole principle, we may find an $x \in F_n$

and $F \subset \{1,\ldots,n-1\}$ with $\#F > 1/\varepsilon^2$ such that $|S_{m,n}(x)| \geq \varepsilon$ and the segments $S_{m,n}|_{[p_n,q_n]}$ are pairwise disjoint for $m \in F$. This contradicts Remark 4.15(ii). ∎

Hence, for every $n \in L$, let $\{S_{m,n}|_{[0,p_n]} : m \in L,\ m < n\} = \{S_1^n,\ldots,S_{\mu(n)}^n\}$ with $\mu(n) \leq \mu$ and set $L_i^n = \{m \in L : m < n \text{ and } S_{m,n}|_{[0,p_n]} = S_i^n\}$ for $1 \leq i \leq \mu(n)$, and $L_i^n = \emptyset$ for $\mu(n) < i \leq \mu$. Notice that $\{m \in L : m < n\} = \bigcup_{i=1}^{\mu} L_i^n$ for all $n \in L$. Passing to a further subsequence we may assume that, for every $1 \leq i \leq \mu$, $(L_i^n)_{n\in L}$ converges pointwise and we denote that limit by L_i. Then it is easy to see that $L = \bigcup_{i=1}^{\mu} L_i$ and hence some L_{i_0} is an infinite subset of L such that, for every $n \in L_{i_0}$, there exists an initial segment S_n such that, for all $m < n$ in L_{i_0}, we have $|S_n^*(x_m)| \geq \varepsilon$ for some $x_m \in F_m$. Then there exist $M \in [L_{i_0}]^\infty$ and a sequence $(x_n)_{n\in M}$ with $x_n \in F_n$ such that $(S_n)_{n\in M}$ converges pointwise to a branch B and $|S_m^*(x_n)| \geq \varepsilon$ for all $m > n$ in M. Hence $|B(x_n)| \geq \varepsilon$ for all $n \in M$, which contradicts Definition 4.17(ii). ∎

Lemma 4.19. *Let $\varepsilon > 0$ and $(F_n)_n$ be a weakly null level block family with respect to $(Q_{[p_n,q_n]})_n$ and assume that $\sup_n \#F_n < \infty$. Then there exist an increasing sequence $(n_k)_k$ in $\mathbb{N}$ and a decreasing sequence $(\varepsilon_k)_k$ of positive reals such that*

(i) *for every $k \in \mathbb{N}$ and every initial segment S there exists at most one $k' > k$ such that $|S^*(x)| \geq \varepsilon_k$ for some $x \in F_{n_{k'}}$,*
(ii) $\sum_{k=1}^{\infty} 2^{q_{n_k}} \sum_{i=k}^{\infty} (i+1)\varepsilon_i < \varepsilon$.

Proof. Let $(\delta_n)_n$ be a sequence of positive reals such that $\sum_{n=1}^{\infty} \delta_n < \varepsilon$. We construct $(n_k)_k$ and $(\varepsilon_k)_k$ by induction on $\mathbb{N}$ as follows. We set $n_1 = 1$ and $L_1 = \mathbb{N}$ and choose ε_1 such that $2^{q_1} 2\varepsilon_1 < \delta_1$. Suppose that $n_1,\ldots,n_k$ and $\varepsilon_1,\ldots,\varepsilon_k$ have been chosen for some k in $\mathbb{N}$. Then Lemma 4.18 yields an $L_k \in [L_{k-1}]^\infty$ such that for every segment S there exists at most one $n \in L_k$ with $|S^*(x)| \geq \varepsilon_k$ for some $x \in F_n$. We then choose $n_{k+1} \in L_k$ with $n_{k+1} > n_k$ and $\varepsilon_{k+1} < \varepsilon_k$ such that

(a) $2^{q_{n_{k+1}}}(k+2)\varepsilon_{k+1} < \delta_{k+1}$,
(b) $2^{q_{n_m}} \sum_{i=m}^{k+1} (i+1)\varepsilon_i < \delta_m$ for every $m \leq k$.

It is easy to see that $(n_k)_k$ and $(\varepsilon_k)_k$ are as desired. ∎

Lemma 4.20. *Let $\varepsilon > 0$ and $(\varepsilon_n)_n$ be a decreasing sequence of positive reals. Let also $(F_n)_n$ be a weakly null level block family with respect to $(Q_{[p_n,q_n]})_n$ and assume that $\sup_n \#F_n < \infty$ and*

(i) *for every $n \in \mathbb{N}$ and every initial segment S there exists at most one $m > n$ such that $|S^*(x)| \geq \varepsilon_n$ for some $x \in F_m$,*
(ii) $\sum_{n=1}^{\infty} 2^{q_n} \sum_{i=n}^{\infty} (i+1)\varepsilon_i < \varepsilon$.

Then for every $n \in \mathbb{N}$ and every choice of $x_1, \dots, x_n$ with $x_i \in F_i$ and scalars $a_1, \dots, a_n$ we have

$$\Big(\sum_{i=1}^n a_i^2\Big)^{1/2} \le \Big\|\sum_{i=1}^n a_i x_i\Big\| \le (\sqrt{2}+\varepsilon)\Big(\sum_{i=1}^n a_i^2\Big)^{1/2}.$$

Proof. First observe that if $(x_n)_n$ is a sequence with each x_n in F_n, then for every $n \in \mathbb{N}$ and every segment S with $|M(x_{n-1})| < |\min S| \le |M(x_n)|$, where for $x \in c_{00}(\mathscr{D})$ we denote $M(x) = \max \operatorname{supp}(x)$, the following hold due to (i):

(a) $\#\{i > n : |S^*(x_i)| \ge \varepsilon_n\} \le 1$,
(b) $\#\{i > n : \varepsilon_{k-1} > |S^*(x_i)| \ge \varepsilon_k\} \le k$ for every $k > n$.

Now for each $1 \le i \le n$, there exist pairwise disjoint segments $S_1^i, \dots, S_{m_i}^i$ such that $S_j^i \subset Q_{[p_i,q_i]}$ and $\sum_{j=1}^{m_i}(S_j^{i*}(x_i))^2 = \|x_i\|$ and hence

$$\Big\|\sum_{i=1}^n a_i x_i\Big\| \ge \Big(\sum_{i=1}^n a_i^2 \sum_{j=1}^{m_i}(S_j^{i*}(x_i))^2\Big)^{1/2} \ge \Big(\sum_{i=1}^n a_i^2\Big)^{1/2}.$$

Pick pairwise disjoint segments $S_1, \dots, S_m$ and reals $b_1, \dots, b_m$ with $\sum_{j=1}^m b_j^2 \le 1$. For given $1 \le j \le m$, denote by $i_{j,1}$ the unique $1 \le i \le n$ such that $|M(x_{i_{j,1}-1})| < |\min S_j| \le |M(x_{i_{j,1}})|$ and also by $i_{j,2}$ the unique, if any, $i_{j,1} < i \le n$ such that $|S_j^*(x_{i_{j,2}})| \ge \varepsilon_{i_{j,1}}$. We set $S_{j,k} = S_j \cap Q_{[p_{i_{j,k}}, q_{i_{j,k}}]}$ for $k = 1, 2$, and $S_{j,3} = S_j \setminus (S_{j,1} \cup S_{j,2})$, and $J_i = \{j : i_{j,1} = i \text{ or } i_{j,2} = i\}$ for $1 \le i \le n$. Note that, by (a), each j appears in J_i for at most two i and so $\sum_{i=1}^n \sum_{j \in J_i} b_j^2 \le 2\sum_{j=1}^m b_j^2$. We thus calculate

$$\begin{aligned}\Big|\sum_{j=1}^m b_j S_{j,1}^*\Big(\sum_{i=1}^n a_i x_i\Big) + \sum_{j=1}^m b_j S_{j,2}^*\Big(\sum_{i=1}^n a_i x_i\Big)\Big| &= \Big|\sum_{i=1}^n a_i \sum_{j \in J_i} b_j S_j^*(x_i)\Big| \\ &\le \Big(\sum_{i=1}^n a_i^2\Big)^{1/2}\Big(\sum_{i=1}^n \big(\sum_{j \in J_i} b_j S_j^*(x_i)\big)^2\Big)^{1/2} \\ &\le \Big(\sum_{i=1}^n a_i^2\Big)^{1/2}\Big(\sum_{i=1}^n \sum_{j \in J_i} b_j^2\Big)^{1/2} \le \sqrt{2}\Big(\sum_{i=1}^n a_i^2\Big)^{1/2}.\end{aligned}$$

Finally, we set $G_i = \{j : |M(x_{i-1})| < |\min S_j| \le |M(x_i)|\}$ and we see that $\{1, \dots, m\} = \bigcup_{i=1}^n G_i$. Notice that $\#G_i \le 2^{q_i}$ and $|S_{j,3}^*(\sum_{k=1}^n x_k)| < \sum_{k=i}^\infty (k+1)\varepsilon_k$ for any $j \in G_i$. Then due to (b) and (ii), it follows that $\sum_{j=1}^m |S_{j,3}^*(\sum_{i=1}^n x_i)| < \varepsilon$ and hence

$$\Big|\sum_{j=1}^m b_j S_{j,3}^*\Big(\sum_{i=1}^n a_i x_i\Big)\Big| = \Big|\sum_{i=1}^n a_i \sum_{j=1}^m b_j S_{j,3}^*(x_i)\Big| < \varepsilon\Big(\sum_{i=1}^n a_i^2\Big)^{1/2}. \ \blacksquare$$

Proof of Theorem 4.16. Let $(x_n^1)_n, \dots, (x_n^l)_n$ in $\mathscr{F}_0(JT)$ be such that $((x_n^i)_n)_{i=1}^l$ generates a sequence $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ as an l-joint spreading model and by a sliding hump argument we may assume that each sequence $(x_n^i)_n$ is block. Hence we may choose $L \in [\mathbb{N}]^\infty$ such that the family $(F_n)_{n\in L}$ with $F_n = \{x_n^1, \dots, x_n^l\}$ is a weakly null level block family in JT which satisfies (i) and (ii) in Lemma 4.20. Then, since $((x_n^i)_{n\in L})_{i=1}^l$ also generates $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ as an l-joint spreading model, $(e_n^i)_{i=1,\, n\in\mathbb{N}}^l$ is $\sqrt{2}$-equivalent to the usual basis of ℓ_2 and therefore any two l-joint spreading models generated by sequences from $\mathscr{F}_0(JT)$ are 2-equivalent. ∎

Remark 4.21. The notion of asymptotic models, which appeared in [HO], also concerns the asymptotic behavior of countably many basic sequences. Although asymptotic models are different from joint spreading models, as B. Sari pointed out, a Banach space admits a uniformly unique asymptotic model with respect to a family $\mathscr{F}$ if and only if it admits a uniformly unique joint spreading model with respect to $\mathscr{F}$.

5. Uniform approximation of bounded operators.

We now pass to the study of the UALS property on certain classes of spaces. First we consider spaces with very few operators, namely spaces with the scalar-plus-compact property. The second class includes spaces admitting a uniformly unique joint spreading model with respect to certain families of Schauder basic sequences. Here, the notion of joint spreading models and the UALS property come together in the sense that the first property yields the second one. The third subsection is devoted to the study of the UALS property under duality. A consequence of the main result, Theorem 5.23, is that the spaces $C(K)$ with K countable compact satisfy the UALS. In the fourth subsection we show that the spaces $L_p[0,1]$ for $1 \le p \le \infty$ and $p \ne 2$, and $C(K)$ for uncountable compact metric spaces K, fail the UALS property. We close with some final remarks and open problems.

