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ON THE STRUCTURE OF SEPARABLE $\mathcal{L}_\infty$-SPACES

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Abstract. Based on a construction method introduced by Bourgain and Delbaen, we give a general definition of a Bourgain–Delbaen space and prove that every infinite-dimensional separable $\mathcal{L}_\infty$-space is isomorphic to such a space. Furthermore, we provide an example of a $\mathcal{L}_\infty$ and asymptotic c_0 space not containing c_0.

§1. *Introduction.* In the late 1960s Lindenstrauss and Pełczyński introduced the class of $\mathcal{L}_\infty$-spaces, which naturally extends the class of L_∞-spaces [**11**]. Whether such spaces always contain a copy of c_0 remained a long standing open problem, which was solved in the negative direction by Bourgain and Delbaen in [**6**]. In particular, they introduced a method for constructing $\mathcal{L}_\infty$-spaces and any example constructed with its use has been customarily called a Bourgain–Delbaen space. This method proved to be very fruitful, as it has been used extensively to construct a wide variety of $\mathcal{L}_\infty$-spaces, including the first example of a Banach space satisfying the "scalar plus compact" property [**5**], as well as to obtain other structural results in the geometry of Banach spaces [**4**, **7**].

The main framework of the Bourgain–Delbaen method concerns the construction of an increasing sequence $(Y_n)_n$ of finite-dimensional subspaces of a $\ell_\infty(\Gamma)$-space, with Γ countably infinite, each one uniformly isomorphic to some ℓ_∞^n. This is achieved by carefully defining a sequence of extension operators $(i_n)_n$, each one defined on $\ell_\infty(\Gamma_n)$ with Γ_n an appropriate finite subset of Γ, and taking Y_n to be the image of i_n. If the sequence $(i_n)_n$ satisfies a certain compatibility property, then the closure of the union of the Y_n, $n \in \mathbb{N}$, is a $\mathcal{L}_\infty$-space whose properties depend on the definition of the aforementioned extension operators.

Based on this method, we give a broad definition of a Bourgain–Delbaen space. We include a brief study of the basic properties of such spaces and, using techniques rooted in the early theory of $\mathcal{L}_\infty$-spaces (i.e. [**9**–**14**]), we prove that every separable infinite-dimensional $\mathcal{L}_\infty$-space is isomorphic to such a space. We use this result to deduce that every separable infinite-dimensional $\mathcal{L}_\infty$-space X has an infinite-dimensional $\mathcal{L}_\infty$-subspace Z, so that the quotient X/Z is isomorphic to c_0.

In the final section of this paper we provide an example of an asymptotic c_0 isomorphic ℓ_1-predual (i.e. a space whose dual is isomorphic to ℓ_1) $\mathfrak{X}_0$ that does not contain c_0. This result in particular yields that the proximity of a Banach space to c_0 in a local setting, in the sense of being $\mathcal{L}_\infty$, as well as in an asymptotic

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setting, in the sense of being asymptotic c_0, does not imply proximity to c_0 in an infinite-dimensional level. We think of this example as a step towards the solution of a problem in [**8**, Question IV.2], namely whether every isomorphic ℓ_1-predual satisfying Pełczyński's property-(u) is isomorphic to c_0. The question of the existence of an asymptotic c_0 $\mathcal{L}_\infty$-space not containing c_0 was asked by B. Sari.

§2. *Bourgain–Delbaen spaces.* In the present section we give a general definition of the spaces that shall be called Bourgain–Delbaen spaces and prove some of their basic properties. It turns out that every infinite-dimensional separable $\mathcal{L}_\infty$-space is isomorphic to a Bourgain–Delbaen space. We recall the definition of a $\mathcal{L}_\infty$-space, which was introduced in [**11**, Definition 3.1].

Definition 2.1. A Banach space X is called a $\mathcal{L}_{\infty,C}$-space, for some $C \geqslant 1$, if for every finite-dimensional subspace F of X there exists a finite-dimensional subspace G of X, containing F, which is C-isomorphic to ℓ_∞^n, where $n = \dim G$. A Banach space X will be called a $\mathcal{L}_\infty$-space if it is a $\mathcal{L}_{\infty,C}$-space for some $C \geqslant 1$.

Remark 2.2. If X is an infinite-dimensional separable Banach space, then it is well known, and not difficult to prove, that X is a $\mathcal{L}_\infty$-space if and only if there exist a constant C and an increasing sequence $(Y_n)_n$ of finite-dimensional subspaces of X, whose union is dense in X, such that Y_n is C-isomorphic to $\ell_\infty^{k_n}$, where $k_n = \dim Y_n$, for all $n \in \mathbb{N}$.

2.1. *The definition of a Bourgain–Delbaen space.* We give a broad definition of what kind of spaces we will refer to as Bourgain–Delbaen spaces.

Definition 2.3. Let Γ_1, Γ be non-empty sets with $\Gamma_1 \subset \Gamma$. A linear operator $i : \ell_\infty(\Gamma_1) \to \ell_\infty(\Gamma)$ will be called an extension operator if, for every $x \in \ell_\infty(\Gamma_1)$ and $\gamma \in \Gamma_1$, we have that $x(\gamma) = i(x)(\gamma)$.

Definition 2.4. Let $(\Gamma_q)_{q=0}^\infty$ be a strictly increasing sequence of non-empty sets and $\Gamma = \bigcup_q \Gamma_q$. A sequence of extension operators $(i_q)_{q=0}^\infty$, with $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$ for all $q \in \mathbb{N} \cup \{0\}$, will be called compatible if, for every $p, q \in \mathbb{N} \cup \{0\}$ with $p < q$ and $x \in \ell_\infty(\Gamma_p)$, the following holds:

$$i_p(x) = i_q(r_q(i_p(x))),$$

i.e. $i_p = i_q \circ r_q \circ i_p$, where $r_q : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma_q)$ denotes the natural restriction operator.

Definition 2.5. Let $(\Gamma_q)_{q=0}^\infty$ be a strictly increasing sequence of non-empty finite sets, $\Gamma = \bigcup_q \Gamma_q$ and $(i_q)_{q=0}^\infty$, with $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$ for all $q \in \mathbb{N} \cup \{0\}$, be a sequence of compatible extension operators such that $C = \sup_q \|i_q\|$ is finite.

(i) We define the sets $\Delta_0 = \Gamma_0$ and $\Delta_q = \Gamma_q \setminus \Gamma_{q-1}$ for $q \in \mathbb{N} \cup \{0\}$.
(ii) For every $\gamma \in \Gamma$, we define d_γ, a vector in $\ell_\infty(\Gamma)$, as follows: if $\gamma \in \Delta_q$ for some $q \in \mathbb{N} \cup \{0\}$, then $d_\gamma = i_q(e_\gamma)$.

The space $\mathfrak{X}_{(\Gamma_q, i_q)_q} = \overline{\langle\{d_\gamma : \gamma \in \Gamma\}\rangle}$, i.e. the closed subspace of $\ell_\infty(\Gamma)$ spanned by the vectors $(d_\gamma)_{\gamma\in\Gamma}$, will be called a Bourgain–Delbaen space.

Remark 2.6. Since the operators $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$ are extension operators, it easily follows that i_q is a C-isomorphism for all $q \in \mathbb{N} \cup \{0\}$, where $C = \sup_q \|i_q\|$. In particular, the following hold:

(i) if $Y_q = i_q[\ell_\infty(\Gamma_q)]$, then Y_q is C-isomorphic to $\ell_\infty(\Gamma_q)$; and
(ii) the vectors $(d_\gamma)_{\gamma\in\Delta_q}$ are C-equivalent to the unit vector basis of $\ell_\infty(\Delta_q)$ for all $q \in \mathbb{N} \cup \{0\}$.

Remark 2.7. The above Remark 2.6 and Proposition 2.12 from §2.2 provide an equivalent definition of a Bourgain–Delbaen space, namely the following.