Definition 5.1. We will say that a Banach space X satisfies the *Uniform Approximation on Large Subspaces* (*UALS*) property if there exists $C > 0$ such that the following is satisfied. For every convex compact subset W of $\mathcal{L}(X)$, every $A \in \mathcal{L}(X)$ and $\varepsilon > 0$ with the property that, for every $x \in B_X$, there is a $B \in W$ such that $\|A(x) - B(x)\| \le \varepsilon$, there exist a finite-codimensional subspace Y of X and a $B \in W$ such that $\|(A-B)|_Y\|_{\mathcal{L}(Y,X)} \le C\varepsilon$.

Definition 5.2. A Banach space X will be called *UALS-saturated* if there exists $C > 0$ such that for every convex compact subset W of $\mathcal{L}(X)$, every $A \in \mathcal{L}(X)$ and $\varepsilon > 0$ with the property that, for every $x \in B_X$, there is a $B \in W$ such that $\|(A-B)x\| \le \varepsilon$, every subspace Y of X contains a further subspace Z such that $\|(A-B)|_Z\|_{\mathcal{L}(Z,X)} \le C\varepsilon$ for some $B \in W$.

5.1. The UALS property for compact operators. The first class of spaces satisfying the UALS includes spaces with very few operators. We prove that Banach spaces with the scalar-plus-compact property satisfy the UALS and are in fact UALS-saturated. Hence, the main result of [AF$^+$] shows that a large class of spaces, which includes all superreflexive spaces, embed into spaces that satisfy the UALS. We start with the following variation of Mazur's theorem [LT, Theorem 1.a.5].

LEMMA 5.3. *Let X be a Banach space, $T \in \mathcal{L}(X)$ and let $\varepsilon > 0$ be such that $\|T|_Y\|_{\mathcal{L}(Y,X)} > \varepsilon$ for every subspace Y of X of finite codimension. Then there exists a normalized sequence $(x_n)_n$ in X such that $(Tx_n)_n$ is seminormalized Schauder basic.*

Proof. Let $\delta > 0$. Pick x_1 in the unit sphere of X with $\|Tx_1\| \geq \varepsilon$ and assume that $x_1, \dots, x_n$ have been chosen for some $n \in \mathbb{N}$. Let G be a finite subset of X^* such that, for every $x \in \operatorname{span}\{Tx_1, \dots, Tx_n\}$, we have $\|x\| \leq (1+\delta)\max\{g(x) : g \in G\}$, and choose x_{n+1} in the unit sphere of $\bigcap_{g\in G} \ker T^*g$ with $\|Tx_{n+1}\| \geq \varepsilon$. It follows quite easily that $(Tx_n)_n$ is a Schauder basic sequence. ∎

PROPOSITION 5.4. *Let X be a Banach space and $T \in \mathcal{L}(X)$ be a compact operator. Then* $\inf \|T|_Y\|_{\mathcal{L}(Y,X)} = 0$*, where the infimum is taken over all subspaces Y of X of finite codimension.*

Proof. If the conclusion is false, then the previous lemma shows that $T[B_X]$ contains a seminormalized Schauder basic sequence, and this contradicts the fact that T is compact. ∎

NOTATION 5.5. Let X be a Banach space. We will denote by $\mathcal{K}(X)$ the ideal of all compact operators in the unital algebra $\mathcal{L}(X)$.

COROLLARY 5.6. *Let X be a Banach space, W be a compact subset of $\mathcal{K}(X)$ and $A \in \mathcal{L}(X)$. Assume that there exists $\varepsilon > 0$ such that, for every $x \in B_X$, there is a $B \in W$ such that $\|A(x) - B(x)\| \leq \varepsilon$. Then, for every $\delta > 0$, there exists a finite-codimensional subspace Y of X such that $\|(A-B)|_Y\|_{\mathcal{L}(Y,X)} \leq \varepsilon + \delta$ for all $B \in W$.*

Proof. Let $\delta > 0$ and $\{B_i\}_{i=1}^n$ be a δ-net of W. Applying Proposition 5.4, we choose a finite-codimensional subspace Y of X such that $\|B_i|_Y\| < \delta$ for every $1 \leq i \leq n$. Note that $\|B|_Y\| \leq 2\delta$ for all $B \in W$. Then, for every x in the unit ball of Y, there is a B in W such that $\|A(x) - B(x)\| \leq \varepsilon$ and hence $\|A(x)\| < \varepsilon + 2\delta$. That is, $\|A|_Y\| \leq \varepsilon + 2\delta$. Therefore $\|(A-B)|_Y\|_{\mathcal{L}(Y,X)} \leq \varepsilon + 4\delta$ for every $B \in W$. ∎

THEOREM 5.7. *Every Banach space with the scalar-plus-compact property satisfies the UALS.*

Proof. Let W, A, ε be as in Definition 5.1, with $A = \lambda_A I + K_A$ and K_A compact. Let $\delta > 0$ and $\{B_i\}_{i=1}^n$ be a δ-net of W with $B_i = \lambda_i I + K_i$ and $K_i \in \mathcal{K}(X)$ for $i = 1, \dots, n$. Proposition 5.4 then yields a finite-codimensional subspace Y of X such that $\|K_A|_Y\| \le \delta$ and $\|K_i|_Y\| \le \delta$ for all $1 \le i \le n$. Pick an $x \in Y$ with $\|x\| = 1$ and $B \in W$ with $B = \lambda_B I + K_B$ and $\|A(x) - B(x)\| \le \varepsilon$. Then, for $1 \le i \le n$ such that $\|B - B_i\| \le \delta$, we have $\|A(x) - B_i(x)\| \le \varepsilon + \delta$ and hence $|\lambda_A - \lambda_i| = \|\lambda_A x - \lambda_i x\| \le \|Ax - B_i x\| + \|K_A x + K_i x\| \le \varepsilon + 2\delta$. Therefore, for every y in the unit ball of Y, we have $\|A(y) - B_i(y)\| \le \varepsilon + 4\delta$, which proves the desired result. ∎

THEOREM 5.8. *Let X be a Banach space such that, for every $A \in \mathcal{L}(X)$, there is a strictly singular operator S and $\lambda \in \mathbb{R}$ such that $A = \lambda I + S$. Then X is UALS-saturated.*

Proof. Let W, A, ε be as in Definition 5.2, with $A = \lambda_A I + S_A$ and S_A a strictly singular operator. Let $\delta > 0$ and $\{B_i\}_{i=1}^n$ be a δ-net of W with $B_i = \lambda_i I + S_i$ and S_i strictly singular, $1 \le i \le n$. Recall that for every infinite-dimensional subspace of X there exists a further subspace Y such that $S_A|_Y$ and $S_i|_Y$, for $1 \le i \le n$, are compact operators. Applying the same arguments used in the previous proof we obtain the desired conclusion. ∎

5.2. Uniformly unique joint spreading models and the UALS property. In this subsection we study spaces that admit uniformly unique l-joint spreading models with respect to families with sufficient stability properties, described in the following definition, to deduce that in certain cases they satisfy the UALS property. Such spaces are, for example, all Asymptotic ℓ_p spaces. This should be compared to the examples of the following subsection that fail the UALS and the proof of this fact is based on the existence of diverse plegma spreading sequences in these spaces.

The families of sequences that we restrict our study to are very rich, in the sense that any sequence has a subsequence whose successive differences are in $\mathscr{F}$, and it is also closed under taking subsequences. Moreover, if a space has a uniformly unique l-joint spreading model with respect to such a family then, as already shown in Proposition 4.9, it has to be at least unconditional and in most cases it has to be some ℓ_p or c_0.

DEFINITION 5.9. Let X be a Banach space. A collection $\mathscr{F}$ of normalized and Schauder basic sequences in X will be called *difference-including* if

(i) for every $(x_n)_n$ in $\mathscr{F}$ any subsequence of $(x_n)_n$ is in $\mathscr{F}$,
(ii) for every sequence $(x_n)_n$ in X without a norm convergent subsequence there exists an infinite subset L of $\mathbb{N}$ such that, for any further infinite subset $M = \{m_k : k \in \mathbb{N}\}$ of L, the sequence $(z_k)_k$ defined by $z_k = \|x_{m_{2k-1}} - x_{m_{2k}}\|^{-1}(x_{m_{2k-1}} - x_{m_{2k}})$ is in $\mathscr{F}$.

REMARK 5.10. A difference-including collection clearly satisfies (a) and (b) of Proposition 4.9. In fact, most naturally defined families of normalized Schauder basic sequences in a Banach space X are difference-including. Such families include

(i) $\mathscr{F}(X)$, the collection of all normalized Schauder basic sequences in X;
(ii) $\mathscr{F}_{(1+\varepsilon)}(X)$, the collection of all normalized $(1+\varepsilon)$-Schauder basic sequences in X for some fixed $\varepsilon > 0$;
(iii) $\mathscr{F}_{0,\mathscr{A}}$ for a countable subset $\mathscr{A}$ of the dual, where

$$\mathscr{F}_{0,\mathscr{A}} = \Big\{(x_n)_n : (x_n)_n \text{ is normalized and } \lim_n f(x_n) = 0 \text{ for all } f \in \mathscr{A}\Big\};$$

(iv) $\tilde{\mathscr{F}}_b(X) = \mathscr{F}_{0,(e_n^*)_n}$ if X has a Schauder basis $(e_n)_n$, where $(e_n^*)_n$ are the biorthogonal functionals associated to the basis; notice that a Banach space X admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_b(X)$ if and only if it does so with respect to $\tilde{\mathscr{F}}_b(X)$;
(v) $\mathscr{F}_0(X)$ if X does not contain ℓ_1;
(vi) $\mathscr{F}_{\mathrm{su}}(X)$, the collection of all normalized Schauder basic sequences that generate a 1-suppression unconditional spreading model.

In certain cases, for X non-separable it is convenient to consider different collections $\mathscr{F}_Z$ for different separable subspaces Z of X. This is included in the statement of the following theorem.

THEOREM 5.11. *Let X be a Banach space and assume that for every separable subspace Z of X we have a difference-including collection $\mathscr{F}_Z$ of normalized Schauder basic sequences in Z. If there exists a uniform $K \geq 1$ such that each such Z admits a K-uniformly unique l-joint spreading model with respect to $\mathscr{F}_Z$, then X satisfies the UALS property.*

We postpone the proof of Theorem 5.11 in order to first state and prove its corollaries. Note that if X is a Banach space and $\mathscr{F}$ is a collection of normalized Schauder basic sequences in X with respect to which it admits a uniformly unique l-joint spreading model, then we may consider, for every separable subspace Z of X, the family $\mathscr{F}_Z = \{(x_n)_n \text{ in } \mathscr{F} : x_i \in Z \text{ for all } i \in \mathbb{N}\}$. This is in fact sufficient to prove most cases stated below.