Let $(\Gamma_q)_q$ be a strictly increasing sequence of finite non-empty sets and $(Y_q)_q$ be an increasing sequence of subspaces of $\ell_\infty(\Gamma)$, where $\Gamma = \bigcup_q \Gamma_q$. If there exists a constant $C > 0$ such that for every $q \in \mathbb{N}$, when restricted onto the subspace Y_q, the restriction operator $r_q : Y_q \to \ell_\infty(\Gamma_q)$ is an onto C-isomorphism, then the space $X = \overline{\bigcup_q Y_q}$ is a Bourgain–Delbaen space.

Indeed, it is straightforward to check that the maps $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$ with $i_q = r_q^{-1} : \ell_\infty(\Gamma_q) \to Y_q \hookrightarrow \ell_\infty(\Gamma)$ are uniformly bounded compatible extension operators and $\mathfrak{X}_{(\Gamma_q, i_q)_q} = X$.

2.2. *Properties of a Bourgain–Delbaen space.* We present some basic properties of a Bourgain–Delbaen space, which can be deduced from Definition 2.5.

PROPOSITION 2.8. *Let $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ be a Bourgain–Delbaen space. For all $q \in \mathbb{N} \cup \{0\}$, we denote $M_q = \langle\{d_\gamma : \gamma \in \Delta_q\}\rangle$. Then $(M_q)_{q=0}^\infty$ forms a finite-dimensional decomposition (FDD) for the space $\mathfrak{X}_{(\Gamma_q, i_q)_q}$. More precisely, for every $p, q \in \mathbb{N} \cup \{0\}$ with $p < q$ and $x_\ell \in M_\ell$ for $\ell = 0, 1, \ldots, q$, the following holds:*

$$\Big\|\sum_{\ell=0}^{p} x_\ell\Big\| \leqslant C \Big\|\sum_{\ell=0}^{q} x_\ell\Big\|, \tag{1}$$

where $C = \sup_q \|i_q\|$.

Proof. Let $x_\ell = i_\ell(y_\ell)$ for some $y_\ell \in \langle\{e_\gamma : \gamma \in \Delta_\ell\}\rangle$ for $\ell = 0, 1, \ldots, q$. Then, by the compatibility of the operators, we have that $x_\ell = i_p(r_p(i_\ell(y_\ell)))$, i.e. $x_\ell = i_p \circ r_p(x_\ell)$ for $\ell = 0, 1, \ldots, p$ and hence

$$\Big\|\sum_{\ell=0}^{p} x_\ell\Big\| = \Big\|i_p \circ r_p\Big(\sum_{\ell=0}^{p} x_\ell\Big)\Big\| \leqslant C \Big\|r_p\Big(\sum_{\ell=0}^{p} x_\ell\Big)\Big\|. \tag{2}$$

On the other hand, once more by the extension property of the operators, we have that $r_p(x_\ell) = 0$ (which also yields that $i_p \circ r_p(x_\ell) = 0$) for $\ell = p+1, \ldots, q$ and therefore we obtain

$$\Big\|\sum_{\ell=0}^{q} x_\ell\Big\| \geqslant \Big\|r_p\Big(\sum_{\ell=0}^{q} x_\ell\Big)\Big\| = \Big\|r_p\Big(\sum_{\ell=0}^{p} x_\ell\Big)\Big\|. \tag{3}$$

The desired inequality immediately follows from (2) and (3). □

Remark 2.9. By the proposition above, for every interval $E = \{p, \ldots, q\}$ of $\mathbb{N} \cup \{0\}$ we can define the projection $P_E : \mathfrak{X}_{(\Gamma_q, i_q)_q} \to M_p + \cdots + M_q$ associated to the FDD $(M_q)_q$ and the interval E. The above proof implies that

$$P_{[0,q]}x = i_q \circ r_q(x) \tag{4}$$

for all $q \in \mathbb{N} \cup \{0\}$ and $x \in \mathfrak{X}_{(\Gamma_q, i_q)_q}$ and hence also

$$P_{(p,q]}x = i_q \circ r_q(x) - i_p \circ r_p(x) \tag{5}$$

for all $p, q \in \mathbb{N} \cup \{0\}$ with $p < q$ and $x \in \mathfrak{X}_{(\Gamma_q, i_q)_q}$. We shall call P_E the Bourgain–Delbaen projection onto E. Note that $\|P_E\| \leqslant 2C$ for every interval E of $\mathbb{N} \cup \{0\}$.

Remark 2.10. Proposition 2.8 in conjunction with Remark 2.6(ii) yield that $((d_\gamma)_{\gamma \in \Delta_q})_{q=0}^{\infty}$ is a Schauder basis of $\mathfrak{X}_{(\Gamma_q, i_q)_q}$. Although in some cases it is more convenient to use the FDD $(M_q)_q$, in §5 we shall indeed use this basis.

Remark 2.11. Let x be a finitely supported vector in $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ with $\operatorname{ran} x = E$. Using Remark 2.9, one can see that if $q = \max E$, there exists $y \in \ell_\infty(\Gamma_q)$ with $\operatorname{supp} y \subset \bigcup_{p \in E} \Delta_p$ such that $x = i_q(y)$.

PROPOSITION 2.12. *Every Bourgain–Delbaen space $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ is a $\mathcal{L}_\infty$-space. More precisely, if $Y_q = i_q[\ell_\infty(\Gamma_q)]$ for all $q \in \mathbb{N} \cup \{0\}$, then $(Y_q)_q$ is a strictly increasing sequence of subspaces of $\mathfrak{X}_{(\Gamma_q, i_q)_q}$, whose union is dense in the whole space and for every $q \in \mathbb{N} \cup \{0\}$ Y_q is C-isomorphic to $\ell_\infty(\Gamma_q)$, where $C = \sup_q \|i_q\|$.*

Proof. By Remark 2.6(i), we have that Y_q is C-isomorphic to $\ell_\infty(\Gamma_q)$ for all $q \in \mathbb{N} \cup \{0\}$. It remains to show that the sequence $(Y_q)_q$ is strictly increasing and its union is dense in $\mathfrak{X}_{(\Gamma_q, i_q)_q}$. We will prove that $Y_q = M_0 + \cdots + M_q$, where $M_q = \langle\{d_\gamma : \gamma \in \Delta_q\}\rangle$ for all $q \in \mathbb{N} \cup \{0\}$. Note that, in conjunction with Proposition 2.8, the previous fact easily implies the desired result.

Note that for $p, q \in \mathbb{N} \cup \{0\}$ with $p \leqslant q$ and $x \in M_p$, there is a $y \in \langle\{e_\gamma : \gamma \in \Delta_p\}\rangle$ with $x = i_p(y)$. Using the compatibility of the extension operators, we obtain that $x = i_q(r_q(x)) \in i_q(\ell_\infty(\Gamma_q)) = Y_q$. We have hence concluded that $M_0 + \cdots + M_q \subset Y_q$. To conclude that the above inclusion cannot be proper, we will show that $\dim(M_0 + \cdots + M_q) = \dim Y_q$. Note that since, by Proposition 2.8,

$(M_q)_{q=0}^\infty$ is an FDD, we have that $\dim(M_0 + \cdots + M_q) = \dim(M_0) + \cdots + \dim(M_q)$. Moreover, Remark 2.6(ii) implies that $\dim(M_p) = \#\Delta_p$ for $p = 0, \ldots, q$. The definition of the sets Δ_p yields that $\dim = (M_0 + \cdots + M_q) = \#\Gamma_q$. Remark 2.6(i) implies that $\dim Y_q = \#\Gamma_q$ and therefore $\dim(M_0 + \cdots + M_q) = \dim Y_q$. □

2.3. *The functionals* $(e_\gamma^*)_{\gamma\in\Gamma}$. Let $\mathfrak{X}_{(\Gamma_q,i_q)_q}$ be a Bourgain–Delbaen space. For every $\gamma \in \Gamma$, we denote by $e_\gamma^* : \mathfrak{X}_{(\Gamma_q,i_q)_q} \to \mathbb{R}$ the evaluation functional on the γth coordinate, defined on $\ell_\infty(\Gamma)$ and then restricted to the subspace $\mathfrak{X}_{(\Gamma_q,i_q)_q}$. Note that $\|e_\gamma^*\| \leqslant 1$ for all $\gamma \in \Gamma$.

PROPOSITION 2.13. *Let $\mathfrak{X}_{(\Gamma_q,i_q)_q}$ be a Bourgain–Delbaen space. Then $(e_\gamma^*)_{\gamma\in\Gamma}$ is C-equivalent to the unit vector basis of $\ell_1(\Gamma)$, where $C = \sup_q \|i_q\|$.*

Proof. Let A be a non-empty finite subset of Γ and $(\lambda_\gamma)_{\gamma\in A}$ be scalars. Choose $q \in \mathbb{N} \cup \{0\}$ such that $A \subset \Gamma_q$ and take the normalized vector $y = \sum_{\gamma\in A} \operatorname{sgn}\lambda_\gamma e_\gamma$ in $\ell_\infty(\Gamma_q)$. Note that $\|i_q(y)\| \leqslant C$. Since i_q is an extension operator and $A \subset \Gamma_q$, we obtain the following estimate:

$$C\sum_{\gamma\in A}|\lambda_\gamma| \geqslant C\left\|\sum_{\gamma\in A}\lambda_\gamma e_\gamma^*\right\| \geqslant \sum_{\gamma\in A}\lambda_\gamma e_\gamma^*(i_q(y)) = \sum_{\gamma\in A}\lambda_\gamma i_q(y)(\gamma)$$
$$= \sum_{\gamma\in A}\lambda_\gamma y(\gamma) = \sum_{\gamma\in A}\lambda_\gamma \operatorname{sgn}\lambda_\gamma = \sum_{\gamma\in A}|\lambda_\gamma|,$$

i.e. $C^{-1}\sum|\lambda_\gamma| \leqslant \|\sum\lambda_\gamma e_\gamma^*\| \leqslant \sum|\lambda_\gamma|$, which is the desired result. □

Definition 2.14. Let $\mathfrak{X}_{(\Gamma_q,i_q)_q}$ be a Bourgain–Delbaen space. For every $\gamma \in \Gamma$, we define two bounded linear functionals $c_\gamma^*, d_\gamma^* : \mathfrak{X}_{(\Gamma_q,i_q)_q} \to \mathbb{R}$ as follows:

(i) if $\gamma \in \Delta_0$, then $c_\gamma^* = 0$ and otherwise if $\gamma \in \Delta_{q+1}$ for some $q \in \mathbb{N} \cup \{0\}$, then $c_\gamma^* = e_\gamma^* \circ i_q \circ r_q$; and
(ii) $d_\gamma^* = e_\gamma^* - c_\gamma^*$ for all $\gamma \in \Gamma$.

Remark 2.15. By Remark 2.9, we obtain that if $q \in \mathbb{N} \cup \{0\}$ and $\gamma \in \Delta_{q+1}$, then $c_\gamma^* = e_\gamma^* \circ P_{[0,q]}$ and hence $d_\gamma^* = e_\gamma^* \circ P_{(q,\infty)}$. Also, using the extension property of the operators i_q, it easily follows from Remark 2.9 that for $p \in \mathbb{N}\cup\{0\}$ and $\gamma \in \Gamma_p$ we have that $e_\gamma^* = e_\gamma^* \circ P_{[0,p]}$, which also implies that if $\gamma \in \Delta_p$, then $d_\gamma^* = e_\gamma^* \circ P_{\{p\}}$. Moreover, for $\gamma \in \Delta_0$, $d_\gamma^* = e_\gamma^*$.

LEMMA 2.16. *Let $\mathfrak{X}_{(\Gamma_q,i_q)_q}$ be a Bourgain–Delbaen space. For all $q \in \mathbb{N}\cup\{0\}$ and $\gamma_0 \in \Delta_{q+1}$, the functional $c_{\gamma_0}^*$ is in the linear span of $\{e_\gamma^* : \gamma \in \Gamma_q\}$.*

Proof. It suffices to show that $\bigcap_{\gamma\in\Gamma_q}\ker e_\gamma^* \subset \ker c_{\gamma_0}^*$ and to that end let $x \in \mathfrak{X}_{(\Gamma_q,i_q)_q}$ with $e_\gamma^*(x) = 0$ for all $\gamma \in \Gamma_q$. Recall that x is also a vector in $\ell_\infty(\Gamma)$ and in particular we have that $x(\gamma) = e_\gamma^*(x) = 0$ for all $\gamma \in \Gamma_q$ and hence $r_q(x) = 0$. By the definition of $c_{\gamma_0}^*$, it easily follows that $c_{\gamma_0}^*(x) = 0$. □

PROPOSITION 2.17. *Let* $\mathfrak{X}_{(\Gamma_q,i_q)_q}$ *be a Bourgain–Delbaen space. The following hold.*

(i) *The functionals* $(d_\gamma^*)_{\gamma\in\Gamma}$ *are biorthogonal to the vectors* $(d_\gamma)_{\gamma\in\Gamma}$.

(ii) *For all* $q \in \mathbb{N}\cup\{0\}$, *we have that* $\langle\{d_\gamma^* : \gamma\in\Gamma_q\}\rangle = \langle\{e_\gamma^* : \gamma\in\Gamma_q\}\rangle$. *In particular, the closed linear span of the functionals* $(d_\gamma^*)_{\gamma\in\Gamma}$ *is* C*-isomorphic to* ℓ_1, *where* $C = \sup_q \|i_q\|$.

(iii) *If moreover the FDD* $(M_q)_q$ *is shrinking, then the closed linear span of the functionals* $(d_\gamma^*)_{\gamma\in\Gamma}$ *is* $\mathfrak{X}^*_{(\Gamma_q,i_q)_q}$. *In particular,* $\mathfrak{X}^*_{(\Gamma_q,i_q)_q}$ *is* C*-isomorphic to* ℓ_1.

Proof. The first statement easily follows from Remark 2.15, in particular the fact that for $q\in\mathbb{N}\cup\{0\}$ and $\gamma\in\Delta_q$, we have that $d_\gamma^* = e_\gamma^*\circ P_{\{q\}}$.

For the proof of (ii), observe that by Lemma 2.16 and $d_\gamma^* = e_\gamma^* - c_\gamma^*$, we have that $\langle\{d_\gamma^* : \gamma\in\Gamma_q\}\rangle \subset \langle\{e_\gamma^* : \gamma\in\Gamma_q\}\rangle$. Moreover, (i) implies that $\dim\langle\{d_\gamma^* : \gamma\in\Gamma_q\}\rangle = \#\Gamma_q$ and hence the inclusion cannot be proper. The last part of statement (ii) follows from Proposition 2.13.

To prove the last statement, note that if $(M_q)_q$ is shrinking, then so is the basis $((d_\gamma)_{\gamma\in\Delta_q})_{q=0}^\infty$. From (ii) the desired result follows. □

§3. *Separable* $\mathcal{L}_\infty$*-spaces are Bourgain–Delbaen spaces.* We combine some simple remarks concerning Bourgain–Delbaen spaces with results from [**9**], [**10**] and [**14**] to prove that whenever X is an infinite-dimensional separable $\mathcal{L}_\infty$-space, then X is isomorphic to a Bourgain–Delbaen space.

LEMMA 3.1. *Let* $(\mu_i)_i$ *be a bounded sequence in* $\mathcal{M}[0,1]$. *Then there exists a sequence* $(t_i)_i$ *of elements of* $[0,1]$ *such that if* $\nu_i = \mu_i + \delta_{t_i}$ *for all* $i\in\mathbb{N}$, *then the following hold:*

(i) *the sequence* $(\nu_i)_i$ *is equivalent to the unit vector basis of* $\ell_1(\mathbb{N})$; *and*

(ii) *the space* $Y = [(\nu_i)_i]$ *is complemented in* $\mathcal{M}[0,1]$.