COROLLARY 5.12. *In the following cases, a Banach space X and all of its subspaces satisfy the UALS property:*

(i) *X has a Schauder basis and it admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_b(X)$.*
(ii) *X is an arbitrary Banach space that admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}(X)$.*
(iii) *X does not contain ℓ_1 and it admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_0(X)$.*

(iv) *X is an arbitrary Banach space and, for some $\varepsilon > 0$, it admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_{(1+\varepsilon)}(X)$.*

Proof. All cases follow from Theorem 5.11. We describe some of the details. Case (i) follows from the fact that such an X admits a uniformly unique l-joint spreading model with respect to $\tilde{\mathscr{F}}_b(X) = \mathscr{F}_{0,(e_i^*)_i}$, which is difference-including. Case (ii) follows directly from the fact that $\mathscr{F}(X)$ is difference-including. In case (iii), by Rosenthal's ℓ_1 theorem [Ro], $\mathscr{F}_0(X)$ is difference-including. For (iv), $\mathscr{F}_{(1+\varepsilon)}$ is difference-including as well. ∎

Corollary 5.13. *The following Banach spaces and all of their subspaces satisfy the UALS property:*

(a) *the space $\ell_p(\Gamma)$ for $1 \le p < \infty$ and any infinite set Γ,*
(b) *the space $c_0(\Gamma)$ for any infinite set Γ,*
(c) *the James Tree space,*
(d) *every Asymptotic ℓ_p space for $1 \le p \le \infty$.*

Proof. The case of $\ell_p(\Gamma)$ follows from item (ii) of Corollary 5.12 for $1 < p < \infty$ and from (i) for $p = 1$, while that of $c_0(\Gamma)$ and the James Tree space follows from item (iii). Moreover, for case (d), if $1 < p \le \infty$ and X is an Asymptotic ℓ_p space, then it does not contain ℓ_1 and admits a uniformly unique l-joint spreading model with respect to $\mathscr{F}_0(X)$, so the result follows from case (iii) as well. Finally, if X is C-Asymptotic ℓ_1, use Proposition 4.11 to choose for every separable subspace Z of X a countable subset $\mathscr{A}_Z$ of Z^* such that Z admits a C^2-uniformly unique l-joint spreading model with respect to $\mathscr{F}_Z = \mathscr{F}_{0,\mathscr{A}_Z}$. ∎

We break up the proof of Theorem 5.11 into several steps.

Lemma 5.14. *Let X be a Banach space that admits a K-uniformly unique l-joint spreading model with respect to a difference-including collection $\mathscr{F}$ of normalized Schauder basic sequences. Then for any $D > 2K^2$ and any sequences $(z_n^i)_n$, $(y_n^i)_n$, $i = 1, \dots, l$, in $\mathscr{F}$, there exists an infinite subset L of $\mathbb{N}$ such that, for any scalars $a_1, \dots, a_l$, $\theta_1, \dots, \theta_l$ and $n_1 < \dots < n_l$ in L,*

$$(1) \qquad \min_{1\le i\le l} |\theta_i| \frac{1}{D} \Big\| \sum_{i=1}^{l} a_i z_{n_i}^i \Big\| \le \Big\| \sum_{i=1}^{l} a_i \theta_i y_{n_i}^i \Big\| \le \max_{1\le i\le l} |\theta_i| D \Big\| \sum_{i=1}^{l} a_i z_{n_i}^i \Big\|.$$

Proof. Choose $C > K$ and pass to an infinite set L so that the l-tuples $((z_n^i)_{n\in L})_{i=1}^l$ and $((y_n^i)_{n\in L})_{i=1}^l$ generate some l-joint spreading models that are K-equivalent to each other. This means that, after perhaps passing to a further subset of L, for any $n_1 < \dots < n_l$ in L, the sequences $(z_{n_i}^i)_{i=1}^l$ and $(y_{n_i}^i)_{i=1}^l$ are C-equivalent and each of them is C-suppression unconditional and hence $2C$-unconditional. For any scalars $a_1, \dots, a_l, \theta_1, \dots, \theta_l$, we calculate

$$\Big\| \sum_{i=1}^{l} a_i \theta_i y^i_{n_i} \Big\| \geq \frac{1}{2C} \min_{1\leq i\leq l} |\theta_i| \Big\| \sum_{i=1}^{l} a_i y^i_{n_i} \Big\| \geq \frac{1}{2C^2} \min_{1\leq i\leq l} |\theta_i| \Big\| \sum_{i=1}^{l} a_i z^i_{n_i} \Big\|.$$

The other inequality is obtained identically. Therefore, any $D > 2K^2$ satisfies the conclusion. ∎

The following is another variant of Mazur's theorem [LT, Theorem 1.a.5].

LEMMA 5.15. *Let X be a separable Banach space. Let also $T_{ij} \in \mathcal{L}(X)$ for $1 \leq i \leq n$, $1 \leq j \leq m_i$ and $c > 0$ be such that, for every $i = 1, \dots, n$ and each finite-codimensional subspace Y of X, there is an $x_i \in Y$ with $\|x_i\| = 1$ and $\|T_{ij} x_i\| > c$ for all $1 \leq j \leq m_i$. Then, for every $\varepsilon > 0$, there exist normalized sequences $(x^i_k)_k$, $i = 1, \dots, n$, in X such that if we set $Z_k = \operatorname{span}\{\{x^i_k\}_{i=1}^n \cup \{T_{ij} x^i_k\}_{i=1, j=1}^{n, m_i}\}$ for $k \in \mathbb{N}$ and $Z = \overline{\operatorname{span}} \bigcup_k Z_k$, then*

(i) *$(Z_k)_k$ forms an FDD for the space Z, with projection constant at most $1 + \varepsilon$,*
(ii) *$\|T_{ij}(x^i_k)\| > c$ for all $i = 1, \dots, n$, $j = 1, \dots, m_i$, and $k \in \mathbb{N}$.*

Proof. Set $\mathcal{A} = \{I\} \cup \{T_{ij}\}_{i=1, j=1}^{n, m_i}$ and, for every $1 \leq i \leq n$, choose a normalized vector x^i_1 in X with $\|T_{ij} x^i_1\| > c$ for all $j = 1, \dots, m_i$. Assume that we have chosen $(x^i_k)_{k=1}^d$ up to some $d \in \mathbb{N}$ for $1 \leq i \leq n$, so that the spaces $(Z_k)_{k=1}^d$ satisfy (i) for the space they generate and (ii) for $1 \leq k \leq d$. Choose a finite subset G of the unit sphere of X^* such that, for all x in the linear span of $\bigcup_{k=1}^d Z_k$, we have $\|x\| \leq (1 + \varepsilon) \max\{g(x) : g \in G\}$, and set $F = \bigcup_{T \in \mathcal{A}} \{T^* g : g \in G\}$. Finally, for $i = 1, \dots, n$, choose x^i_{d+1} in the unit sphere of $\bigcap_{f \in F} \ker f$ such that $\|T_{ij} x^i_{d+1}\| > c$ for all $1 \leq j \leq m_i$. It then follows quite easily that the sequences are as desired. ∎

LEMMA 5.16. *Let X be a Banach space and assume that for every separable subspace Z of X we have a difference-including collection $\mathscr{F}_Z$ of normalized Schauder basic sequences in Z. Let $T_1, \dots, T_l$ be bounded linear operators on X and assume that there is $c > 0$ such that, for every finite-codimensional subspace Y of X and every $i = 1, \dots, l$, we have $\|T_i|_Y\|_{\mathcal{L}(Y,X)} \geq c$. Assume moreover that, for some $0 < \delta < c$ and $i = 1, \dots, l$, we have $T_{i1}, \dots, T_{im_i}$ in $\mathcal{L}(X)$ with $\|T_{ij} - T_i\| \leq \delta$ for $j = 1, \dots, m_i$. If $\tilde{c} = c - \delta$, then there exist a separable subspace Z of X and normalized sequences $(z^i_k)_k$, $i = 1, \dots, l$, in $\mathscr{F}_Z$ such that*

(i) *for any $n_1 < \dots < n_l$, the sequence $(z^i_{n_i})_{i=1}^l$ is $9/8$-Schauder basic,*
(ii) *$\|T_{ij} z^i_k\| > \tilde{c}/3$ for $i = 1, \dots, n$, $j = 1, \dots, m_i$ and $k \in \mathbb{N}$,*
(iii) *if $y^{ij}_k = \|T_{ij} z^i_k\|^{-1} T_{ij} z^i_k$ then $(y^{ij}_k)_k$ is in $\mathscr{F}_Z$ for $i = 1, \dots, n$ and $j = 1, \dots, m_i$.*

Proof. Note that, for any finite-codimensional subspace Y of X, we may choose x_i in the unit ball of Y such that $\|T_i x_i\| > c - (c-\delta)/4$, which means that, for $j = 1,\dots,m_i$, we have $\|T_{ij}x_i\| > 3(c-\delta)/4 = 3\tilde{c}/4$. Apply Lemma 5.15 to find normalized sequences $(x_k^i)_k$, $i = 1,\dots,n$, such that $\|T_{ij}x_k^i\| > 3\tilde{c}/4$ for all $k \in \mathbb{N}$, $i = 1,\dots,n$ and $j = 1,\dots,m_i$, and the sequence $(Z_k)_k$ defined in Lemma 5.15 is an FDD with constant 9/8. Let $Z = \overline{\operatorname{span}}\bigcup_k Z_k$. Choose L so that if we set $z_k^i = \|x^i_{m_{2k-1}} - x^i_{m_{2k}}\|^{-1}(x^i_{m_{2k-1}} - x^i_{m_{2k}})$, then $(z_k^i)_k$ as well as $(\|T_{ij}z_k^i\|^{-1}T_{ij}z_k^i)_k$ are in $\mathscr{F}_Z$ for $i = 1,\dots,n$, $j = 1,\dots,m_i$. By the fact that $(T_{ij}x_k^i)_k$ is 9/8-Schauder basic we obtain

$$\begin{aligned}\|T_{ij}z_k^i\| &= \frac{1}{\|x^i_{m_{2k-1}} - x^i_{m_{2k}}\|}\|T_{ij}x^i_{m_{2k-1}} - T_{ij}x^i_{m_{2k}}\| \\ &\geq \frac{1}{2}\,\frac{1}{9/8}\|T_{ij}x^i_{m_{2k-1}}\| > \frac{3\tilde{c}/4}{9/4}.\end{aligned}$$

Statement (i) follows from the fact $(Z_k)_k$ is an FDD with constant 9/8 and $(z^i_{n_i})_{i=1}^l$ is a block sequence. ∎

S. Kakutani [Ka] proved the finite-dimensional analog of the following theorem, also known as Kakutani's Fixed Point Theorem. We present the infinite-dimensional case by H. F. Bohnenblust and S. Karlin [BK], which as already mentioned is a key ingredient in the proof of Theorem 5.11. Recall that a multivalued mapping $\phi : X \twoheadrightarrow Y$ between topological spaces has closed graph if for every $(x_n)_n \in X$ with $\lim x_n = x$ and $(y_n)_n \in Y$ with $y_n \in \phi(x_n)$ and $\lim y_n = y$, we have $y \in \phi(x)$.

THEOREM 5.17. *Let X be a Banach space, K a non-empty compact convex subset of X and let $\phi : K \twoheadrightarrow K$ have closed graph and non-empty convex values. Then ϕ has a fixed point, i.e. there exists $x \in X$ such that $x \in \phi(x)$.*

Proof of Theorem 5.11. Let $D > 2K^2$, i.e. a constant for which the conclusion of Lemma 5.14 can be applied to all families $\mathscr{F}_Z$. Set $C = 7D$. Let W be a convex and compact subset of $\mathcal{L}(X)$, $A \in \mathcal{L}(X)$, and $\varepsilon > 0$ such that, for all x in the unit ball of X, there is $T \in W$ with $\|A(x) - T(x)\| \leq \varepsilon$. We claim that there is a finite-codimensional subspace Y of X and $T \in W$ such that $\|(A-T)|_Y\|_{\mathcal{L}(Y,X)} < C\varepsilon$. Assume that the conclusion is false. Set $c = C\varepsilon$, $\delta = c/2$, and $\tilde{c} = c - \delta = C/2$. Choose a maximal δ-separated subset $(T_i)_{i=1}^l$ of W, set $\eta = \varepsilon/(27l)$, and for $i = 1,\dots,l$ choose a maximal η-separated subset $(T_{ij})_{j=1}^{m_i}$ of $B_W(T_i,\delta) = \{T \in W : \|T_i - T\| \leq \delta\}$. Apply Lemma 5.16 to the operators $A - T_i$ and $A - T_{ij}$ for $i = 1,\dots,l$ and $j = 1,\dots,m_i$ to find a separable subspace Z and normalized 9/8-Schauder basic sequences $(z_k^i)_k$ in $\mathscr{F}_Z$ such that for $i = 1,\dots,l$, $j = 1,\dots,m_i$ if $y_k^{ij} = \|(A - T_{ij})z_k^i\|^{-1}(A - T_{ij})z_k^i$ for $k \in \mathbb{N}$, then $\|(A - T_{ij})z_k^i\| \geq \tilde{c}/3$ and the sequence $(y_k^{ij})_k$ is in $\mathscr{F}_Z$. Iterate Lemma 5.14 to find an infinite subset L

of $\mathbb{N}$ such that (1) is satisfied for $(z_k^i)_{k\in L}$ and $(y_k^{ij_i})_{k\in L}$ for all $i = 1,\dots,l$ and for any choice of $1 \le j_i \le m_i$.

Fix $k_1 < \dots < k_l$ in L and take a partition of unity $f_1,\dots,f_l$ of W subordinate to $T_1,\dots,T_l$. That is, $f_i : W \to [0,1]$ is continuous, $\sum_{i=1}^l f_i(T) = 1$ for all T in W and $f_i(T_j) = \delta_{ij}$. We define a continuous mapping $x : W \to X$ by

$$x(T) = \frac{\sum_{i=1}^l f_i(T) z^i_{n_i}}{\|\sum_{i=1}^l f_i(T) z^i_{n_i}\|}.$$

Let T be an arbitrary element of W, and if $I_T = \{i = 1,\dots,l : \|T - T_i\| \le \delta\}$, then for $i \in I_T$ choose $1 \le j_i \le m_i$ such that $\|T - T_{ij_i}\| \le \eta$. Recall that $(z^i_{n_i})_i$ is 9/8-Schauder basic and therefore

$$\Big\|\sum_{i=1}^l f_i(T) z^i_{n_i}\Big\| \ge \frac{4}{9l}\sum_{i=1}^l |f_i(T)| = \frac{4}{9l}. \tag{2}$$

We observe that

$$\begin{aligned}
\|(A-T)x(T)\| &= \frac{1}{\|\sum_{i\in I_T} f_i(T) z^i_{n_i}\|}\Big\|\sum_{i\in I_T} f_i(T)(A-T) z^i_{n_i}\Big\| \qquad (3)\\
&\ge \frac{\|\sum_{i\in I_T} f_i(T)(A-T_{ij_i}) z^i_{n_i}\|}{\|\sum_{i\in I_T} f_i(T) z^i_{n_i}\|} - \frac{\sum_{i\in I_T} |f_i(T)|\,\|T - T_{ij_i}\|}{\|\sum_{i\in I_T} f_i(T) z^i_{n_i}\|}\\
&\ge \frac{\|\sum_{i\in I_T} f_i(T)(A-T_{ij_i}) z^i_{n_i}\|}{\|\sum_{i\in I_T} f_i(T) x^i_{n_i}\|} - \eta\frac{\sum_{i\in I_T} |f_i(T)|}{(4/(9l))\sum_{i\in I_T} |f_i(T)|} \qquad \text{(by (2))}\\
&= \frac{\|\sum_{i\in I_T} f_i(T)\|(A-T_{ij_i}) z^i_{n_i}\| y^{ij}_{n_i}\|}{\|\sum_{i\in I_T} f_i(T) x^i_{n_i}\|} - \frac{9l\eta}{4}\\
&\ge \min_{1\le i\le l}\|(A-T_{ij_i}) z^i_{n_i}\|\frac{1}{D}\frac{\|\sum_{i\in I_T} f_i(T) x^i_{n_i}\|}{\|\sum_{i\in I_T} f_i(T) x^i_{n_i}\|} - \frac{9l\eta}{4} \qquad \text{(by (1))}\\
&\ge \frac{\tilde c}{3D} - \frac{9l\eta}{4} = \frac{7D}{6D}\varepsilon - \frac{1}{12}\varepsilon = \frac{13}{12}\varepsilon.
\end{aligned}$$

We now define a multivalued mapping $\phi : W \twoheadrightarrow W$ by

$$\phi(T) = \{S \in W : \|(A-S)x(T)\| \le \varepsilon\}.$$

By assumption, the values of ϕ are non-empty and they are also closed and convex. It also easily follows that ϕ has closed graph. Hence, from Theorem 5.17, there exists $T \in W$ with $T \in \phi(T)$, i.e., $\|(A-T)x(T)\| \le \varepsilon$. This contradicts (3), which completes the proof of Theorem 5.11. ∎

The following lemma shows that if X is a Banach space with a shrinking FDD that satisfies the UALS property, then the finite-codimensional subspaces of X on which the approximations happen can be assumed to be tail subspaces.

LEMMA 5.18. *Let X be a Banach space with a shrinking FDD $(X_n)_n$ and Y be a finite-codimensional subspace of X. Then, for every $\varepsilon > 0$, there exists a tail subspace Z of X such that $B_Z \subset B_Y + \varepsilon B_X$.*

Proof. Let $x_1, \dots, x_n \in B_X$ with $X = Y \oplus \operatorname{span}\{x_1, \dots, x_n\}$ and $x_1^*, \dots, x_n^* \in X^*$ be such that $x_i^*(x_j) = \delta_{ij}$ for all $1 \le i, j \le n$. Notice that $Y = \bigcap_{i=1}^n \ker x_i^*$. Since $(X_n)_n$ is a shrinking FDD, we may choose $n_0 \in \mathbb{N}$ such that $\|x_i^* - P_{n_0}^*(x_i^*)\| < \varepsilon/l\|x_i\|$ for all $1 \le i \le n$, and set $Z = \operatorname{span} \bigcup_{n > n_0} X_n$. Pick $z \in B_Z$ and set $x = \sum_{i=1}^l x_i^*(z) x_i/\varepsilon$. Then $|x_i^*(z)| < \varepsilon/l\|x_i\|$ and $x_i^*(x) = x_i^*(z)/\varepsilon$ for all $1 \le i \le n$. Hence $\|x\| < 1$ and $z - \varepsilon x \in \bigcap_{i=1}^n \ker x_i^*$, from which it follows that $z \in B_Y + 2\varepsilon B_X$. ∎

The next example demonstrates that a shrinking FDD is necessary above to assume that the uniform approximation happens on tail subspaces. Let us recall that the basis of ℓ_1 is not shrinking.

EXAMPLE 5.19. Let $(e_n)_n$ denote the unit vector basis of ℓ_1 and consider the operator $A : \ell_1 \to \ell_1$ with

$$A\big((x_n)_n\big) = \sum_{n=1}^{\infty} x_{2n-1} e_1 + \sum_{n=1}^{\infty} x_{2n} e_2,$$

and for $z \in \ell_1$ the operators $B_z^+, B_z^- : \ell_1 \to \ell_1$ with

$$B_z^+((x_n)_n) = \sum_{n=1}^{\infty} x_n z \quad \text{and} \quad B_z^-((x_n)_n) = \Big(\sum_{n=1}^{\infty} x_{2n-1} - \sum_{n=1}^{\infty} x_{2n}\Big) z.$$

Set $W = \operatorname{co}\{B_z^{\pm} : z \in \operatorname{span}\{e_1, e_2\} \text{ and } \|z\| \le 1\}$.

Let $x \in \ell_1$ with $\|x\| \le 1$ and $A(x) = a_1 e_1 + a_2 e_2$, where $a_1 = \sum_{n=1}^{\infty} x_{2n-1}$ and $a_2 = \sum_{n=1}^{\infty} x_{2n}$. Suppose $A(x) \ne 0$ and set $a = \max\{|a_1 + a_2|, |a_1 - a_2|\}$. Notice that $a = |a_1| + |a_2|$. If $a = |a_1 + a_2|$, then setting $z = \frac{1}{a_1 + a_2} A(x)$ we have $\|z\| = 1$ and $B_z^+(x) = A(x)$. If $a = |a_1 - a_2|$, then the same hold for $z = \frac{1}{a_1 - a_2} A(x)$. Hence, for every $x \in \ell_1$ with $\|x\| \le 1$, there is a $B \in W$ such that $\|(A - B)x\| = 0$.

Pick any $B \in W$ and $n_0 \in \mathbb{N}$. Then there exists a convex combination in W such that $B = \sum_{i=1}^n a_i B_{y_i}^+ + \sum_{i=1}^m b_i B_{z_i}^-$ and, for every $k \in \mathbb{N}$, we have $B(e_{2k-1}) = \sum_{i=1}^n a_i y_i + \sum_{i=1}^m b_i z_i$ and $B(e_{2k}) = \sum_{i=1}^n a_i y_i - \sum_{i=1}^m b_i z_i$ and hence

$$\left\|(A - B)\frac{e_{2k-1} + e_{2k}}{2}\right\| = \left\|\frac{e_1 + e_2}{2} - \sum_{i=1}^n a_i y_i\right\| \ge 1 - \sum_{i=1}^n a_i.$$

Similarly, $\big\|(A - B)\frac{e_{2k-1} - e_{2k}}{2}\big\| \ge \sum_{i=1}^n a_i$ and thus, for any $k_0 \in \mathbb{N}$ with $n_0 \le 2k_0 - 1$, either $\big\|(A - B)\frac{e_{2k_0-1} + e_{2k_0}}{2}\big\| \ge 1/2$ or $\big\|(A - B)\frac{e_{2k_0-1} - e_{2k_0}}{2}\big\| \ge 1/2$. Therefore, $\|(A - B)|_{\operatorname{span}\{e_n : n \ge n_0\}}\| \ge 1/2$ while, for every x in the unit ball of ℓ_1, there exists a $B \in W$ such that $\|A(x) - B(x)\| = 0$.

5.3. The UALS property and duality. We make a connection between the UALS property of a space and its dual. In particular, for reflexive spaces with an FDD we show that the UALS for X is equivalent to the UALS for X^*. We also show that if X has an FDD and X^* has a unique l-joint spreading model with respect to a difference-including family, then X must satisfy the UALS as well. This allows us to show indirectly that certain spaces, such as $\mathscr{L}_\infty$ spaces with separable dual, satisfy the UALS.

Proposition 5.20. *Let X be a Banach space, $A \in \mathcal{L}(X)$ and W be a convex and WOT-compact subset of $\mathcal{L}(X)$. If there is an $\varepsilon > 0$ such that W ε-pointwise approximates A, then the set $W^* = \{T^* : T \in W\}$ ε-pointwise approximates A^*.*

Proof. If the conclusion is false, then there exists $x^* \in S_{X^*}$ and $\delta > 0$ such that if $W^*x^* = \{T^*x^* : T \in W\}$, then $\operatorname{dist}(A^*x^*, W^*x^*) \geq \varepsilon + \delta$. As W^*x^* is a convex and w^*-compact subset of X^*, a separation theorem shows that there exists $x \in S_X$ such that $x(A^*x^*) + (\varepsilon + \delta/2) \leq \inf_{T\in W} x(T^*x^*)$, so

$$\|Ax - Tx\| \geq x^*(Tx - Ax) \geq \varepsilon + \delta/2$$

for all $T \in W$. ∎

Remark 5.21. The compactness of W is necessary in Proposition 5.20. To see this, consider the case when $X = \ell_1$, A is the identity operator, and W is the closed convex hull of all natural projections onto finite subsets of $\mathbb{N}$ with respect to the unit vector basis.