Proof. Find a sequence $(t_i)_i$ of elements of $[0,1]$ so that $\mu_j(\{t_i\}) = 0$ for all $i,j\in\mathbb{N}$. Setting $\nu_i = \mu_i + \delta_{t_i}$, it can be shown that $(\nu_i)_i$ is equivalent to the basis of ℓ_1 and that the map $S\mu = \sum_{i=1}^\infty \mu(\{t_i\})\nu_i$ is a bounded projection onto the space $[(\nu_i)_i]$. □

LEMMA 3.2. *Let* X *be an infinite-dimensional and separable* $\mathcal{L}_\infty$*-space. Then there exists a sequence* $(x_i^*)_i$ *in* X^* *satisfying the following:*

(i) *the sequence* $(x_i^*)_i$ *is equivalent to the unit vector basis of* $\ell_1(\mathbb{N})$;

(ii) *there exists a constant* $\theta > 0$ *such that* $\theta\|x\| \leqslant \sup_i |x_i^*(x)|$ *for all* $x\in X$; *and*

(iii) *the space* $Y = [(x_i^*)_i]$ *is complemented in* X^*.

Proof. As is shown in [**14**], the dual of X is isomorphic either to $\ell_1(\mathbb{N})$ or to $\mathcal{M}[0,1]$ (see also [**10**]). In the first case, just choose a Schauder basis $(x_i^*)_i$ of X^* which is equivalent to the unit vector basis of $\ell_1(\mathbb{N})$.

In the second case, let $T : X^* \to \mathcal{M}[0,1]$ be an onto isomorphism and choose a normalized sequence $(z_i^*)_i$ in X^* such that $\|x\| = \sup_i |z_i^*(x)|$ for all $x \in X$. Fix $C > \|T^{-1}\|$ and apply Lemma 3.1 to the sequence $(\mu_i)_i$ with $\mu_i = CTz_i^*$ for all $i \in \mathbb{N}$ to find a sequence $(t_i)_i$ in $[0,1]$ such that if $\nu_i = \mu_i + \delta_{t_i}$ for all $i \in \mathbb{N}$, then $(\nu_i)_i$ satisfies the conclusion of that lemma. Setting $x_i^* = T^{-1}\nu_i$ for all $i \in \mathbb{N}$, it is not hard to check that, for $\theta = C - \|T^{-1}\|$, the sequence $(x_i^*)_i$ is the desired one. □

LEMMA 3.3. *Let X be a $\mathcal{L}_{\infty,\lambda}$-space, Y be a subspace of X^* and assume that there exists a constant $\theta > 0$ so that for all $x \in X$, $\theta\|x\| \leqslant \sup\{|y^*(x)| : y^* \in B_{Y^*}\}$. Then for every finite-dimensional subspace F of X and every $\varepsilon > 0$ there exists a finite-rank operator $T : X \to X$ satisfying the following:*

(i) $\|Tx - x\| \leqslant \varepsilon\|x\|$ *for all* $x \in F$;
(ii) $\|T\| \leqslant \lambda/\theta$;
(iii) $T^*[X^*] \subset Y$.

In particular, if X is separable, then there exists a sequence of finite-rank operators $T_n : X \to X$ with $T_n x \to x$ for all $x \in X$ and $T_n^*[X^*] \subset Y$ for all $n \in \mathbb{N}$.

Proof. Let F be a finite-dimensional subspace of X and $\varepsilon > 0$. By passing to a larger subspace, we may assume that $F = \langle\{y_i : i = 1, \ldots, n\}\rangle$, where the map $A : \ell_\infty^n \to F$ with $Ae_i = y_i$ is invertible with $\|A\|\|A^{-1}\| \leqslant \lambda$. By the Hahn–Banach theorem, there is a sequence $(y_i^*)_{i=1}^n$ in X^*, biorthogonal to $(y_i)_{i=1}^n$, such that $\|y_i^*\| \leqslant \|A^{-1}\|$ for $i = 1, \ldots, n$. A separation theorem yields that $B_{X^*} \subset \theta^{-1}\overline{B_Y}^{w^*}$ and hence we may choose $\tilde{y}_i^* \in (\theta^{-1}\|A^{-1}\|)B_Y$ such that $\|(\tilde{y}_i^* - y_i^*)|_F\| < \varepsilon/\|A\|$ for $i = 1, \ldots, n$. Define $T : X \to X$ with $Tx = \sum_{i=1}^n \tilde{y}_i^*(x)y_i$. Some standard calculations yield that T is the desired operator. □

The following terminology is from [9]. If E_1, E_2 are subspaces of a Banach space X and $\varepsilon > 0$, we say that E_2 is ε-close to E_1 if there is an invertible operator T from E_1 onto E_2 with $\|Tx - x\| \leqslant \varepsilon\|x\|$ for all $x \in E_1$. Note that if E_2 is ε-close to E_1, then E_1 is $\varepsilon/(1-\varepsilon)$-close to E_2.

LEMMA 3.4. *Let X be a Banach space and $(x_i^*)_{i=1}^\infty$ be a sequence in X^* which is equivalent to the unit vector basis of $\ell_1(\mathbb{N})$. Assume moreover that there exists a sequence of bounded finite-rank projections $(Q_n)_n$ satisfying the following.*

(i) $Q_n : X \to X$, $Q_n[X] \subset Q_{n+1}[X]$ *for all* $n \in \mathbb{N}$ *and* $Q_n x \to x$ *for all* $x \in X$.
(ii) *There are a strictly increasing sequence of natural numbers $(k_n)_n$ and $0 < \varepsilon < 1$ with $\varepsilon/(1-\varepsilon) < (\sup_n \|Q_n\|)^{-1}$ such that $Q_n^*[X^*]$ is ε-close to the space $\langle\{x_i^* : i = 1, \ldots, k_n\}\rangle$ for all $n \in \mathbb{N}$.*

Then X is isomorphic to a Bourgain–Delbaen space.

Proof. Note that (i) implies that $\bigcup_n Q_n[X]$ is dense in X. Define a linear operator $U : X \to \ell_\infty(\mathbb{N})$ with $Ux = (x_i^*(x))_i$ for all $x \in X$; evidently U is bounded. Set $\Gamma_n = \{1, \dots, k_n\}$ for all $n \in \mathbb{N}$ and $Y_n = UQ_n[X]$. We shall prove that U is an isomorphic embedding and that the restriction operators onto the first k_n coordinates $r_{k_n} : Y_n \to \ell_\infty(\{1, \dots, k_n\})$ are onto C-isomorphisms for all $n \in \mathbb{N}$ for a uniform constant C. Given the aforementioned facts and Remark 2.7, the desired result follows easily.

Since $\bigcup_n Q_n[X]$ is dense in X, to show that U is an isomorphic embedding, it is enough to find a uniform constant $c > 0$ such that $\|Ux\| \geqslant c\|x\|$ for every $x \in Q_n[X]$ and for every $n \in \mathbb{N}$. To this end, let $S : [(x_i^*)_i] \to \ell_1(\mathbb{N})$ be the onto isomorphism with $Sx_i^* = e_i$ for all $i \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $x \in Q_n[X]$. The Hahn–Banach theorem and the fact that $\langle\{x_i^* : i = 1, \dots, k_n\}\rangle$ is $\varepsilon/(1-\varepsilon)$-close to $Q_n^*[X^*]$, in conjunction with some tedious computations, yield that there exists an $i_0 \in \{1, \dots, k_n\}$ such that

$$|x_{i_0}^*(x)| \geqslant (1-\varepsilon)\frac{1 - (\varepsilon/(1-\varepsilon))\|Q_n\|}{\|S\|\|Q_n\|}\|x\|. \tag{6}$$

Setting $c = (1-\varepsilon)(1 - (\varepsilon/(1-\varepsilon))\sup_k \|Q_k\|)/(\|S\| \sup_k \|Q_k\|)$, we conclude that $\|Ux\| \geqslant c\|x\|$.