We state the main results and prove them afterwards.

Theorem 5.22. *Let X be a reflexive Banach space with an FDD. Then X satisfies the UALS if and only if X^* does.*

Theorem 5.23. *Let X be a Banach space with an FDD. Assume that there exist a uniform constant $C > 0$ and, for every separable subspace Z of X^*, a difference-including family $\mathscr{F}_Z$ of normalized sequences in X^* such that Z admits a C-uniformly unique l-joint spreading model with respect to $\mathscr{F}_Z$. Then X satisfies the UALS property.*

Recall that results from [H], [HS], [LS], and [S] imply that if X is an infinite-dimensional $\mathscr{L}_\infty$ space with separable dual then X^* is isomorphic to ℓ_1. Also, this is the case if and only if ℓ_1 is not isomorphic to a subspace of X. As proved in [FOS], every Banach space with separable dual embeds in an $\mathscr{L}_\infty$ space with separable dual.

Corollary 5.24. *Every $\mathscr{L}_\infty$ space with separable dual satisfies the UALS property. In particular,*

(i) *every hereditarily indecomposable $\mathscr{L}_\infty$ space satisfies the UALS,*

(ii) *every Banach space with separable dual embeds in a space that satisfies the UALS,*
(iii) *for every countable compact metric space K, the space $C(K)$ satisfies the UALS.*

COROLLARY 5.25. *If X is Banach space such that X^* is an Asymptotic ℓ_p space for some $1 \le p \le \infty$, then every quotient of X with an FDD satisfies the UALS.*

LEMMA 5.26. *Let X be a Banach space, let $R : X \to X$ be a finite rank operator, and let $Q = I - R$. If T is in $\mathcal{L}(X)$, then there exists a subspace Y of X of finite codimension such that $\|T|_Y\|_{\mathcal{L}(Y,X)} \le \|QT\|$.*

Proof. Since RT is a finite rank operator, the subspace $Y = \ker RT$ is of finite codimension. So, $\|T|_Y\| \le \|RT|_Y\| + \|QT|_Y\| \le \|QT\|$. ∎

Proof of Theorem 5.22. It is clearly enough to show one implication. Let us assume that X^* satisfies the UALS with constant $C > 0$ and let $A \in \mathcal{L}(X)$ and W be a compact and convex subset of $\mathcal{L}(X)$ that ε-approximates A. Then by Proposition 5.20 the set $W^* = \{T^* : T \in W\}$ ε-approximates A^* and so there exists a subspace Z of X^* of finite codimension such that $\|(T^* - A^*)|_Z\| \le C\varepsilon$. By Lemma 5.18, and perhaps with some additional error, we may assume that Z is a tail subspace with an associated projection Q_n^* and hence

$$\|Q_n(T - A)\| = \|(T^* - A^*)Q_n^*\| \le C\|Q_n\|\varepsilon.$$

Applying Lemma 5.26, we may find a subspace Y of X of finite codimension such that $\|(T - A)|_Y\| \le \|Q_n(T - A)\|$ and hence $\|(T - A)|_Y\| \le C\|Q_n\|\varepsilon$. ∎

LEMMA 5.27. *Let X be a Banach space with a bimonotone FDD and let $(Q_n)_n$ denote the basis tail projections (i.e. $Q_n = I - P_n$ for all $n \in \mathbb{N}$). Assume that for every separable subspace Z of X^* we have a difference-including collection $\mathscr{F}_Z$ of normalized Schauder basic sequences in Z. Let $T_1, \ldots, T_l$ be bounded linear operators on X and assume that there is $c > 0$ such that, for every $n \in \mathbb{N}$ and every $i = 1, \ldots, l$, we have $\|T_i^* Q_n^*\| \ge c$. Assume moreover that, for some $0 < \delta < c$ and every $i = 1, \ldots, l$, we have $T_{i1}, \ldots, T_{im_i}$ in $\mathcal{L}(X)$ with $\|T_{ij} - T_i\| \le \delta$ for $j = 1, \ldots, m_i$. Then, if $\tilde{c} = c - \delta$, there exist a separable subspace Z of X^* and normalized sequences $(z_k^i)_k$, $i = 1, \ldots, l$, in $\mathscr{F}_Z$ such that*

(i) *for any $n_1 < \cdots < n_l$, the sequence $(z_{n_i}^i)_{i=1}^l$ is $9/8$-Schauder basic,*
(ii) *$\|T_{ij}^* z_k^i\| > \tilde{c}/3$ for $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ and $k \in \mathbb{N}$,*
(iii) *if $y_k^{ij} = \|T_{ij}^* z_k^i\|^{-1} T_{ij}^* z_k^i$, then $(y_k^{ij})_k$ is in $\mathscr{F}_Z$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$.*

Proof. For every $i = 1, \ldots, l$, we choose a normalized sequence $(x_n^{i*})_n$ such that $\|T_i^* Q_n^* x_n^{i*}\| \ge c - (c - \delta)/4$. Since the FDD is bimonotone, we may assume that $\min \operatorname{supp}(x_n^{i*}) > n$ for all $n \in \mathbb{N}$ and $1 \le i \le l$ and that $\|T_i^* x_n^{i*}\| \ge c - (c-\delta)/4$. This means that all sequences $(x_n^{i*})_n, (T^* x_n^{i*})_n$ for $1 \le i \le l$ are w^*-null (by w^*-continuity). We may now reason as for Lemmas 5.15 and 5.16 to achieve the desired conclusion. ∎

Proof of Theorem 5.23. We renorm the space X so that its FDD is bimonotone. Let $D > 2K^2$, i.e. a constant for which the conclusion of Lemma 5.14 can be applied to all families $\mathscr{F}_Z$ for all separable subspaces Z of X^*. Set $C = 14D$. We will show that X satisfies the UALS with constant C. Let $A \in \mathcal{L}(X)$ and W be a convex compact subset of $\mathcal{L}$ of X that ε-approximates A. It is sufficient to find $T \in W$ and $n_0 \in \mathbb{N}$ such that $\|(T^* - A^*)Q_{n_0}^*\| < C\varepsilon$. Indeed, then $\|Q_{n_0}(A-T)\| = \|(T^* - A^*)Q_{n_0}^*\| < C\varepsilon$ and by Lemma 5.26 we will be done. If we assume that the conclusion is false, we may follow the proof of Theorem 5.11 to the letter, only replacing Lemma 5.16 with Lemma 5.27, to reach the desired conclusion. ∎

5.4. Spaces failing the UALS property. In this section we present an archetypal example of a reflexive Banach space $\mathcal{X}$ that fails the UALS and admits a unique spreading model isometric to ℓ_2. The proof that $\mathcal{X}$ fails the property is based on the fact that it does not admit a uniformly unique joint spreading model. This reasoning may then be modified and utilized to show that classical spaces such as $L_p[0,1]$ for $1 \le p \le \infty$ and $p \ne 2$, and $C(K)$ for uncountable compact metric spaces K, fail the UALS.

Definition 5.28. For each $n \in \mathbb{N}$, we set $X_n = (\sum_{i=1}^{2n} \oplus \ell_2)_1$ and $Y_n = (\sum_{i=1}^{2n} \oplus \ell_2)_\infty$ and let $\mathcal{X} = (\sum \oplus X_n \oplus Y_n)_2$.

For a vector x in $\mathcal{X}$, we write $x = \sum_{n=1}^{\infty} x_n + y_n$ to mean that $x_n \in X_n$ and $y_n \in Y_n$, and $x_n = \sum_{j=1}^{n} x_{n(j)}$, $y_n = \sum_{j=1}^{2n} y_{n(j)}$ to denote the coordinates of x_n and y_n with respect to the natural decomposition of X_n and Y_n respectively. Under this notation we compute the norm of x as follows:

$$\|x\|^2 = \sum_{n=1}^{\infty} \Big(\Big(\sum_{j=1}^{2n} \|x_{n(j)}\| \Big)^2 + \Big(\max_{1 \le j \le 2n} \|y_{n(j)}\| \Big)^2 \Big).$$

By taking an orthonormal basis for each ℓ_2-component of X_n as well as of Y_n and taking the union over all $n \in \mathbb{N}$ and for $j = 1, \dots, 2n$, we obtain a 1-unconditional basis for $\mathcal{X}$. Henceforth, a block sequence in $\mathcal{X}$ will be understood to be with respect to a fixed enumeration of the aforementioned basis.

Proposition 5.29. *The space $\mathcal{X}$ fails the UALS property.*

Proof. Assume that $\mathcal{X}$ satisfies the UALS with constant $C > 0$ and pick $n \in \mathbb{N}$ with $C/n < 1/2$. For $G \subset \{1, \dots, 2n\}$, consider the bounded operator $I_G : X_n \to Y_n$ with $I_G(\sum_{i=1}^{2n} x_i) = \sum_{i \in G} x_i$ and set $A_n = I_{\{1,\dots,2n\}}$ and

$W_n = \mathrm{co}\{I_G : \#G = n\}$. Let $x \in X_n$ with $x = \sum_{i=1}^{2n} x_i$ and $\|x\| = 1$, that is, $\sum_{i=1}^{2n} \|x_i\| = 1$, and σ be a permutation of $\{1, \ldots, 2n\}$ such that $\|x_{\sigma(1)}\| \geq \cdots \geq \|x_{\sigma(2n)}\|$. Then notice that $\|x_{\sigma(n+1)}\| \leq \frac{1}{n+1}$ and hence $\|A_n(x) - I_G(x)\| \leq \frac{1}{n+1}$ for $G = \{\sigma(1), \ldots, \sigma(n)\}$.

The basis $(e_n)_n$ of $\mathcal{X}$ is shrinking, since $\mathcal{X}$ is reflexive, and therefore Lemma 5.18 yields a tail subspace $Y = \mathrm{span}\{e_n : n \geq n_0\}$ of X such that $\|(A_n - B)|_Y\| < C/n$ for some $B \in W$. Then B is a convex combination $B = \sum_{i=1}^{k} \lambda_i I_{G_i}$ and we have $\int \sum_{i=1}^{k} \lambda_i \chi_{G_i} = 1/2$, where the integral is with respect to the normalized counting measure on $\{1, \ldots, 2n\}$. Hence there exists a $1 \leq j \leq 2n$ such that $\sum_{i=1}^{k} \lambda_i \chi_{G_i}(j) \leq \frac{1}{2}$. Pick any $x \in X_{n(j)}$ with $\|x\| = 1$ and $\mathrm{supp}(x) \geq n_0$ and notice that $\|A_n(x) - B(x)\| \geq 1 - \sum_{i=1}^{k} \lambda_i \chi_{G_i}(j)$. Thus $\|(A_n - B)|_Y\| \geq \frac{1}{2}$, a contradiction. ∎

The space $\mathcal{X}$ is a first example of a space failing the UALS property. As we show next, it admits a uniformly unique spreading model while it fails to admit a uniformly unique l-joint spreading model. We start with the following lemmas.