It remains to show that there a exists a constant $C > 0$ such that $r_{k_n} : Y_n \to \ell_\infty(\{1, \dots, k_n\})$ is an onto C-isomorphism for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $z \in Y_n$. Set $x = U^{-1}z \in Q_n[X]$. Then, by (6), there is an $i_0 \in \{1, \dots, k_n\}$ such that

$$\|z\| \geqslant \|r_{k_n}(z)\| \geqslant |x_{i_0}^*(x)| \geqslant c\|x\| \geqslant \frac{c}{\|U\|}\|z\|.$$

Setting $C = \|U\|/c$, we conclude that $r_{k_n}|_{Y_n}$ is a C-isomorphic embedding. Finally, observe that $\dim Y_n = \dim Q_n[X] = \dim Q_n^*[X^*] = k_n$ and hence $r_{k_n}|_{Y_n}$ is also onto. □

The following lemma and its proof can be found in [9, Lemma 4.3].

LEMMA 3.5. *Let X be a separable Banach space and Y be a separable subspace of X^*. Assume moreover that $(P_n)_n$, $(T_n)_n$ are bounded finite-rank operators satisfying the following conditions.*

(a) $P_n : X \to X$, $P_n^*[X^*] \subset Y$ *and* $T_n : X^* \to Y$ *for all* $n \in \mathbb{N}$.
(b) $P_n x \to x$ *for all* $x \in X$, $T_n y \to y$ *for all* $y \in Y$ *and* $\sup_n \|T_n\| < \infty$.
(c) *The operators* $(T_n)_n$ *are projections.*

If E, F are finite-dimensional subspaces of X, Y respectively and $0 < \varepsilon < 1$, then there exists a projection $Q : X \to X$ with finite-dimensional range such that:

(i) $Qe = e$ *for all* $e \in E$;
(ii) $Q^* f = f$ *for all* $f \in F$;
(iii) $Q^*[X^*] \subset Y$;
(iv) $\|Q\| \leqslant 2c + 2K + 4cK$, *where* $c = \sup_n \|P_n\|$ *and* $K = \sup_n \|T_n\|$;
(v) $Q^*[X^*]$ *is* ε*-close to* $T_n[X^*]$ *for some integer* n.

THEOREM 3.6. *Every separable infinite-dimensional $\mathcal{L}_\infty$-space is isomorphic to a Bourgain–Delbaen space.*

Proof. Let X be a separable infinite-dimensional $\mathcal{L}_\infty$-space. By Lemma 3.2, there exist a sequence $(x_i^*)_i$ in X^* which is equivalent to the unit vector basis of $\ell_1(\mathbb{N})$, a constant $\theta > 0$ such that $\theta\|x\| \leqslant \sup_i |x_i^*(x)|$ for all $x \in X$ and a bounded linear projection $P : X^* \to Y = [(x_i^*)_i]$. For $n \in \mathbb{N}$, define $T_n : X^* \to Y$ with $T_n = S_n \circ P$, where $S_n : Y \to Y$ denotes the basis projection of $(x_i^*)_i$ onto the first n coordinates. Also apply Lemma 3.3 to find a sequence of finite-rank operators $P_n : X \to X$ such that $P_n x \to x$ for all $x \in X$ and $P_n^*[X^*] \subset Y$ for all $n \in \mathbb{N}$. Assumptions (a), (b) and (c) of Lemma 3.5 are evidently satisfied.

Choose $\varepsilon > 0$ with $\varepsilon/(1-\varepsilon) < 1/(2c+2K+4cK)$, where $c = \sup_n \|P_n\|$ and $K = \sup_n \|T_n\|$. Recursively, setting $E_n = P_n[X] + Q_{n-1}[X]$, where Q_0 is the zero operator on X, and $F_n = \{0\}$, using Lemma 3.5, choose $(Q_n)_n$, a sequence of bounded linear projections on X, such that $P_n[X] \subset Q_n[X] \subset Q_{n+1}[X]$, $\|Q_n\| \leqslant 2c+2K+4cK$ for all $n \in \mathbb{N}$ and there exist natural numbers $(k_n)_n$ such that $Q_n^*[X^*]$ is ε-close to $T_{k_n}[X^*] = \langle\{x_i^* : i = 1, \ldots, k_n\}\rangle$ for all $n \in \mathbb{N}$. Note that $\dim Q_n[X] \to \infty$ and hence, by passing to a subsequence, we may assume that $(k_n)_n$ is strictly increasing. We conclude that the sequences $(x_i^*)_{i=1}^\infty$ and $(Q_n)_n$ witness the fact that the space X satisfies the assumptions of Lemma 3.4 and therefore it is isomorphic to a Bourgain–Delbaen space. □

§4. *Separable infinite-dimensional $\mathcal{L}_\infty$-spaces contain $\mathcal{L}_\infty$-subspaces of infinite co-dimension.* Using the main result of §3, we prove that every infinite-dimensional $\mathcal{L}_\infty$-space X contains an infinite-dimensional $\mathcal{L}_\infty$-subspace Z, so that the quotient X/Z is isomorphic to c_0.

LEMMA 4.1. *Let $\mathfrak{X}_{(\Gamma_q,i_q)_q}$ be a Bourgain–Delbaen space and assume that there exists a decreasing sequence of positive real numbers $(\varepsilon_q)_{q=1}^\infty$, with $2C^2\sum_q \varepsilon_q < 1$, where $C = \sup_q \|i_q\|$, so that for every $q \in \mathbb{N}$ there exist $\gamma_1^q, \gamma_2^q \in \Delta_q$ with $\gamma_1^q \neq \gamma_2^q$ satisfying $\|e^*_{\gamma_1^q} \circ i_p \circ r_p - e^*_{\gamma_2^q} \circ i_p \circ r_p\| < \varepsilon_q$ for $p = 0, 1, \ldots, q-1$. Then $\mathfrak{X}_{(\Gamma_q,i_q)_q}$ contains an infinite-dimensional $\mathcal{L}_\infty$-subspace Z.*

Proof. Define $R_q = \{\gamma_1^q, \gamma_2^q\}$, $S_q = \bigcup_{p=1}^q R_p$ for $q \in \mathbb{N}$ and $S = \bigcup_{q=1}^\infty S_q$. Define $N_0 = M_0 = \langle\{d_\gamma : \gamma \in \Delta_0\}\rangle$ and

$$N_q = \langle\{d_\gamma : \gamma \in \Delta_q \setminus R_q\} \cup \{d_{\gamma_1^q} + d_{\gamma_2^q}\}\rangle$$

for $q \in \mathbb{N}$. Note that N_q is a subspace of M_q of co-dimension one for every $q \in \mathbb{N}$. Define

$$Z_q = \langle\{d_\gamma : \gamma \in \Gamma_q \setminus S_q\} \cup \{d_{\gamma_1^p} + d_{\gamma_2^p} : p = 1, \ldots, q\}\rangle$$

for all $q \in \mathbb{N}$. Observe that $Z_q = N_0 + \cdots + N_q$ for all $q \in \mathbb{N}$ and that $(Z_q)_q$ is an increasing sequence of finite-dimensional spaces whose union is dense in the space

$$Z = \overline{\langle\{d_\gamma : \gamma \in \Gamma \setminus S\} \cup \{d_{\gamma_1^q} + d_{\gamma_2^q} : q \in \mathbb{N}\}\rangle}.$$

We shall prove that Z is the desired subspace. To begin, Z is clearly infinite dimensional. More precisely, the spaces $(N_q)_q$ form an FDD for the space Z with a projection constant C.

To conclude that Z is indeed a $\mathcal{L}_\infty$-space, it suffices to show that for each $q \in \mathbb{N}$ the space Z_q is $((1+\varepsilon)/(1-\varepsilon))C$-isomorphic to $\ell_\infty^{n_q}$, where $\varepsilon = 2C^2 \sum_{q=1}^{\infty} \varepsilon_q$ and $n_q = \dim Z_q = \#\Gamma_q - q$. For $q \in \mathbb{N}$, we define a subspace of $\ell_\infty(\Gamma_q)$ as follows:

$$W_q = \langle \{e_\gamma : \gamma \in \Gamma_q \setminus S_q\} \cup \{e_{\gamma_1^p} + e_{\gamma_2^p} : p = 1, \ldots, q\} \rangle.$$

Also define $\widetilde{W}_q = i_q[W_q]$. Observe that W_q is isometric to $\ell_\infty^{n_q}$ and, therefore, by Remark 2.6, $\widetilde{W}_q$ is C-isomorphic to $\ell_\infty^{n_q}$. Hence, if we find a linear map $T_q : Z_q \to \widetilde{W}_q$ with $\|T_q x - x\| \leqslant \varepsilon \|x\|$ for all $x \in Z_q$, the proof will be complete.