LEMMA 5.30. *Let $(x^k)_k$ be a block sequence in $\mathcal{X}$ with $x^k = \sum_{n=n_0}^{n_1} x_n^k + y_n^k$ and assume that $\|x_{n(j)}^{k_1}\| = \|x_{n(j)}^{k_2}\|$ and $\|y_{n(j)}^{k_1}\| = \|y_{n(j)}^{k_2}\|$ for all $k_1, k_2 \in \mathbb{N}$, $n_0 \leq n \leq n_1$ and $1 \leq j \leq 2n$. Set $\varepsilon = \|x^k\|$ for $k \in \mathbb{N}$. Then, for all $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, we have $\|\sum_{k=1}^{m} \lambda_k x^k\| = \varepsilon(\sum_{k=1}^{m} \lambda_k^2)^{1/2}$.*

Proof. Let $k_0 \in \mathbb{N}$. For every $k \in \mathbb{N}$ and $n_0 \leq n \leq n_1$, since $(x^k)_k$ is block, we have

$$\Big\|\sum_{k=1}^{m} \lambda_k x_{n(j)}^k\Big\| = \Big(\sum_{k=1}^{m} \lambda_k^2 \|x_{n(j)}^k\|^2\Big)^{1/2} = \|x_{n(j)}^{k_0}\| \Big(\sum_{k=1}^{m} \lambda_k^2\Big)^{1/2}.$$

We thus calculate

$$\Big\|\sum_{k=1}^{m} \lambda_k x_n^k\Big\| = \sum_{j=1}^{2n} \Big\|\sum_{k=1}^{m} \lambda_k x_{n(j)}^k\Big\| = \|x_n^{k_0}\| \Big(\sum_{k=1}^{m} \lambda_k^2\Big)^{1/2} \tag{4}$$

and similarly

$$\Big\|\sum_{k=1}^{m} \lambda_k y_n^k\Big\| = \max_{1 \leq j \leq 2n} \Big\|\sum_{k=1}^{m} \lambda_k y_{n(j)}^k\Big\| = \|y_n^{k_0}\| \Big(\sum_{k=1}^{m} \lambda_k^2\Big)^{1/2}. \tag{5}$$

Finally, using (4) and (5), we conclude that

$$\begin{aligned} \Big\|\sum_{k=1}^{m} \lambda_k x^k\Big\|^2 &= \sum_{n=n_0}^{n_1} \Big(\Big\|\sum_{k=1}^{m} \lambda_k x_n^k\Big\|^2 + \Big\|\sum_{k=1}^{m} \lambda_k y_n^k\Big\|^2\Big) \\ &= \sum_{k=1}^{m} \lambda_k^2 \sum_{n=n_0}^{n_1} (\|x_n^{k_0}\|^2 + \|y_n^{k_0}\|^2) = \sum_{k=1}^{m} \lambda_k^2 \|x^{k_0}\|^2. \end{aligned} \tag{6}$$ ∎

LEMMA 5.31. *Let $(n_k)_{k\geq 0}$ be an increasing sequence of natural numbers and $(x^k)_k$ be a block sequence in $\mathcal{X}$ such that*

(i) *there exist $c_1, c_2 > 0$ such that $c_1 \leq \|x^k\| \leq c_2$ for every $k \in \mathbb{N}$,*
(ii) *$x^k = \sum_{n=n_0}^{n_1}(x_n^k + y_n^k) + \sum_{n=n_k+1}^{n_{k+1}}(x_n^k + y_n^k)$ for every $k \in \mathbb{N}$,*
(iii) *$\|x_{n(j)}^{k_1}\| = \|x_{n(j)}^{k_2}\|$ and $\|y_{n(j)}^{k_1}\| = \|y_{n(j)}^{k_2}\|$ for all $k_1, k_2 \in \mathbb{N}$, $n_0 \leq n \leq n_1$ and $1 \leq j \leq 2n$.*

Then, for all $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$,

$$c_1\Big(\sum_{k=1}^m \lambda_k^2\Big)^{1/2} \leq \Big\|\sum_{k=1}^m \lambda_k x^k\Big\| \leq c_2\Big(\sum_{k=1}^m \lambda_k^2\Big)^{1/2}.$$

Proof. Using (6), we have

$$\begin{aligned}\Big\|\sum_{k=1}^m \lambda_k x^k\Big\|^2 &= \sum_{n=n_0}^{n_1}\Big(\Big\|\sum_{k=1}^m \lambda_k x_n^k\Big\|^2 + \Big\|\sum_{k=1}^m \lambda_k y_n^k\Big\|^2\Big) + \sum_{k=1}^m \sum_{n=n_k+1}^{n_{k+1}} (\|\lambda_k x_n^k\|^2 + \|\lambda_k y_n^k\|^2)\\ &= \sum_{k=1}^m \lambda_k^2 \sum_{n=n_0}^{n_1} (\|x_n^k\|^2 + \|y_n^k\|^2) + \sum_{k=1}^m \lambda_k^2 \sum_{n=n_k+1}^{n_{k+1}} (\|x_n^k\|^2 + \|y_n^k\|^2)\\ &= \sum_{k=1}^m \lambda_k^2\Big(\sum_{n=n_0}^{n_1} (\|x_n^k\|^2 + \|y_n^k\|^2) + \sum_{n=n_k+1}^{n_{k+1}} (\|x_n^k\|^2 + \|y_n^k\|^2)\Big) = \sum_{k=1}^m \lambda_k^2\|x^k\|^2,\end{aligned}$$

which, due to (i), yields the desired result. ∎

PROPOSITION 5.32. *Let $(x^k)_k$ be a normalized block sequence in $\mathcal{X}$. For every $\varepsilon > 0$, $(x^k)_k$ has a subsequence $(x^{k_i})_i$ such that for every $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$,*

$$(1-\varepsilon)\Big(\sum_{i=1}^m \lambda_i^2\Big)^{1/2} \leq \Big\|\sum_{i=1}^m \lambda_i x^{k_i}\Big\| \leq (1+\varepsilon)\Big(\sum_{i=1}^m \lambda_i^2\Big)^{1/2}.$$

Proof. We choose $L \in [\mathbb{N}]^\infty$ with $\lim_{k\in L}\|x_{n(j)}^k\| = a_{n,j}$ and $\lim_{k\in L}\|y_{n(j)}^k\| = b_{n,j}$ for all $n \in \mathbb{N}$ and $1 \leq j \leq 2n$. Set $\lim_{k\in L}\|x_n^k\| = a_n$ and $\lim_{k\in L}\|y_n^k\| = b_n$. As $\sum_{n=1}^\infty(\|x_n^k\|^2 + \|y_n^k\|^2) \leq 1$ for all $k \in \mathbb{N}$, we deduce that $\sum_{n=1}^\infty a_n^2 + b_n^2 \leq 1$.

Let $(\varepsilon_i)_i$ and $(\delta_i)_i$ be sequences of positive reals such that $\sum_{i=1}^\infty \varepsilon_i < \varepsilon$ and $\sum_{i=1}^\infty \delta_i < \varepsilon$. We then choose, by induction, increasing sequences $(n_i)_i \subset \mathbb{N}$ and $(k_i)_i \subset L$ such that for every $i \in \mathbb{N}$,

(i) $n_i > \max\{n : x_n^{k_{i-1}} \neq 0 \text{ or } y_n^{k_{i-1}} \neq 0\}$ when $i > 1$,
(ii) $\sum_{n>n_i} a_n^2 + b_n^2 < \varepsilon_i$,
(iii) $\sum_{n=1}^{n_i}\sum_{j=1}^{2n} \big|\|x_{n(j)}^{k_i}\| - a_{n,j}\big|^2 + \big|\|y_{n(j)}^{k_i}\| - b_{n,j}\big|^2 < \delta_i$.

For each $i\in\mathbb{N}$, due to (iii), we may assume that $\|x^{k_i}_{n(j)}\| = a_{n,j}$ and $\|y^{k_i}_{n(j)}\| = b_{n,j}$ for all $1\le n\le n_i$ and $1\le j\le 2n$, with an error δ_i. Then Lemma 5.30 yields

$$\Big\|\sum_{i=2}^m\sum_{j=i}^m\lambda_j\sum_{n=n_{i-1}+1}^{n_i}(x^{k_j}_n+y^{k_j}_n)\Big\|\le\Big(\sum_{i=2}^m\varepsilon_{i-1}\sum_{j=i}^m\lambda_j^2\Big)^{1/2}+\Big(\sum_{i=2}^m\delta_i\sum_{j=i}^m\lambda_j^2\Big)^{1/2}.$$

Hence, applying Lemma 5.31, we calculate

$$\begin{aligned}\Big\|\sum_{i=1}^m\lambda_i x^{k_i}\Big\| &\ge \Big\|\sum_{i=1}^m\lambda_i\Big(\sum_{n=1}^{n_1}(x^{k_i}_n+y^{k_i}_n)+\sum_{n=n_i+1}^{n_{i+1}}(x^{k_i}_n+y^{k_i}_n)\Big)\Big\|\\ &\quad-2\sqrt{\varepsilon}\Big(\sum_{i=1}^m\lambda_i^2\Big)^{1/2}\\ &\ge\sqrt{1-4\varepsilon}\Big(\sum_{i=1}^m a_i^2\Big)^{1/2}-2\sqrt{\varepsilon}\Big(\sum_{i=1}^m a_i^2\Big)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\Big\|\sum_{i=1}^m\lambda_i x^{k_i}\Big\| &\le \Big\|\sum_{i=1}^m\lambda_i\Big(\sum_{n=1}^{n_1}(x^{k_i}_n+y^{k_i}_n)+\sum_{n=n_i+1}^{n_{i+1}}(x^{k_i}_n+y^{k_i}_n)\Big)\Big\|\\ &\quad+2\sqrt{\varepsilon}\Big(\sum_{i=1}^m\lambda_i^2\Big)^{1/2}\\ &\le\sqrt{1+4\varepsilon}\Big(\sum_{i=1}^m \lambda_i^2\Big)^{1/2}+2\sqrt{\varepsilon}\Big(\sum_{i=1}^m \lambda_i^2\Big)^{1/2}.\ \blacksquare\end{aligned}$$

Corollary 5.33. *Every spreading model generated by a basic sequence in $\mathcal{X}$ is isometric to ℓ_2 and hence $\mathcal{X}$ admits a uniformly unique spreading model with respect to $\mathscr{F}(X)$.*

Remark 5.34. Using similar arguments we may show that every l-joint spreading model generated by a basic sequence in $\mathcal{X}$ is isomorphic to ℓ_2, while this does not happen with a uniform constant and (as already shown) $\mathcal{X}$ fails the UALS property. This exhibits a strong connection between the UALS and spaces with uniformly unique joint spreading models, which fails when the space only admits a uniformly unique spreading model.

As mentioned in Section 3, the space from [AM1] is another example of a space that admits a uniformly unique spreading model and fails to have a uniform constant for which all of its l-joint spreading models, for every $l\in\mathbb{N}$, are equivalent. This space however satisfies the stronger property that none of its subspaces admits a uniformly unique l-joint spreading model, in contrast to the space $\mathcal{X}$ which contains ℓ_2.

Motivated by the definition of $\mathcal{X}$, we modify the above arguments to show that every $L_p[0,1]$, for $1 \le p \le \infty$, as well as $C(K)$ for an uncountable compact metric space K, fail the UALS.

PROPOSITION 5.35. *For every $1 < p < q < \infty$, the spaces $(\sum \oplus \ell_p)_q$ and $(\sum \oplus \ell_q)_p$ fail the UALS property.*

Proof. For each $n \in \mathbb{N}$, we set $X_n = (\sum_{i=1}^{2n} \oplus \ell_p)_p$, $Y_n = (\sum_{i=1}^{2n} \oplus \ell_p)_q$ and $X = (\sum X_n \oplus Y_n)_q$. Assume that X satisfies the UALS property with constant $C > 0$ and pick $n \in \mathbb{N}$ with $C/n^r < 1/2$, where $r = (q-p)/(pq)$.