Let us fix $q \in \mathbb{N}$ and observe that if $x \in N_p$, for some $0 \leqslant p \leqslant q$, then $x = i_p(y)$, where $y \in \langle \{e_\gamma : \gamma \in \Delta_q \setminus S_q\} \cup \{e_{\gamma_1^q} + e_{\gamma_2^q}\} \rangle$. For $p = 0, \ldots, q$, we define $T_{q,p} : N_p \to W_q$ as follows:

$$T_{q,p}(x)(\gamma) = \begin{cases} y(\gamma) & \text{if } \operatorname{rank}(\gamma) \leqslant p, \\ i_p(y)(\gamma) & \text{if } \operatorname{rank}(\gamma) > p \text{ and } \gamma \notin S_q, \\ i_p(y)(\gamma_1^s) & \text{if } \operatorname{rank}(\gamma) > p \text{ and } \gamma = \gamma_1^s \text{ for some } s \in (p, q], \\ i_p(y)(\gamma_1^s) & \text{if } \operatorname{rank}(\gamma) > p \text{ and } \gamma = \gamma_2^s \text{ for some } s \in (p, q]. \end{cases}$$

Observe that the map is linear and well defined. Also observe that if $x(\gamma) \neq T_{q,p}(x)(\gamma)$, for some $\gamma \in \Gamma_q$, then necessarily there is an $s \in (p, q]$ so that $\gamma = \gamma_2^s$ and $T_{q,p}(x)(\gamma) = i_p(y)(\gamma_1^s) = e_{\gamma_1^s}^* \circ i_p \circ r_p(x)$. Hence, for such a $\gamma \in \Gamma_q$,

$$|x(\gamma) - T_{q,p}(x)(\gamma)| = |e_{\gamma_2^s}^* \circ i_p \circ r_p(x) - e_{\gamma_1^s}^* \circ i_p \circ r_p(x)| < \varepsilon_s \|x\| \leqslant \varepsilon_{p+1} \|x\|.$$

We conclude that $\|r_q(x) - T_{q,p}(x)\| \leqslant \varepsilon_{p+1} \|x\|$ (actually observe that if $p = q$, then $r_q(x) = T_{q,q}(x)$) and hence if we define $\widetilde{T}_{q,p} : N_p \to \widetilde{W}_q$ with $\widetilde{T}_{q,p} = i_q \circ T_{q,p}$, then

$$\|x - \widetilde{T}_{q,p}(x)\| \leqslant C\varepsilon_{p+1} \|x\|$$

for all $x \in N_p$ and $p = 0, \ldots, q$. As we previously mentioned, $(N_p)_p$ is an FDD for Z with a projection constant C, so we may consider the associated projections $Q_{\{p\}} : Z \to N_p$ for all p. Define $T_q : Z_q \to \widetilde{W}_q$ with $T_q = \sum_{p=0}^{q} \widetilde{T}_{q,p} \circ Q_{\{p\}}$. Some simple calculations using $\|Q_{\{p\}}\| \leqslant 2C$ for all p yield that T_q is the desired operator. □

We recall that it is known that separable $\mathcal{L}_\infty$-spaces have c_0 as a quotient [2, 13].

LEMMA 4.2. *Let $\mathfrak{X}_{(\Gamma_q, i_q)_q}$ be a Bourgain–Delbaen space satisfying the assumptions of Lemma 4.1. If Z is the subspace constructed in that lemma, then the quotient $\mathfrak{X}_{(\Gamma_q, i_q)_q}/Z$ is isomorphic to $c_0(\mathbb{N})$.*

Proof. Following the notation of the proof of Lemma 4.1, let Z be the constructed subspace and let also W be the closed linear span of the vectors $d_{\gamma_1^q} - d_{\gamma_2^q}$, $q \in \mathbb{N}$. Let $Q : \mathfrak{X}_{(\Gamma_q,i_q)_q} \to \mathfrak{X}_{(\Gamma_q,i_q)_q}/Z$ denote the corresponding quotient map and for $q \in \mathbb{N}$ define $y_q = Q(d_{\gamma_1^q} - d_{\gamma_2^q})$. It is not very difficult to prove that $(y_q)_q$ is a Schauder basis of $\mathfrak{X}_{(\Gamma_q,i_q)_q}/Z$ with projection constant $C = \sup_q \|i_q\|$. Let $w_q^* = 1/2(d^*_{\gamma_1^q} - d^*_{\gamma_2^q})$ for all $q \in \mathbb{N}$ and $(y_q^*)_q \subset (\mathfrak{X}_{(\Gamma_q,i_q)_q}/Z)^*$ be the biorthogonal sequence of $(y_q)_q$. It follows that the sequences $(w_q^*)_q$ and $(y_q^*)_q$ are naturally isometrically equivalent and that $\|w_q^* - 2(e^*_{\gamma_1^q} - e^*_{\gamma_2^q})\| \leqslant 2\varepsilon_q$ for all q. Proposition 2.13 yields that $(y_q^*)_q$ is equivalent to the unit vector basis of ℓ_1, which indeed implies that $(y_q)_q$ is equivalent to the unit vector basis of $c_0(\mathbb{N})$. □

PROPOSITION 4.3. *Every separable infinite-dimensional $\mathcal{L}_\infty$-space X contains an infinite-dimensional $\mathcal{L}_\infty$-subspace Z, so that the quotient X/Z is isomorphic to $c_0(\mathbb{N})$. In other words, every separable infinite-dimensional $\mathcal{L}_\infty$-space X is the twisted sum of a $\mathcal{L}_\infty$-space Z and $c_0(\mathbb{N})$.*

Proof. By virtue of Theorem 3.6, we may assume that X is a Bourgain–Delbaen space $\mathfrak{X}_{(\Gamma_q,i_q)_q}$. We start by observing that for every strictly increasing sequence $(q_s)_{s=0}^\infty$ of $\mathbb{N} \cup \{0\}$, the Bourgain–Delbaen spaces $\mathfrak{X}_{(\Gamma_q,i_q)_q}$ and $\mathfrak{X}_{(\Gamma_{q_s},i_{q_s})_s}$ are actually equal. This follows from Proposition 2.12, in particular the fact that $\bigcup_q Y_q$ is dense in $\mathfrak{X}_{(\Gamma_q,i_q)_q}$. It is therefore sufficient to find an appropriate increasing sequence $(q_s)_{s=0}^\infty$ so that the space $\mathfrak{X}_{(\Gamma_{q_s},i_{q_s})_s}$ satisfies the assumptions of Lemma 4.1 (and hence also those of Lemma 4.2).

Fix a decreasing sequence of positive real numbers $(\varepsilon_q)_{q=1}^\infty$, with $\sum_q \varepsilon_q < 1/(2C^2)$, where $C = \sup_q \|i_q\|$. A compactness argument yields that for every $\varepsilon > 0$ and $q \in \mathbb{N} \cup \{0\}$, there exist non-equal $\gamma_1, \gamma_2 \in \Gamma \setminus \Gamma_q$ so that $\|e^*_{\gamma_1} \circ i_p \circ r_p - e^*_{\gamma_2} \circ i_p \circ r_p\| < \varepsilon$ for $p = 0, \dots, q$. Set $q_0 = 0$ and, using the above, recursively choose a strictly increasing sequence $(q_s)_{s=0}^\infty$ so that for every $s \in \mathbb{N}$ there are distinct $\gamma_1^s, \gamma_2^s \in \Gamma_{q_s} \setminus \Gamma_{q_{s-1}}$ with $\|e^*_{\gamma_s^1} \circ i_{q_t} \circ r_{q_t} - e^*_{\gamma_s^2} \circ i_{q_t} \circ r_{q_t}\| \leqslant \varepsilon_s$ for $t = 0, \dots, s-1$. □

§5. *A $\mathcal{L}_\infty$ and asymptotic c_0 space not containing c_0.* We employ the Bourgain–Delbaen method to define an isomorphic ℓ_1-predual $\mathfrak{X}_0$, which is asymptotic c_0 and does not contain a copy of c_0. We follow notation similar to that used in [**6**] and [**5**].