For every $G \subset \{1, \dots, 2n\}$, consider the operator $I_G : X_n \to Y_n$ such that $I_G(\sum_{i=1}^{2n} a_i x_i) = \sum_{i \in G} a_i x_i$ and set $A_n = I_{\{1,\dots,2n\}}$ and $W_n = \mathrm{co}\{I_G : \#G = n\}$. Let $x \in X_n$ with $x = \sum_{i=1}^{2n} x_i$ and $\sum_{i=1}^{2n} \|x_i\|^p = 1$ and let σ be a permutation of $\{1, \dots, 2n\}$ such that $\|x_{\sigma(1)}\|^p \ge \dots \ge \|x_{\sigma(2n)}\|^p$. Hence for $G = \{\sigma(1), \dots, \sigma(n)\}$ we have $\|A_n(x) - I_G(x)\| < 1/n^r$, and using the same arguments as in the proof of Proposition 5.29, we derive a contradiction. The case of $(\sum \oplus \ell_q)_p$ is similar. ∎

REMARK 5.36. It is immediate that if some infinite-dimensional complemented subspace of Banach space X fails the UALS property, then the same holds for X.

PROPOSITION 5.37. *The space $L_p[0,1]$ for $1 < p < \infty$ and $p \neq 2$ fails the UALS property.*

Proof. Recall that, as follows from Khinchin's inequality, ℓ_2 embeds isomorphically as a complemented subspace into $L_p[0,1]$ for all $1 < p < \infty$. If $p > 2$, for each $n \in \mathbb{N}$ set $X_n = (\sum_{i=1}^{2n} \oplus \ell_2)_2$ and $Y_n = (\sum_{i=1}^{2n} \oplus \ell_2)_p$, and if $p < 2$, set $X_n = (\sum_{i=1}^{2n} \oplus \ell_2)_p$ and $Y_n = (\sum_{i=1}^{2n} \oplus \ell_2)_2$. Then, by the proof of Proposition 5.29, $X = (\sum \oplus X_n \oplus Y_n)_p$ fails the UALS, and since it is complemented in $L_p[0,1] = (\sum \oplus L_p[0,1])_p$, the latter also fails that property. ∎

PROPOSITION 5.38. *The space $L_1[0,1]$ fails the UALS property.*

Proof. Assume that $L_1[0,1]$ satisfies the UALS with constant $C > 0$ and pick $n \in \mathbb{N}$ with $C/n < 1/9$. Set $X_n = (\sum_{i=1}^{2n} \oplus \ell_1)_1$ and $Y_n = (\sum_{i=1}^{2n} \oplus \ell_2)_2$. Since ℓ_1 is isometric to a complemented subspace of $L_1[0,1]$, the same holds for $(\sum \oplus X_n)_1$. Moreover, Khinchin's inequality shows that ℓ_2 embeds isomorphically into $L_1[0,1]$ and hence so does $(\sum \oplus Y_n)_2$.

For every $G \subset \{1, \dots, 2n\}$, consider the operator $I_G : X_n \to Y_n$ such that $I_G(\sum_{i=1}^{2n} a_i x_i) = \sum_{i \in G} a_i x_i$ and set $A_n = I_{\{1,\dots,2n\}}$ and $W_n = \mathrm{co}\{I_G : \#G = n\}$. As above, for all $x \in X_n$ with $\|x\| \le 1$, we may find $G \subset \{1, \dots, 2n\}$ such that $\|A_n(x) - I_G(x)\| < 1/n$. Let $B = \sum_{i=1}^{k} \lambda_i I_{G_i}$ in W_n and Y be a finite-codimensional subspace of $L_1[0,1]$ such that $\|(A_n - B)|_Y\| < C/n$ and choose, as in the proof of Proposition 5.29, $1 \le j \le 2n$ with $\sum_{i=1}^{k} \lambda_i \chi_{G_i}(j) \le 1/2$.

Let $x_1^*,\dots,x_l^*\in L_\infty[0,1]$ with $Y=\bigcap_{i=1}^l \ker x_i^*$. Denote by $(e_m)_m$ the basis of $X_{n(j)}$ and choose $M\in[\mathbb{N}]^\infty$ such that $(x_i^*(e_m))_{m\in M}$ converges for all $1\le i\le l$. Using Lemma 2.1 we choose $m_1,m_2\in M$ such that $d(x,Y)<1/8$ for $x=(e_{m_1}-e_{m_2})/2$. Then $\|A_n(x)-B(x)\|\ge 1/4$ and hence $\|(A_n-B)|_Y\|\ge 1/9$, a contradiction. ∎

Proposition 5.39. *The space $L_\infty[0,1]$ fails the UALS property.*

Proof. Fix $n\in\mathbb{N}$. The σ-algebra $\mathcal{B}[0,1]$ of all Borel sets of $[0,1]$ is homeomorphic to that of $[0,1]^{2n}$ and hence $L_\infty[0,1]$ is isometric to $L_\infty[0,1]^{2n}$. For $1\le i\le 2n$, denote by $\mathcal{B}_i$ the σ-algebra generated by $\{B\in\prod_{i=1}^{2n}\mathcal{B}[0,1]: B_j=[0,1] \text{ for } j>i\}$ and for $f\in L_\infty[0,1]^{2n}$ set $E_i(f)=E[f|\mathcal{B}_i]$ and consider the operator $\Delta_i : L_\infty[0,1]^{2n}\to L_2([0,1]^i,\otimes_{j\le i}\lambda)$ with $\Delta_i(f)=E_i(f)-E_{i-1}(f)$, where $E_0(f)=0$ and λ denotes the Lebesgue measure on $[0,1]$.

For every $G\subset\{1,\dots,2n\}$, let

$$\Delta_G : L_\infty[0,1]^{2n}\to\Big(\sum_{i=1}^{2n}\oplus L_2([0,1]^i,\otimes_{j\le i}\lambda)\Big)_\infty$$

with $\Delta_G=\sum_{i\in G}\Delta_i$ and set $A_n=\Delta_{\{1,\dots,2n\}}$ and $W_n=\mathrm{co}\{\Delta_G:\#G=n\}$. Observe that $(\sum_{i=1}^{2n}\oplus L_2([0,1]^i,\otimes_{j\le i}\lambda))_\infty$ embeds isometrically into $L_\infty[0,1]^{2n}$ and hence $\Delta_G : L_\infty[0,1]\to L_\infty[0,1]$. Let $f\in L_\infty[0,1]^{2n}$ and notice that $(E_i(f))_{i=1}^{2n}$ is a martingale, since $\mathcal{B}_i$ is a subalgebra of $\mathcal{B}_j$ for every $1\le i<j\le 2n$. Then for the martingale differences $(\Delta_i(f))_{i=1}^{2n}$ the Burkholder inequality [B] yields a $c_2>0$ such that

$$\Big(\int_0^1\sum_{i=1}^{2n}\Delta_i(f)^2\Big)^{1/2}\le c_2\|f\|_2. \tag{7}$$

Claim 1. *For every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that, for every $n\ge n_0$ and $f\in L_\infty[0,1]^{2n}$, there is a $B\in W_n$ such that $\|(A_n-B)f\|\le\varepsilon\|f\|$.*

Proof of Claim 1. Pick $n_0\in\mathbb{N}$ such that $c_2/\sqrt{n_0}<\varepsilon$. Let $n\ge n_0$ and f be in $L_\infty[0,1]^{2n}$ with $\|f\|=1$. Then, as a direct consequence of (7), we see that $\#\{i:\|\Delta_i(f)\|_2>c_2/\sqrt{n+1}\}\le n$. Let σ be permutation of $\{1,\dots,2n\}$ such that $\|\Delta_{\sigma(1)}(f)\|_2\ge\cdots\ge\|\Delta_{\sigma(2n)}(f)\|_2$. Hence for $G=\{\sigma(1),\dots,\sigma(n)\}$, we conclude that $\|(A_n-\Delta_G)f\|<c_2/\sqrt{n}$ and this yields the desired result. ∎

Claim 2. *For every $n\in\mathbb{N}$, every finite-codimensional subspace Y of $L_\infty[0,1]^{2n}$ and $B\in W_n$, we have $\|(A-B)|_Y\|\ge 1/9$.*

Proof of Claim 2. There exist $x_1^*,\dots,x_l^*\in(L_\infty[0,1]^{2n})^*$ with $Y=\bigcap_{i=1}^l\ker x_i^*$ and also B is a convex combination $\sum_{i=1}^k\lambda_i\Delta_{G_i}$ in W_n. Then, as in Proposition 5.29, choose $1\le j\le 2n$ with $\sum_{i=1}^k\lambda_i\chi_{G_i}(j)\le 1/2$. Denote by $(R_m)_m$ the Rademacher system and consider a natural extension $(\tilde{R}_m)_m$

into $L_\infty[0,1]^{2n}$ such that $\tilde{R}_m(t_1,\dots,t_{2n}) = R_m(t_j)$. We choose $M \in [\mathbb{N}]^\infty$ such that $(x_i^*(\tilde{R}_m))_{m\in M}$ converges for all $1 \le i \le l$, and applying Lemma 2.1 we find $m_1, m_2 \in M$ such that $d(f,Y) < 1/8$ for $f = (\tilde{R}_{m_1} - \tilde{R}_{m_2})/2$. We recall that $(\tilde{R}_m)_m$ is isomorphic to the unit vector basis of ℓ_1 in the L_∞-norm and hence $\|f\|_\infty = 1$. Notice that, for every $m \in M$, we have $\Delta_i(\tilde{R}_m) = \delta_{ij}\tilde{R}_m$ and $\|\tilde{R}_m\|_2 = 1$, and since R_m are orthogonal, $\|\tilde{R}_{m_1} - \tilde{R}_{m_2}\|_2 = (\|\tilde{R}_{m_1}\|^2 + \|\tilde{R}_{m_2}\|^2)^{1/2}$. Hence $\|(A_n - B)f\| \ge 1/4$ and so we conclude that $\|(A_n - B)|_Y\| \ge 1/9$, since $d(f,Y) < 1/8$. ■

Assume that $L_\infty[0,1]$ satisfies the UALS with constant $C > 0$ and pick $\varepsilon > 0$ such that $C\varepsilon < 1/9$. The first claim yields an $n \in \mathbb{N}$ such that, for every $f \in L_\infty[0,1]^{2n}$ with $\|f\| \le 1$, there exists $B \in W_n$ with $\|(A_n - B)f\| < \varepsilon$. Hence there exist a subspace Y of $L_\infty[0,1]^{2n}$ of finite codimension and a $B \in W_n$ such that $\|(A - B)|_Y\| < C\varepsilon$, and this contradicts our second claim, since $C\varepsilon < 1/9$. ■

Proposition 5.40. *Let K be an uncountable compact metrizable space. Then the space $C(K)$ fails the UALS property.*

Proof. We set $\Omega = \{-1,1\}^{\mathbb{N}}$, and Milyutin's theorem [M] implies that the space $C(K)$ is isomorphic to $C(\Omega)$ for every K uncountable compact metrizable. We now fix $n \in \mathbb{N}$, consider a partition of $\mathbb{N}$ into disjoint infinite sets $N_1,\dots,N_{2n}$ and set $\Omega_i = \{-1,1\}^{N_i}$ for $1 \le i \le 2n$. Clearly $C(\Omega)$ is isometric to $C(\prod_{i=1}^{2n}\Omega_i)$.

In a similar manner to the previous proposition, for every $1 \le i \le 2n$, we define $E_i, \Delta_i : C(\prod_{i=1}^{2n}\Omega_i) \to L_2(\prod_{j\le i}\Omega_j, \otimes_{j\le i}\mu_j)$, where by μ_j we denote the Haar probability measure on Ω_j. Moreover, for every $G \subset \{1,\dots,2n\}$, we define the operator

$$\Delta_G : C\Big(\prod_{i=1}^{2n}\Omega_i\Big) \to \Big(\sum_{i=1}^{2n} L_2\Big(\prod_{j\le i}\Omega_j, \otimes_{j\le i}\mu_j\Big)\Big)_\infty$$

with $\Delta_G = \sum_{i\in G}\Delta_i$. Observe that $(\sum_{i=1}^{2n} L_2(\prod_{j\le i}\Omega_j, \otimes_{j\le i}\mu_j))_\infty$ is isometric to a subspace of $C(\Omega)$ and hence $\Delta_G : C(\Omega) \to C(\Omega)$. Also set $A_n = \Delta_{\{1,\dots,2n\}}$ and $W_n = \mathrm{co}\{\Delta_G : \#G = n\}$.