5.1. *Definition of the space $\mathfrak{X}_0$.* We fix a natural number $N \geqslant 3$ and a constant $1 < \theta < N/2$. Define $\Delta_0 = \{0\}$ and assume that we have defined the sets $\Delta_0, \Delta_1, \dots, \Delta_q$. We set $\Gamma_p = \bigcup_{i=0}^p \Gamma_i$ and, for each $\gamma \in \Gamma_q$, we denote by $\text{rank}(\gamma)$ the unique p so that $\gamma \in \Delta_p$. Assume that to each $\gamma \in \Gamma_q \setminus \Gamma_0$ we have assigned a natural number in $\{1, \dots, n\}$, called the age of γ and denoted by $\text{age}(\gamma)$.

Assume moreover that we have defined extension functionals $(c^*_\gamma)_{\gamma \in \Delta_p}$ and extension operators $i_{p-1,p} : \ell_\infty(\Gamma_{p-1}) \to \ell_\infty(\Gamma_p)$ so that

$$i_{p-1,p}(x)(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \in \Gamma_{p-1}, \\ c_\gamma^*(x) & \text{if } \gamma \in \Delta_p \end{cases} \tag{7}$$

for $p = 1, \dots, q$. If $0 \leqslant p < s \leqslant q$, we define $i_{p,s} = i_{s-1,s} \circ \cdots \circ i_{p,p+1}$ and we also denote by $i_{p,p}$ the identity operator on $\ell_\infty(\Gamma_p)$.

We define Δ_{q+1} to be the set of all tuples of one of the two forms described below:

$$(q+1, n, (\varepsilon_i)_{i=1}^k, (E_i)_{i=1}^k, (\eta_i)_{i=1}^k), \tag{8a}$$

where $n \leqslant (\#\Gamma_q)^2$, $1 \leqslant k \leqslant n$, $\varepsilon_i \in \{-1, 1\}$ for $i = 1, \dots, k$, $(E_i)_{i=1}^k$ is a sequence of successive intervals of $\{0, \dots, q\}$ and $\eta_i \in \Gamma_q$ with $\text{rank}(\eta_i) \in E_i$ for $i = 1, \dots, k$, or

$$(q+1, \xi, n, (\varepsilon_i)_{i=1}^k, (E_i)_{i=1}^k, (\eta_i)_{i=1}^k), \tag{8b}$$

where $\xi \in \Gamma_{q-1} \setminus \Gamma_0$ with $\text{age}(\xi) < N$, $(\#\Gamma_{\text{rank}(\xi)})^2 \leqslant n \leqslant (\#\Gamma_q)^2$, $1 \leqslant k \leqslant n$, $\varepsilon_i \in \{-1, 1\}$ for $i = 1, \dots, k$, $(E_i)_{i=1}^k$ is a sequence of successive intervals of $\{\text{rank}(\xi) + 1, \dots, q\}$ and $\eta_i \in \Gamma_q$ with $\text{rank}(\eta_i) \in E_i$ for $i = 1, \dots, k$.

To each $\gamma \in \Delta_{q+1}$, we assign an extension functional $c_\gamma^* : \ell_\infty(\Delta_q) \to \mathbb{R}$. If $0 \leqslant p \leqslant q$, we define the projection $P_{[0,p]}^q x = i_{p,q} \circ r_p(x)$ while if $0 \leqslant p \leqslant s \leqslant q$, we define the projection $P_{(p,s]}^q = P_{[0,s]}^q - P_{[0,p]}^q$. If $\gamma \in \Delta_{q+1}$ is of the form (8*a*), we set $\text{age}(\gamma) = 1$ and define the extension functional c_γ^* as follows:

$$c_\gamma^* = \frac{\theta}{N} \frac{1}{n} \sum_{i=1}^k \varepsilon_i e_{\eta_i}^* \circ P_{E_i}^q. \tag{9a}$$

If $\gamma \in \Delta_{q+1}$ is of the form (8*b*), we set $\text{age}(\gamma) = \text{age}(\xi) + 1$ and define the extension functional $c_\gamma^* : \ell_\infty(\Delta_q) \to \mathbb{R}$ as follows:

$$c_\gamma^* = e_\xi^* + \frac{\theta}{N} \frac{1}{n} \sum_{i=1}^k \varepsilon_i e_{\eta_i}^* \circ P_{E_i}^q. \tag{9b}$$

The inductive construction is complete. We set $\Gamma = \bigcup_{q=1}^\infty \Delta_q$ and, for each $q \in \mathbb{N} \cup \{0\}$, we define the extension operator $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$ by the rule

$$i_q(x)(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \in \Gamma_q, \\ i_{q,p}(x)(\gamma) & \text{if } \gamma \in \Delta_p \text{ for some } p > q. \end{cases}$$

Standard arguments yield that $(i_q)_{q=0}^\infty$ is a compatible sequence of extension operators with $\sup_q \|i_q\| \leqslant N/(N - 2\theta)$. We denote by $\mathfrak{X}_0$ the resulting Bourgain–Delbaen space $\mathfrak{X}_{(\Gamma_q, i_q)_q}$. We shall use the notation from §2.1.

Remark 5.1. By Remark 2.9, we obtain that for every interval E of the natural numbers, $\|P_E\| \leqslant 2N/(N - 2\theta)$. It follows that for every $\gamma \in \Gamma$ and such E, $\|e_\gamma^* \circ P_E\| \leqslant 2N/(N - 2\theta)$, in particular $\|d_\gamma^*\| \leqslant 2N/(N - 2\theta)$.

Remark 5.2. We enumerate the set $\Gamma = \bigcup_{q=0}^{\infty} \Delta_q$ in such a manner that the sets Δ_q correspond to successive intervals of $\mathbb{N}$. If we denote this enumeration by $\Gamma = \{\gamma_i :\in \mathbb{N}\}$, according to Remark 2.10, $(d_{\gamma_i})_i$ forms a Schauder basis for $\mathfrak{X}_0$. It is with respect to this basis that we show that the space $\mathfrak{X}_0$ is asymptotic c_0. However, when we write P_E we shall mean the Bourgain–Delbaen projection onto E as defined in Remark 2.9.

Arguing as in [5, Proposition 4.5], each e_γ^* admits an analysis.

PROPOSITION 5.3. *Let $\gamma \in \Gamma$ with $\operatorname{rank}(\gamma) > 0$. The functional e_γ^* admits a unique analysis of the following form:*

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + \frac{\theta}{N} \sum_{r=1}^{a} \frac{1}{n_r} \sum_{i=1}^{k_r} \varepsilon_{r,i} e_{\eta_{r,i}}^* \circ P_{E_{r,i}}, \tag{10}$$

where $a = \operatorname{age}(\gamma) \leqslant N$, the ξ_i are in $\Gamma \setminus \Gamma_0$ with $\xi_a = \gamma$, the $E_{r,i}$ are intervals of $\mathbb{N} \cup \{0\}$ with $E_{1,1} < \cdots < E_{1,k_1} < \operatorname{rank}(\xi_1) < E_{2,1} < \cdots < E_{a,k_a} < \operatorname{rank}(\xi_a)$, the $\eta_{r,i}$ are in Γ with $\operatorname{rank}(\eta_{r,i}) \in E_{r,i}$, the $\varepsilon_{r,i}$ are in $\{-1, 1\}$, $n_r > (\#\Gamma_{\operatorname{rank}(\xi_{r-1})})^2$ for $r = 2, \ldots, a$ and $1 \leqslant k_r \leqslant n_r$ for $r = 1, \ldots a$.

Remark 5.4. Note that if in (10) we set $f_r = (1/n_r) \sum_{i=1}^{k_r} \varepsilon_{r,i} e_{\eta_{r,i}}^* \circ P_{E_{r,i}}$, then $(f_r)_{r=1}^{a}$ constitutes a very fast growing sequence of α-averages, in the sense of [3].

5.2. *The main property of the space $\mathfrak{X}_0$.*

LEMMA 5.5. *Let $x_1 < \cdots < x_m$ be blocks of $(d_{\gamma_i})_i$ of norm at most one, $E_1 < \cdots < E_k$ be intervals of $\mathbb{N} \cup \{0\}$, $(\eta_i)_{i=1}^k$ be a sequence in Γ with $\operatorname{rank}(\eta_i) \in E_i$, for $i = 1, \ldots, k$, $(\varepsilon_i)_{i=1}^k$ be a sequence in $\{-1, 1\}$ and let $n \geqslant \max\{m^2, k\}$. Then*

$$\left| \left(\frac{1}{n} \sum_{i=1}^{k} \varepsilon_i e_{\eta_i}^* \circ P_{E_i} \right) \left(\sum_{j=1}^{m} x_j \right) \right| \leqslant \frac{4N}{N - 2\theta}. \tag{11}$$

Proof. We shall consider the support and range of vectors with respect to the basis $(d_{\gamma_i})_i$. Set $x_i^* = \varepsilon_i e_{\eta_i}^* \circ P_{E_i}$ for $i = 1, \ldots, k$. Note that $(x_i^*)_{i=1}^k$ is a successive block sequence of $(d_{\gamma_i}^*)_i$ and, by Remark 5.1, $\|x_i^*\| \leqslant 2N/(N - 2\theta)$ for $i = 1, \ldots, k$. Let I_1 be the set of those $i \leqslant k$ such that the support of x_i^* intersects the range of at least two of the x_j. Then $\#I_1 \leqslant m$ and so $|((1/n) \sum_{i \in I_1} x_i^*)(\sum_{j=1}^m x_j)| \leqslant (2N/(N - 2\theta))m^2/n \leqslant 2N/(N - \theta)$. Let $I_2 = \{i \leqslant n : i \notin I_1\}$. It is clear that $|((1/n) \sum_{i \in I_2} x_i^*)(\sum_{j=1}^m x_j)| \leqslant 2N/(N - 2\theta)$ and the proof is complete. □

PROPOSITION 5.6. *Set $K_{N,\theta} = (2N^3 + 4\theta N^2 - 4\theta N)/(N^2 - 3\theta N + 2\theta^2)$ and let $u_1 < \cdots < u_m$ be blocks of $(d_{\gamma_i})_i$ of norm at most equal to one so that if we consider the support of the vectors u_i with respect to the basis $(d_{\gamma_i})_i$, then $m \leqslant \min \operatorname{supp} u_1$. Then $\|\sum_{i=1}^m u_i\| \leqslant K_{N,\theta}$.*

Proof. Set $u = \sum_{i=1}^{m} u_i$. We use induction on rank(γ) to show that for every $\gamma \in \Gamma$ and every interval E of $\mathbb{N}$, $|e_\gamma^* \circ P_E(u)| \leqslant K_{N,\theta}$. This assertion is easy when rank$(\gamma) = 0$. Assume that $q \in \mathbb{N} \cup \{0\}$ is such that the assertion holds for every $\gamma \in \Gamma_q$ and E interval of $\mathbb{N}$ and let $\gamma \in \Gamma_{q+1} \setminus \Gamma_q$ and E be an interval of $\mathbb{N}$. Applying Proposition 5.3, write

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + \frac{\theta}{N} \sum_{r=1}^{a} \frac{1}{n_r} \sum_{i=1}^{k_r} \varepsilon_{r,i} e_{\eta_{r,i}}^* \circ P_{E_{r,i}},$$

so that the conclusion of that proposition is satisfied. For $r = 1, \dots, a$, define $y_r^* = (1/n_r) \sum_{i=1}^{k_r} \varepsilon_{r,i} e_{\eta_{r,i}}^* \circ P_{E_{r,i} \cap E}$ and $G = \{r : \text{rank}(\xi_r) \in E\}$. Observe that $e_\gamma^* \circ P_E = \sum_{r \in G} d_{\xi_r}^* + (\theta/N) \sum_{r=1}^{a} y_r^*$.

If $\xi_r = \gamma_{i_r}$, note that $i_1 < \cdots < i_a$. Considering the support of u with respect to the basis $(d_{\gamma_i})_i$, set $r_0 = \min\{r : i_r \geqslant \min \operatorname{supp} u\}$. The inductive assumption yields that $|y_{r_0}^*(u)| \leqslant K_{N,\theta}$, while the growth condition on the n_r implies that $n_r > m^2$ for $r > r_0$. Lemma 5.5 yields

$$\frac{\theta}{N} \sum_{r=1}^{a} |y_r^*(u)| \leqslant \frac{\theta}{N} \left(K_{N,\theta} + (N-1) \frac{4N}{N - 2\theta} \right), \tag{12}$$

while Remark 5.1 implies that

$$\sum_{r \in G} |d_{\xi_r}^*(u)| \leqslant N \frac{2N}{N - 2\theta}. \tag{13}$$

Some calculations combining (12) and (13) yield the desired result. □

PROPOSITION 5.7. *The space $\mathfrak{X}_0$ is a $\mathcal{L}_\infty$-space with a Schauder basis $(d_{\gamma_i})_i$ satisfying the following properties.*

(i) *The basis $(d_{\gamma_i})_i$ is shrinking, in particular $\mathfrak{X}_0^*$ is isomorphic to ℓ_1.*
(ii) *The space $\mathfrak{X}_0$ is asymptotic c_0 with respect to the basis $(d_{\gamma_i})_i$.*
(iii) *The space $\mathfrak{X}_0$ does not contain an isomorphic copy of c_0.*

Proof. The fact that $\mathfrak{X}_0$ is asymptotic c_0 with respect to $(d_{\gamma_i})_i$ follows directly from Proposition 5.6. We obtain in particular that $(d_{\gamma_i})_i$ is shrinking and hence, by Proposition 2.17, the dual of $\mathfrak{X}_0$ is isomorphic to ℓ_1. To show that $\mathfrak{X}_0$ does not contain an isomorphic copy of c_0, let us consider the FDD $(M_q)_{q=0}^\infty$ as it was defined in Proposition 2.8. Let $(x_k)_k$ be a sequence of skipped block vectors, with respect to the FDD $(M_q)_{q=0}^\infty$, all of which have norm at least equal to one. Then, for every $\varepsilon > 0$, there exist a finite subset I_1 of $\mathbb{N}$ and $\gamma \in \Gamma$ so that $e_\gamma^*(\sum_{k \in I_1} x_k) \geqslant \theta - \varepsilon$. It follows from this that for all $n \in \mathbb{N}$, we can find a $\gamma \in \Gamma$ and find a finite subset J of $\mathbb{N}$ so that $e_\gamma^*(\sum_{i \in J} x_k) \geqslant (\theta - \varepsilon)^n$. Therefore, c_0 is indeed not isomorphic to a subspace of $\mathfrak{X}_0$. □

Remark 5.8. In [1], Alspach proved that the $\mathcal{L}_\infty$-space with separable dual defined in [6] has Szlenk index ω. We note that the space $\mathfrak{X}_0$ has Szlenk index ω as well. Indeed, if this were not the case, then, by [2, Theorem 1], we would conclude that $\mathfrak{X}_0$ has a quotient isomorphic to $C(\omega^\omega)$, which would imply that $\mathfrak{X}_0$ admits an ℓ_1 spreading model.

Remark 5.9. Every skipped block sequence, with respect to the FDD $(M_q)_{q=1}^\infty$, is boundedly complete and hence the space $\mathfrak{X}_0$ is reflexively saturated and also every block sequence in $\mathfrak{X}_0$ contains a further block sequence which is unconditional. We also note that a similar method can be used to construct a reflexive asymptotic c_0 space with an unconditional basis. This space is related to Tsirelson's original Banach space [15].

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