The family $(\pi_n)_n$ of the projections of Ω onto its coordinates corresponds to the Rademacher system in $L_\infty[0,1]$. Therefore, assuming that $C(\Omega)$ satisfies the UALS property, we arrive at a contradiction applying the corresponding arguments of Proposition 5.39. ■

5.5. Final remarks. This last subsection contains some final remarks and open problems concerning the UALS property. We start with the following example suggested by W. B. Johnson which shows that in the definition of the UALS we cannot expect the uniform approximation to happen on the whole space.

Example 5.41. Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and for $x, x^* \in \mathbb{R}^2$ define the operator $x^* \otimes x : \mathbb{R}^2 \to \mathbb{R}^2$ with $x^* \otimes x(y) = x^*(y)x$ and set

$$W = \mathrm{co}\{x^* \otimes x : x, x^* \in \mathbb{R}^2 \text{ and } \|x\|, \|x^*\| \leq 1\}.$$

Let $y \in \mathbb{R}^2$ with $\|y\| \leq 1$ and $x^* \in \mathbb{R}^2$ with $\|x^*\| = 1$ be such that $x^*(y) = \|y\|$. Then for $x = y/\|y\|$, we have $x^* \otimes x \in W$ and $\|x^* \otimes x(y) - I(y)\| = 0$, where I denotes the identity operator.

For any $B \in W$, there exists a convex combination $\sum_{i=1}^5 a_i B_i$ in W such that $B = \sum_{i=1}^5 a_i B_i$. Then $a_{i_0} \geq 1/5$ for some $1 \leq i_0 \leq 5$, and for $x \in \ker B_{i_0}$ with $\|x\| = 1$ we have $\|x - \sum_{i=1}^4 a_i B_i(x)\| \geq 1 - \sum_{i=1}^4 a_i$. Hence $\|I - B\| \geq 1/5$ for all $B \in W$.

This example is extended to every Banach space of dimension greater than two by the following easy modification.

Proposition 5.42. *Let X be a Banach space with* $\dim X \geq 2$. *There exist $C > 0$ and a convex compact subset W of $\mathcal{L}(X)$ with the property that, for every $x \in B_X$, there exists a $B \in W$ such that $\|x - B(x)\| = 0$ whereas $\|I - B\| \geq C$ for all $B \in W$, where $I : X \to X$ denotes the identity operator.*

Proof. Let e_1, e_2 be linearly independent vectors in X, denote by Y their linear span and let Z be a subspace of X such that $X = Y \oplus Z$. Set

$$W = \mathrm{co}\{x^* \otimes x|_Y + I|_Z : x, x^* \in Y \text{ and } \|x\|, \|x^*\| \leq 1\}$$

and notice that, using similar arguments to those in the previous example, we obtain the desired result. ∎

Remark 5.43. I. Gasparis pointed out that in the case of c_0, the UALS can be proved without the use of Kakutani's theorem. This is a consequence of the following fact. Let $T \in \mathcal{L}(c_0)$ and $(x_n^i)_n$, $1 \leq i \leq l$, be normalized block sequences such that for some $\varepsilon > 0$, we have $\|T(x_n^1)\| \geq \varepsilon$ for all $n \in \mathbb{N}$. Then, for every $\delta > 0$, there exists a choice $n_1 < \cdots < n_l$ such that $\|T(\sum_{i=1}^l x_{n_i}^i)\| > \varepsilon - \delta$. Assume now that $T_1, \ldots, T_l \in \mathcal{L}(c_0)$ and $\varepsilon > 0$ are such that, for every x in the unit ball of c_0, there exists $1 \leq i \leq l$ such that $\|T_i(x)\| \leq \varepsilon$. Then, for every $\varepsilon' > \varepsilon$, there exist $1 \leq i \leq l$ and $n_0 \in \mathbb{N}$ such that $\|T_i|_{\mathrm{span}\{e_n : n \geq n_0\}}\| \leq \varepsilon'$. If not, we may choose for each $i = 1, \ldots, l$ a normalized block sequence $(x_n^i)_n$ such that $\|T_i(x_n^i)\| \geq \varepsilon'$ for all $n \in \mathbb{N}$. Then applying simultaneously the above observation for the operators $T_1, \ldots, T_l$, we may select $n_1 < \cdots < n_l$ such that

$$\Big\|T_i\Big(\sum_{i=1}^l x_{n_i}^i\Big)\Big\| > \varepsilon \quad \text{for all } i = 1, \ldots, l,$$

and this yields a contradiction.

REMARK 5.44. There exist Banach spaces which satisfy the UALS while this is not true for all of their subspaces. As already shown, every $L_p[0,1]$ for $1 < p < \infty$ and $p \neq 2$ fails the UALS whereas item (ii) of Corollary 5.24 implies that it embeds in a space satisfying this property.

Another open problem in a similar context is the following. Notice that all spaces in the previous subsection failing the UALS contain a subspace which satisfies that property.

PROBLEM 2. *Does there exist a Banach space such that none of its subspaces satisfies the UALS property?*

Acknowledgements. We express our thanks to I. Gasparis, W. B. Johnson and B. Sari for their comments and remarks that allowed us to improve the content of the paper.

The fourth named author was supported by the National Science Foundation under Grant Numbers DMS-1600600 and DMS-1912897.

References

[AI] I. Amemiya and T. Ito, *Weakly null sequences in James spaces on trees*, Kodai Math. J. 4 (1981), 418–425.

[AF$^+$] S. A. Argyros, D. Freeman, R. Haydon, E. Odell, Th. Raikoftsalis, Th. Schlumprecht and D. Zisimopoulou, *Embedding uniformly convex spaces into spaces with very few operators*, J. Funct. Anal. 262 (2012), 825–849.

[AH] S. A. Argyros and R. G. Haydon, *A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar-plus-compact problem*, Acta Math. 206 (2011), 1–54.

[AKT] S. A. Argyros, V. Kanellopoulos and K. Tyros, *Finite order spreading models*, Adv. Math. 234 (2013), 574–617.

[AM1] S. A. Argyros and P. Motakis, *On the complete separation of asymptotic structures in Banach spaces*, arXiv:1902.10092 (2019).

[AM2] S. A. Argyros and P. Motakis, *A dual method of constructing hereditarily indecomposable Banach spaces*, Positivity 20 (2016), 625–662.

[BL] B. Beauzamy et J. T. Lapresté, *Modèles étalés des espaces de Banach*, Travaux en Cours, Hermann, Paris, 1984, iv+210 pp.

[Be] G. Berg, *On James spaces*, Ph.D. thesis, Univ. of Texas, Austin, TX, 1996.

[BK] H. F. Bohnenblust and S. Karlin, *On a theorem of Ville*, in: Contributions to the Theory of Games, Ann. of Math. Stud. 24, Princeton Univ. Press, Princeton, NJ, 1950, 155–160.

[BS] A. Brunel and L. Sucheston, *On B-convex Banach spaces*, Math. Systems Theory 7 (1973), 294–299.

[B] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Prob. 1 (1973), 19–42.

[CN] W. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer, 1974.

[FG] H. Fetter and B. Gamboa de Buen, *Spreading sequences in JT*, Studia Math. 125 (1997), 57–66.

[FJ] T. Figiel and W. B. Johnson, *A uniformly convex Banach space which contains no ℓ_p*, Compos. Math. 29 (1974), 179–190.

[FOS] D. Freeman, E. Odell and Th. Schlumprecht, *The universality of ℓ_1 as a dual space*, Math. Ann. 351 (2011), 149–186.

[GM] W. T. Gowers and B. Maurey, *Banach spaces with small spaces of operators*, Math. Ann. 307 (1997), 543–568.

[H] J. Hagler, *Some more Banach spaces which contain l^1*, Studia Math. 46 (1973), 35–42.

[HS] J. Hagler and C. Stegall, *Banach spaces whose duals contain complemented subspaces isomorphic to $C[0,1]$*, J. Funct. Anal. 13 (1973), 233–251.

[HO] L. Halbeisen and E. Odell, *On asymptotic models in Banach spaces*, Israel J. Math. 139 (2004), 253–291.

[J1] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. 52 (1950), 518–527.

[J2] R. C. James, *A separable somewhat reflexive Banach space with nonseparable dual*, Bull. Amer. Math. Soc. 80 (1974), 738–743.

[Ka] S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. 8 (1941), 457–459.

[Kr] J. L. Krivine, *Sous espaces de dimension finie des espaces de Banach réticulés*, Ann. of Math. 104 (1976), 1–29.

[L] H. Lemberg, *Nouvelle démonstration d'un théorème de J. L. Krivine sur la finie représentation de l_p dans un espace de Banach*, Israel J. Math. 39 (1981), 341–348.

[LS] D. R. Lewis and C. Stegall, *Banach spaces whose duals are isomorphic to $l_1(\Gamma)$*, J. Funct. Anal. 12 (1973), 177–187.

[LR] J. Lindenstrauss and H. P. Rosenthal, *The $\mathscr{L}_p$ spaces*, Israel J. Math. 7 (1969), 325–349.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces*, Springer, Berlin, 1977.

[MMT] B. Maurey, V. D. Milman and N. Tomczak-Jaegermann, *Asymptotic infinite-dimensional theory of Banach spaces*, in: Geometric Aspects of Functional Analysis (Israel, 1992–1994), Oper. Theory Adv. Appl. 77, Birkhäuser, Basel, 1995, 149–175.

[MR] B. Maurey and H. P. Rosenthal, *Normalized weakly null sequence with no unconditional subsequence*, Studia Math. 61 (1977), 77–98.

[MT] V. D. Milman and N. Tomczak-Jaegermann, *Asymptotic l_p spaces and bounded distortions*, in: Banach Spaces (Mérida, 1992), Contemp. Math. 144, Amer. Math. Soc., Providence, RI, 1993, 173–195.

[M] A. A. Milyutin, *Isomorphisms of spaces of continuous functions on compacts of power continuum*, Teoriya Funktsiĭ (Kharkov) 2 (1966), 150–156 (in Russian).

[Ra] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. 30 (1929), 264–286.

[Ro] H. P. Rosenthal, *A characterization of Banach spaces containing l^1*, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 2411–2413.

[S] C. Stegall, *Banach spaces whose duals contain $l_1(\Gamma)$ with applications to the study of dual $L_1(\mu)$ spaces*, Trans. Amer. Math. Soc. 176 (1973), 463–477.

[T] B. S. Tsirelson, *Not every Banach space contains an imbedding of l_p or c_0*, Funct. Anal. Appl. 8 (1974), 138–141.

S. A. Argyros, A. Georgiou
Department of Mathematics
Faculty of Applied Sciences
National Technical University of Athens
Zografou Campus
157 80, Athens, Greece
E-mail: sargyros@math.ntua.gr
ale.grgu@gmail.com

A.-R. Lagos
School of Electrical and Computer Engineering
National Technical University of Athens
Zografou Campus
157 80 Athens, Greece
E-mail: lagosth993@gmail.com

Pavlos Motakis
Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, IL 61801, U.S.A.
E-mail: pmotakis@illinois.edu